NON-RELATIVISTIC EQUATIONS OF MOTION FOR ARBITRARY SPIN PARTICLES WITH ANOMALOUS INTERACTIONS

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The Galilei-invariant first order differential equations are obtained. They describe the spin-orbit coupling of arbitrary spin particle with the external electromagnetic field. It is thus shown that such a coupling is not a purely relativistic effect and may be described within the framework of non-relativistic quantum mechanics. The equations obtained generalize those proposed by Levy-Leblond and Hurley.

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It is well known that from the group-theoretical point of view the concept of the spin of a particle arises naturally within the framework of non-relativistic quantum mechanics [1]. At the same time the opinion is widespread that the Galilei-invariant equations do not give a complete description of the spinning particle's motion in an external electromagnetic field, since such equations (of Levy-Leblond [2], of Hagen and Hurley [3, 4]) do not take into account the spin-orbit coupling of a particle with a field.

In this note the Levy-Leblond-Hagen-Hurley (LHG) equations have been generalized to the case of an anomalous interaction of a particle with an external field. The equations obtained are Galilei-invariant and describe the spin-orbit coupling. It is shown that the spin-orbit coupling may be also described by the Galilei-invariant equations within the framework of the minimal coupling principle.

The LHG equations for a particle of mass m and spin s have the form [4]

$$L(\vec{p}, p_0)\Psi(t, \vec{x}) \equiv [\beta_\mu p^\mu + (1-\beta_0)2m]\Psi(t, \vec{x}) = 0, \tag{1}$$

where

$$p_0 = i\frac{\partial}{\partial t}, \quad p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1, 2, 3, \quad \mu = 0, 1, 2, 3,$$

$$\beta_0 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta_a = \frac{1}{s}\begin{pmatrix} 0 & S_a & K_a^\dagger \\ S_a & 0 & 0 \\ K_a & 0 & 0 \end{pmatrix}. \tag{2}$$

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I is the $(2s+1)$-row unit matrix, S_a are the matrices of dimensionality $(2s+1)\times(2s+1)$, which realize the representation $D(s)$ of the $O(3)$ algebra, i.e. satisfy the conditions:

$$[S_a, S_b] = i\varepsilon_{abc}S_c, \; S_aS_a = s(s+1). \tag{3}$$

K_a are matrices of dimensionality $(2s-1)\times(2s+1)$, which are determined by the relations

$$K_aS_b - S'_bK_a = i\varepsilon_{abc}K_c,$$
$$S_aS_b + K_a^\dagger K_b = is\varepsilon_{abc}S_c + s^2\delta_{ab}, \tag{4}$$

where S'_a are the generators of the representation $D(s-1)$ of the group $O(3)$ (i.e. the $(2s-1)\times(2s-1)$ dimensional matrices, satisfying the relations (3)), where $s \to s' = s-1$. The explicit form of the matrices S_a, K_a and S'_a (which is not used here) is given in [4].

The equation which describes the motion of the charged non-relativistic particle in the external electromagnetic field may be obtained from (1) by the substitution $p_\mu \to \pi_\mu = p_\mu - eA_\mu$, where A_μ is the vector-potential of the electromagnetic field [4]. As a result one obtains

$$L(\vec{\pi}, \pi_0)\Psi(t, \vec{x}) \equiv [\beta_\mu\pi^\mu + (1-\beta_0)2m]\Psi(t, \vec{x}) = 0. \tag{5}$$

Equation (5), as well as equation (1), is invariant under the Galilei transformations

$$\vec{x} \to \vec{x}' = R\vec{x} + \vec{v}t + \vec{b}, \quad t \to t' = t + b_0,$$
$$\Psi(t, \vec{x}) \to \Psi'(t', \vec{x}') = \exp\,[if(t', \vec{x}')]D(R, \vec{v})\Psi(t, \vec{x})$$
$$A_0 \to A'_0 = A_0 + \vec{V}\cdot\vec{A}, \quad \vec{A} \to \vec{A}' = R\vec{A}, \tag{6}$$

where R is the operator of spatial rotation, $\vec{v}$, $\vec{b}$, b_0 are arbitrary real parameters, $D(R, \vec{v})$ are the matrices, which realize the representation of the homogeneous Galilei group

$$D(R, \vec{v}) = [1 + \tfrac{1}{2}(1-\beta_0)\vec{\beta}\cdot\vec{v}]\begin{pmatrix} D^s(R) & 0 & 0 \\ 0 & D^s(R) & 0 \\ 0 & 0 & D^{s-1}(R) \end{pmatrix}, \tag{7}$$

