A NOTE ON THE STRUCTURE OF THE GENERALIZED FIELD THEORY (GFT)

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The metric hypothesis of the unifield field theory of gravitation and electromagnetism is derived alongside the other field equations from a variational principle.

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The structure and interpretation of the nonsymmetric unified field theory of gravitation and electromagnetism (Ref. [1]) have been extensively modified by the adjunction of the metric hypothesis (Ref. [2]). Although it is now known (Ref. [3]) that the latter is a consequence of more fundamental considerations, a theorem rather than an hypothesis, its physical meaning whereby basic measurements are related to macrophysical field laws is that of a law. As such, it should be incorporated in the variational principle from which the theory is derived. This leads to a modification of the field equations of the same type as that by which Einstein and Straus replaced the original, strong field equations. Unlike their result however, which alone is compatible with the principle of charge conjugation, there are good reasons to believe that the present modification is not likely to be of much practical significance. It should nevertheless be recorded.

Using semicolon to denote covariant differentiation with respect to the symmetric part $\tilde{\Gamma}^{\lambda}_{(\mu\nu)}$ of the Schrödinger affine connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ (for which $\tilde{\Gamma}_{\mu} = \tilde{\Gamma}^{\sigma}_{[\mu\sigma]}$ identically vanishes), which is identified with a Christoffel bracket formed from the components $a_{\mu\nu}$ of the space-time metric, we write the extended variational principle in the form

$$\delta \int (\mathfrak{g}^{\mu\nu}R_{\mu\nu}+\Lambda^{\mu\nu\lambda}a_{\mu\nu;\lambda})d^4x = 0. \tag{1}$$

Here $\Lambda^{\mu\nu\lambda}$ are Lagrange multipliers which must be eliminated once the field equations are found. From the transformation point of view, they are densities, and because $a_{\mu\nu}$ is symmetric

$$\Lambda^{\mu\nu\lambda} = \Lambda^{\nu\mu\lambda} \tag{2}$$

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as well. As in Ref. [4] the Ricci tensor $R_{\mu\nu}$ is constructed from the most general quasi--connection $W^{\lambda}_{\mu\nu}$ in terms of which it is automatically conjugation (or transposition) invariant. $W^{\lambda}_{\mu\nu}$ is then related to the geometrical connection $\Gamma^{\lambda}_{\mu\nu}$ by

$$\Gamma^{\lambda}_{\mu\nu} = W^{\lambda}_{\mu\nu}+(2\alpha_1+\tfrac{1}{3})\delta^{\lambda}_{\nu}W^{\sigma}_{(\mu\sigma)}-\tfrac{1}{3}\,\delta^{\lambda}_{\nu}W_{\mu}-(3\alpha_1+1)\delta^{\lambda}_{\mu}W^{\sigma}_{(\nu\sigma)}+(3\alpha_1+2\alpha_2+1)\delta^{\lambda}_{\mu}W_{\nu}, \tag{3}$$

where α_1 and α_2 are numerical parameters (from which the final theory is in any case independent) which satisfy the condition

$$(15\alpha_1+4)\,(9\alpha_1+6\alpha_2+2) \neq 0. \tag{4}$$

The geometrical connection defines $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ by

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}+\tfrac{2}{3}\,\delta^{\lambda}_{\mu}\Gamma_{\nu}. \tag{5}$$

In terms of the "twiddled" connection we have

$$W^{\lambda}_{(\mu\nu)} = \tilde{\Gamma}^{\lambda}_{(\mu\nu)}-\frac{6\alpha_1+1}{2(15\alpha_1+4)}(\delta^{\lambda}_{\nu}\tilde{\Gamma}^{\sigma}_{(\mu\sigma)}+\delta^{\lambda}_{\mu}\tilde{\Gamma}^{\sigma}_{(\nu\sigma)}),$$

