LETTERS TO THE EDITOR

REMARK ON AN APPLICATION OF THE BANACH METRIC METHOD TO COSMOLOGY

By M. Szydłowski and M. Heller*

Vatican Observatory, Vatican City State

(Received February 8, 1982)

If the cosmological equations can be reduced to the form of a dynamic system, the space of all their solutions is a Banach space. The influence of different parameters on the dynamics of the world models can be easily studied by means of the Banach metric. The method is tested for the Friedman cosmological models "perturbed" by the bulk viscosity.

PACS numbers: 98.80.Dr

0. The aim of this note is to propose a method with the help of which, in certain cases, one could easily study the influence of dissipative processes on the dynamics of cosmological models. The method applies to those world models which can be presented as dynamical systems. It turns out that in such cases there is an isomorphism between the space of all possible world models and a Banach space. The metric can be naturally defined on the Banach space, and a "perturbation" of the original model by a dissipative parameter can be studied in terms of this metric. The method is tested for the case of the Friedman models "perturbed" by the existence of bulk viscosity.

1. Let us consider a vector field V on a compact subset $D \subset \boldsymbol{R}^2$. A vector $X \in V$ defines (locally) the autonomous dynamical system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where x, y are local Cartesian coordinates, and $P(x, y)$ and $Q(x, y)$ are supposed to be C^k functions with $k \geqslant 1$. The vector field V is represented by $[P(x, y), Q(x, y)]$. Let $U_1, \ldots, U_k$

* Address for correspondence: Michał Heller, ul. Powstańców Warszawy 13/94, 33-110 Tarnów, Poland.

be a covering of D, then one can define a norm $||X||$, $X \in V$, in the following way

$$||X|| = \max\left\{\sup_{x,y\in U_i}(|P(x,y)|),\ \sup_{x,y\in U_i}(|Q(x,y)|),\right.$$
$$\sup_{x,y\in U_i}\left(\left|\frac{\partial P(x,y)}{\partial x}\right|\right),\ \sup_{x,y\in U_i}\left(\left|\frac{\partial P(x,y)}{\partial y}\right|\right),$$
$$\left.\sup_{x,y\in U_i}\left(\left|\frac{\partial Q(x,y)}{\partial x}\right|\right),\ \sup_{x,y\in U_i}\left(\left|\frac{\partial Q(x,y)}{\partial y}\right|\right)\right\}. \tag{2}$$

One can show that the vector field V, with the metric $d(X, Y) = ||X—Y||$, $X, Y \in V$, is a Banach space. See [1; 2, p. 60—61].

2. As it is well known (see e. g. [3]), Einstein's field equations with the Robertson--Walker metric and the equation of state $p = (\varepsilon, H)$ may be reduced to the form of the autonomous dynamic system

$$\frac{dH}{dt} \equiv \dot{H} = -H^2 - \tfrac{1}{6}(\varepsilon + 3p - 2\Lambda)$$

$$\frac{d\varepsilon}{dt} \equiv \dot{\varepsilon} = -3H(\varepsilon + p) \tag{3}$$

where $H \equiv \dot{R}/R$ is the Hubble "constant", ε — energy density, p — pressure, Λ — the cosmological canstant. Dependence $p = p(H)$ implies the existence of the bulk viscosity coefficient $\zeta = \frac{1}{3}\frac{\partial p}{\partial H}$ (see [4]).

Definition: A parameter S will be said to modify the dynamics of the model at a point (H_0, ε_0) if $d(X, Y) \to C \neq 0$ as $\varepsilon \to \varepsilon_0$ and $H \to H_0$, where X and Y are vector fields perturbed and non-perturbed by the parameter S, respectively. If the parameter S modifies the dynamics of the model at any point (H, ε) of a region $A \subset \mathbf{R}^2$ it will be said to modify the dynamics of the model in the region A.

As regions A we shall consider such compact subsets $D \subset \mathbf{R}^2$ that the boundary ∂D of D is "arc with no contact", i. e. no singular points are situated on ∂D, and no phase curve passing through D is tangent to ∂D (see, [5]). This condition guarantees that everything we shall say about stability will be valid for structural stability.

3. Let $p = A\varepsilon^m + BH^n$, and $\bar{p} = A\varepsilon^m$, where $\varepsilon \geqslant 0$, $A = \text{const} \geqslant 0$, $B = \text{const}$; $m, n \in \mathbf{R}^2$, $n \geqslant 1$; let us also assume: $D = \{(H, \varepsilon)\colon 0 < \varepsilon < \infty,\ 0 < H < \delta$, where δ is an arbitrarily small real number$\}$. In such a case $d(X, Y)$, where X and Y are defined by Eqs (3) with p (perturbed field) and with $\bar{p}$ (non-perturbed field), respectively, is given by

$$d(X, Y) = \max\left\{\sup_{H,\varepsilon\in D}(|\tfrac{1}{2}BH^n|),\ \sup_{H,\varepsilon\in D}(|3BH^{n+1}|),\right.$$

$$\sup_{H,\varepsilon\in D} (|\tfrac{1}{2} nBH^{n-1}|), \ \sup_{H,\varepsilon\in D} (|3(n+1)BH^n|)\}$$

$$= \begin{cases} \frac{1}{2}|B| & \text{for} \quad n = 1 \\ \frac{1}{2}|nB\delta^{n-1}| & \text{for} \quad n > 1 \end{cases} \tag{4}$$

The case $n = 1$ corresponds to a constant bulk viscosity coefficient $\xi = \frac{1}{3} B$ and we can see that any constant bulk viscosity modifies the dynamics of the model within D. It can be seen that the bulk viscosity effects are unstable within D (in the sense of metric (2)).

The case $n > 1$ corresponds to a variable bulk viscosity coefficient (parametrized by H or ε). We can see that the bulk viscosity does not modify the dynamics when $H \to 0\,(\delta \to 0)$, i. e. when the model approaches to, or recedes from, the static state, and this property is stable: solutions of (3), which are close to each other in the sense of metric (2), correspond to "nearby" values of the parameter H.

Let us notice that the above results are valid for a very general form of the equation of state.

4. By using this method one could also investigate the influence of other "parameters", such as the cosmological constant, pressure, etc., on the dynamics of cosmic evolution.

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