$D^s(R)$ $(D^{s-1}(R))$ are the matrices which realize the representation $D(s)$ $(D(s-1))$ of the $O(3)$ group, $f(t', \vec{x}')$ is the phase multiplicator

$$f(t', \vec{x}') = m\vec{v}\cdot\vec{x}' + \tfrac{1}{2}\,m\vec{v}^2t'. \tag{8}$$

Invariance under the Galilei transformations means that the transformed function $\Psi'(t', \vec{x}')$ satisfies the same equation as $\Psi(t, \vec{x})$ does

$$L(\vec{\pi}', \pi'_0)\Psi'(t', \vec{x}') = 0, \tag{9}$$

where $L(\vec{\pi}', \pi'_0)$ is the operator, which is obtained from $L(\vec{\pi}, \pi_0)$ (5) by the substitution $\pi_\mu \to \pi'_\mu = -i\dfrac{\partial}{\partial x'_\mu} - eA'_\mu$. Equation (9) actually follows from (5)–(8), as soon as the operator $L(\vec{\pi}, \pi_0)$ satisfies the condition

$$\exp\,[if(t', \vec{x}')]D^\dagger(R, \vec{v})L(\vec{\pi}, \pi_0)D(R, \vec{v})\exp\,[-if(t', \vec{x}')] = L(\vec{\pi}', \pi'_0) \tag{10}$$

It is shown in [4] that equation (5) describes the dipole interaction of a particle with the electromagnetic field, but does not take into account the spin-orbit coupling.

However, the substitution $p_\mu \to \pi_\mu$ in the equations of motion is not the only way to describe the charged particle's interaction with the external electromagnetic field. A more general approach is to take into account also the anomalous interaction which is described by introducing into the equation of motion terms which depend on the electromagnetic field strength. Let us consider, together with (5), a generalized LHG equation of the form

$$\hat{L}(\vec{\pi}, \pi_0)\Psi(t, \vec{x}) = 0,$$

$$\hat{L}(\vec{\pi}, \pi_0) = \beta_\mu \pi^\mu + (1-\beta_0)2m + \frac{e}{m}(\vec{A}\cdot\vec{E} + \vec{B}\cdot\vec{H}), \tag{11}$$

where $\vec{A} = (A_1, A_2, A_3)$ and $\vec{B} = (B_1, B_2, B_3)$ are numerical matrices, $\vec{E}$ and $\vec{H}$ are the vectors of the electric and magnetic fields strength,

$$\vec{E} = -i[\pi_0, \vec{\pi}], \qquad H = -i\vec{\pi}\times\vec{\pi}. \tag{12}$$

Let us require the operator $\hat{L}(\vec{\pi}, \pi_0)$ to satisfy the Galilei invariance condition (10). Using (7), (8), (3), (4) and taking into account the fact that according to (6), (12) the Galilei transformation law for $\vec{E}$ and $\vec{H}$ has the form

$$\vec{E} \to \vec{E}' = R\vec{E} - \vec{v}\times H, \qquad \vec{H} \to \vec{H}' = R\vec{H} \tag{13}$$

one obtains the following equations for the matrices A_a, B_a, S_a and $\eta_a = \frac{1}{2}(1-\beta_0)\beta_a$

$$[S_a, A_b] = i\varepsilon_{abc}A_c; \qquad [S_a, B_b] = i\varepsilon_{abc}B_c;$$

$$\eta_a^\dagger A_b - A_b\eta_a = 0; \qquad \eta_a^\dagger B_b - B_b\eta_a = -i\varepsilon_{abc}A_c$$

$$\eta_a^\dagger A_b\eta_c + \eta_c^\dagger A_b\eta_a = \eta_a^\dagger B_b\eta_c + \eta_c^\dagger B_b\eta_a = 0 \tag{14}$$

The general solution of the equations (14) has the form

$$\vec{A} = \frac{ik_1 S}{4}\beta_0\vec{\beta}\times\vec{\beta}, \qquad \vec{B} = \frac{ik_2 S}{4}\beta_0\vec{\beta}\times\vec{\beta} + \frac{k_1}{2}(1-2\beta_0)\vec{\beta} \tag{15}$$

where k_1 and k_2 are arbitrary constants.