$$W^{\lambda}_{[\mu\nu]} = \tilde{\Gamma}^{\lambda}_{[\mu\nu]}+\frac{1}{6(9\alpha_1+6\alpha_2+2)}\,[3(\delta^{\lambda}_{\nu}\tilde{\Gamma}^{\sigma}_{(\mu\sigma)}-\delta^{\lambda}_{\mu}\tilde{\Gamma}^{\sigma}_{(\nu\sigma)})+4(\delta^{\lambda}_{\mu}\Gamma_{\nu}-\delta^{\lambda}_{\nu}\Gamma_{\mu})] \tag{6}$$

whence

$$W^{\sigma}_{(\mu\sigma)} = \frac{5}{2(15\alpha_1+4)}\,\tilde{\Gamma}^{\sigma}_{(\mu\sigma)} \quad \text{and} \quad W_{\mu} = \frac{1}{2(9\alpha_1+6\alpha_2+2)}\,[3\tilde{\Gamma}^{\sigma}_{(\mu\sigma)}-4\Gamma_{\mu}].$$

Because of the first of the relations (which can be easily solved for $\tilde{\Gamma}^{\lambda}_{(\mu\nu)}$ though of course, $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ itself cannot be expressed in terms of W alone), the variation (1) results in the field equations

$$\begin{aligned} R_{\mu\nu} &= 0,\\ \Lambda^{\mu\nu\lambda}{}_{;\lambda} &= 0,\\ \mathcal{N}^{[\mu\nu]}{}_{\lambda} &= 0 \end{aligned} \tag{7}$$

and

$$\mathcal{N}^{(\mu\nu)}_{\lambda}-(\Lambda^{\mu\varrho\nu}+\Lambda^{\nu\varrho\mu})a_{\varrho\lambda}-\tfrac{1}{3}\,(1+6\alpha_1)\,\{(\Lambda^{\mu\varrho\sigma}+\Lambda^{\varrho\sigma\mu})\delta^{\nu}_{\lambda}+(\Lambda^{\nu\varrho\sigma}+\Lambda^{\varrho\sigma\mu})\delta^{\mu}_{\lambda} \; = 0.$$

Here $\mathcal{N}^{\mu\nu}_{\lambda}$ (the variational derivative of the Lagrangian with respect to $W^{\lambda}_{\mu\nu}$) is the same as the left hand side of equation (12) in Ref. [4]. Contracting the last of the equations (7) over ν and λ, we can rewrite it in the form

$$\begin{aligned} \Lambda^{\mu\alpha\nu}+\Lambda^{\nu\alpha\mu} &= \mathcal{N}^{(\mu\nu)\alpha}-\frac{6\alpha_1+1}{2(15\alpha_1+4)}\,\{\mathcal{N}^{(\mu\sigma)}_{\sigma}a^{\sigma\nu}+\mathcal{N}^{(\nu\sigma)}_{\sigma}a^{\mu\alpha}\},\\ \mathcal{N}^{\mu\nu\alpha} &= a^{\alpha\lambda}\mathcal{N}^{\mu\nu}_{\lambda}. \end{aligned} \tag{8}$$

A straight forward calculation shows that if after carrying out the variation we put

$$a_{\mu\nu;\lambda} = 0 \tag{9}$$

(i.e. introduce the metric hypothesis

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}_a = \tilde{\Gamma}^{\lambda}_{(\mu\nu)})$$

then

$$\Lambda^{\mu\nu\alpha} = \tfrac{1}{2}\left[(a^{\mu\sigma}\mathfrak{g}^{(\alpha\nu)} + a^{\nu\sigma}\mathfrak{g}^{(\mu\alpha)} - a^{\alpha\sigma}\mathfrak{g}^{(\mu\nu)} - a^{\mu\nu}\mathfrak{g}^{(\alpha\sigma)})_{;\sigma} - \mathfrak{g}^{[\varrho\sigma]}\tilde{\Gamma}^{\alpha}_{[\varrho\sigma]}a^{\mu\nu} + L^{\mu\nu\alpha}\right], \tag{10}$$