Substituting (15) into (11), one obtains the equation

$$[\beta_0\pi_0 - \vec{\beta}\cdot\vec{\pi} + (1-\beta_0)2m + \frac{ik_1 eS}{4m}\beta_0\vec{\beta}\times\vec{\beta}\cdot\vec{E}$$

$$+ \frac{ik_2 eS}{4m}\beta_0\vec{\beta}\times\vec{\beta}\cdot\vec{H} + \frac{k_1 e}{2m}(1-2\beta_0)\vec{\beta}\cdot\vec{H}]\Psi(t, \vec{x}) = 0 \tag{16}$$

Equation (16) is invariant under the Galilei transformations (6) and under the gauge transformations. Let us show that this equation describes the dipole coupling of a spin particle with an external field as well as the spin-orbit and Darwin interactions. For this purpose (in analogy with the decomposition of the Kemmer-Duffin-Petiau equation [5]) we single out from (16) the equation for the $(2s+1)$-component wave function $\Phi = \beta_0\Psi$.

Multiplying (16) by β_0 and $(1-\beta_0)$ and using the relations

$$\beta_0^2 = \beta_0, \quad (1-\beta_0)\beta_a = \beta_a\beta_0,$$

one obtains the equivalent system

$$\left[\pi_0 + \frac{ieS}{4m}\vec{\beta}\times\vec{\beta}\cdot(k_1\vec{E}+k_2\vec{H})\right]\beta_0\Psi = \left(\vec{\beta}\cdot\vec{\pi}+\frac{ek_1}{2m}\vec{\beta}\cdot\vec{H}\right)(1-\beta_0)\Psi, \tag{17a}$$

$$(1-\beta_0)\Psi = \frac{1}{2m}\left(\vec{\beta}\cdot\vec{\pi}-\frac{ek_1}{2m}\vec{\beta}\cdot\vec{H}\right)\beta_0\Psi. \tag{17b}$$

Substituting (17b) on the right-hand side of equation (17a), one obtains

$$\left\{\pi_0 + \frac{ek_1}{4m}\vec{\beta}\times\vec{\beta}\cdot\vec{E} + \frac{ek_2}{4m}\vec{\beta}\times\vec{\beta}\cdot\vec{H} - \frac{1}{2m}(\vec{\beta}\cdot\vec{\pi})^2 \right.$$

$$\left. - \frac{k_1 e}{4m^2}[\vec{\beta}\cdot\vec{\pi}, \vec{\beta}\cdot\vec{H}] - \frac{e^2k_1^2}{4m^2}(\vec{\beta}\cdot\vec{H})^2\right\}\beta_0\Psi = 0. \tag{18}$$

Using the explicit form of the β-matrices (2) and taking into account relations (3), (4), one may write equation (18) in the form

$$i\frac{\partial}{\partial t}\Phi = H_S\Phi, \quad \Phi = \beta_0\Psi,$$

$$H_S = \frac{\vec{\pi}^2}{2m} + eA_0 + \frac{e(1+k_2)}{2ms}\vec{S}\cdot\vec{H} + \frac{ek_1}{2ms}\vec{S}\cdot\vec{E}$$

$$- \frac{ek_1}{2m^2s}[\vec{S}\cdot\vec{\pi}, \vec{S}\cdot\vec{H}] - \frac{e^2k_1^2}{4m^2}\vec{H}^2. \tag{19}$$

Let us demonstrate that equation (19) may be interpreted as the equation for the particle in the external electromagnetic field, which described the spin-orbit coupling. For this purpose we carry out the unitary transformation

$$\Phi \to U\Phi, H_S \to H_S' = UH_SU^\dagger + i\frac{\partial U}{\partial t}U^\dagger,$$

where

$$U = \exp\left(i\frac{k_1}{2ms}\vec{S}\cdot\vec{\pi}\right).$$

As a result one obtains the Hamiltonian

$$H'_S = \frac{\pi^2}{2m} + eA_0 + \frac{e(1+k_1)}{2ms}\vec{S}\cdot\vec{H} - \frac{ik_1^2}{8m^2s^2}[\vec{S}\cdot\vec{\pi}, \vec{S}\cdot\vec{E}]$$

$$-\frac{e(1+k_2)k_1}{8m^2s^2}[\vec{S}\cdot\vec{\pi}, \vec{S}\cdot\vec{H}] + o\left(\frac{1}{m^3}\right) + o(e^2) = \frac{\vec{\pi}^2}{2m} + eA_0 + \frac{ae}{2mS}\vec{S}\cdot\vec{H}$$

$$+\frac{b^2e}{16m^2s^2}\left[\vec{S}\cdot(\vec{\pi}\times\vec{E}-\vec{E}\times\vec{\pi}) + \tfrac{2}{3}Q_{ab}\frac{\partial E_a}{\partial x_b} + \tfrac{2}{3}S(S+1)\,\mathrm{div}\,\vec{E}\right]$$

$$+\frac{abe}{4m^2s^2}\left[\tfrac{1}{2}\vec{S}\cdot(\vec{\pi}\times\vec{H}-\vec{H}\times\vec{\pi}) + \tfrac{1}{3}Q_{ab}\frac{\partial H_a}{\partial x_b^b}\right] + o\left(\frac{1}{m^2}\right) + o(e^2), \qquad (20)$$

where $a = 1+k_2$, $b = k_1$, Q_{ab} is the tensor of the quadrupole coupling

$$Q_{ab} = \tfrac{1}{2}[3(S_aS_b+S_bS_a) - 2\delta_{ab}s(s+1)].$$

The Hamiltonian (20) possesses terms which correspond to the coupling of a point particle with the external electromagnetic field $\left(\sim eA_0 + \frac{\vec{\pi}^2}{2m}\right)$, and to the dipole $(\sim\vec{S}\cdot\vec{H})$, quadrupole $\left(\sim Q_{ab}\frac{\partial E_a}{\partial x_b}\right)$, spin-orbit $(\sim\vec{S}\cdot(\vec{\pi}\times\vec{E}-\vec{E}\times\vec{\pi}))$ and Darwin $(\sim\mathrm{div}\,\vec{E})$ interactions. Two last terms in (20) (which are P-noninvariant and may be reduced to zero by appropriately choosing the parameters k_1 and k_2) may be interpreted as those corresponding to the magnetic quadrupole and spin-orbit coupling.

An analogous structure is obtained for the approximate Hamiltonians which may be obtained by diagonalizing the relativistic equations of motion [6, 7]. For $k_1 = 1$, $k_2 = 0$, $s = \frac{1}{2}$ the first six terms in (20) coincide with the Hamiltonian which is obtained by the diagonalization of the Dirac equation [8].

So the Galilei invariant equations found above (16) actually may be interpreted as the equations of motion of the non-relativistic particle with any spin in the external electromagnetic field. These equations take into account the spin-orbit coupling which is therefore not a purely relativistic effect and may be described by the Galilei-invariant equations.

In conclusion we note, that the spin-orbit coupling in non-relativistic quantum mechanics is not a certain anomalous effect, which may be described only by equations possessing the "anomalous" terms, which depend on the electromagnetic field strength. For instance, the Galilei-invariant equation

$$[\tfrac{1}{2}(\beta_0-\beta_4)\pi_0 + (\beta_0+\beta_4+2I)m - \beta_a\pi_a]\Psi(t,\vec{x}) = 0$$

where β_k are the 10×10 Kemmer-Duffin matrices, I is the unit matrix, describes the spin-orbit coupling of the spin-one particle with the external field [9].

The general form of the Galilei-invariant interaction is discussed in [10–12] (see also [13, 14]). The results of the present paper are in accordance with the ones obtained in [12–14].

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REFERENCES

[1] V. Bargman, *Ann. Math.* **59**, 1 (1954).
[2] J.-M. Levy-Leblond, *Commun. Math. Phys.* **6**, 286 (1967).
[3] C. R. Hagen, W. J. Hurley, *Phys. Rev. Lett.* **26**, 1381 (1970).
[4] W. J. Hurley, *Phys. Rev.* **D3**, 2339 (1971); **D4**, 3605 (1971).
[5] R. A. Krajcik, M. M. Nieto, *Phys. Rev.* **D11**, 1442 (1975).
[6] K. R. James, *Proc. Phys. Soc.* **1**, 334 (1968).
[7] A. G. Nikitin, V. I. Fushchich, *Teor. Mat. Fiz.* **34**, 319 (1978) (in Russian); *Theor. Math. Phys.* **34**, 203 (1978) (in English); V. I. Fushchich, A. G. Nikitin, *Fiz. Elem. Chastits At. Yadra* **9**, 501 (1978) (in Russian).
[8] L. L. Foldy, S. A. Wouthuysen, *Phys. Rev.* **78**, 29 (1950).
[9] A. G. Nikitin, V. I. Fushchich, *Teor. Mat. Fiz.* **44**, 34 (1980) (in Russian).
[10] P. Roman, J. P. Leveille, *J. Math. Phys.* **10**, 1760 (1974).
[11] E. Celeghini, L. Lusanna, E. Sorace, *Nuovo Cimento* **31A**, 89 (1976).
[12] K. Kraus, *Ann. Phys.* **37**, 82 (1980).
[13] V. I. Fushchich, A. G. Nikitin, *Lett. Nuovo Cimento* **16**, 81 (1976).
[14] V. I. Fushchich, A. G. Nikitin, *Fiz. Elem. Chastils At. Yadra* **12**, No 5 (1981).