where

$$L^{\mu\nu\alpha} = a^{\mu\lambda}(\mathfrak{g}^{[\nu\sigma]}\tilde{\Gamma}^{\alpha}_{[\lambda\sigma]} + \mathfrak{g}^{[\alpha\sigma]}\tilde{\Gamma}^{\nu}_{[\lambda\sigma]}) + a^{\nu\lambda}(\mathfrak{g}^{[\mu\sigma]}\tilde{\Gamma}^{\alpha}_{[\lambda\sigma]} + \mathfrak{g}^{[\alpha\sigma]}\tilde{\Gamma}^{\mu}_{[\lambda\sigma]})$$
$$- a^{\alpha\lambda}(\mathfrak{g}^{[\mu\sigma]}\tilde{\Gamma}^{\nu}_{[\lambda\sigma]} + \mathfrak{g}^{[\nu\sigma]}\tilde{\Gamma}^{\mu}_{[\lambda\sigma]}). \tag{11}$$

Similarly

$$\mathcal{N}^{[\mu\nu]}{}_{\lambda} = \mathfrak{g}^{[\mu\nu]}{}_{;\lambda} + \mathfrak{g}^{(\mu\sigma)}\tilde{\Gamma}^{\nu}_{[\lambda\sigma]} + \mathfrak{g}^{(\nu\sigma)}\tilde{\Gamma}^{\mu}_{[\sigma\lambda]} = 0. \tag{12}$$

Hence the multipliers $\Lambda^{\mu\nu\lambda}$ can be eliminated from the field equations (7) which are seen to be independent of α_1 and α_2. Of course, as in the standard theory, equations (12) immediately imply four identities

$$\mathfrak{g}^{[\mu\nu]}{}_{,\lambda} = 0. \tag{13}$$

We now end up with 50 equations (the first three of the equations (7)) for the 50 unknowns $\mathfrak{g}^{\mu\nu}$ ($= \sqrt{-g}\, g^{\mu\nu}$ field densities), $a_{\mu\nu}$ and $\tilde{\Gamma}^{\lambda}_{[\mu\nu]}$, subject to the conditions (13) with which they are formally compatible because of the variational derivation of the whole set. The reason why they are impracticable in obtaining concrete solutions is that any symmetry restrictions which we may impose on the field $g_{\mu\nu}$ (and perhaps we should strictly speaking impose such restrictions on the metric $a_{\mu\nu}$ which would only make the situation worse) do not imply, as in the stronger standard GFT, that similar restrictions will hold for $a_{\mu\nu}$. Thus the equations are considerably more complicated then either the GFT or the general relativistic field equations.

We may note finally that since it is necessary to assume that

$$g = \det g_{\mu\nu} \neq 0,$$

equations (12) can be easily solved for $\tilde{\Gamma}^{\lambda}_{[\mu\nu]}$. In fact, let

$$\mathfrak{g}_{(\mu\varrho)}\mathfrak{g}^{(\mu\sigma)} = \delta_{\varrho}{}^{\sigma}$$

existence of the density $\mathfrak{g}_{(\mu\nu)}$ being ensured by the nonvanishing of the determinent of the field tensor. Then equations (12) are equivalent to

$$\mathfrak{g}_{(\mu\alpha)}\mathfrak{g}_{(\nu\beta)}\mathfrak{g}^{[\mu\nu]}{}_{;\lambda} + \mathfrak{g}_{(\nu\beta)}\tilde{\Gamma}^{\nu}_{[\lambda\alpha]} + \mathfrak{g}_{(\mu\alpha)}\tilde{\Gamma}^{\mu}_{[\beta\lambda]} = 0,$$

whence

$$\tilde{\Gamma}^{\gamma}_{[\alpha\beta]} = \frac{1}{2}\,\mathfrak{g}^{(\gamma\lambda)}[k_{\alpha\beta\lambda}-k_{\beta\lambda\alpha}-k_{\lambda\alpha\beta}], \tag{14}$$

where

$$k_{\alpha\beta\lambda} = \mathfrak{g}_{(\alpha\mu)}\mathfrak{g}_{(\beta\nu)}\mathfrak{g}^{[\mu\nu]}{}_{;\lambda} = -k_{\beta\alpha\lambda}. \tag{15}$$

Editorial note. This article was proofread by the editors only, not by the author.

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