ON THE EFFECTIVE LAGRANGIAN OF QED AT HIGH TEMPERATURE AND DENSITY AND THE JBW FUNCTION $F^{[1]}$*

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Effective lagrangian of QED at high temperature in the one-loop approximation is studied. The connection with the Johnson-Baker-Willey function $F^{[1]}$ is emphasized. The connection between the effective lagrangian at high density and the renormalized photon propagator at small distances is also postulated.

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Recently I have studied the temperature corrections to the effective lagrangian of QED. In particular I have shown [1] that at low temperatures ($T \ll m$, where m is the electron rest mass) these corrections are extremely small and are proportional to $\exp(-m/T)$. This means that one can use the zero-temperature QED very quietly in the vicinity of our Solar System. At high temperatures ($T \gg m$) the behavior of the effective lagrangian is diametrically opposed to the low-temperature case since the temperature corrections dominate over the lagrangian of the Maxwell theory [2]. A similar situation we encounter considering the renormalized photon propagator at small distances. But it is known that in the Johnson-Baker-Willey (JBW) model of QED the photon propagator approaches a constant value α_0 [3]. In this publication I argue that in the JBW model of QED the temperature logarithms sum up giving a constant factor which multiplies the lagrangian of the Maxwell theory[1]. Moreover, I show that the effective lagrangian at high temperature determines the same Johnson-Baker-Willey function $F^{[1]}$ as both the photon propagator at small distances and the effective lagrangian at strong magnetic fields. I also conjecture that all these statements are fulfilled for QED at high density.

Let me consider the case of weak magnetic field, i.e., $B \ll B_{cr} = m^2/e$, and high temperature.

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[1] This is in the close analogy with the behavior of the effective lagrangian at strong magnetic field [4].

Under such circumstances the effective lagrangian takes the form:

$$\mathscr{L}(B, T) = 2T^4 I\left(-3, \frac{m^2}{4\pi T^2}\ ; 0, 1\right) - \frac{B^2}{2}\eta(T^2/m^2, \alpha) + \frac{\alpha B^4}{90\pi B_{cr}^2}\zeta(T/m), \tag{1}$$

where [2]

$$I(z, y; a, b) = \int_a^b \frac{dk}{k}\left|\frac{KE'+K'(E-K)}{k'^2K^2}\right|\left(\frac{K'}{K}\right)^z \exp\left(-y\frac{K'}{K}\right)\left(1-\left(\frac{2}{\pi}kK'\right)^{1/2}\right). \tag{2}$$

The functions $\zeta(T/m)$ and $\eta(T^2/m^2, \alpha)$ are equal to:

$$\zeta(T/m) = \left(\frac{m^2}{4\pi T^2}\right)^2 \lim_{\delta k\to 0}\left(\tfrac{1}{2}\left(\frac{2}{\pi}\ln\frac{2}{\delta k}\right)^2 - I(1, 0; 2\delta k, 1)\right), \tag{3}$$

and [2]

$$\eta(T^2/m^2, \alpha) = 1 - \frac{\alpha}{3\pi}\ln\left(\frac{a_2 T^2}{m^2}\right), \tag{4}$$

where a_2 is a constant of order 1. Moreover, $\eta(T^2/m^2, \alpha)$ fulfils the same renormalization condition as the photon propagator $d(-q^2/m^2, \alpha)$. Therefore, broadening the Ritus arguments [5]dealing with the asymptotic behavior of the effective lagrangian at zero temperature I find:

$$\eta(T^2/m^2, \alpha) \underset{T\gg m}{\rightarrow} \frac{\alpha}{\alpha_0}, \tag{5}$$

where α_0 is the zero of both the Gell-Mann-Low function ψ [6] and the Johnson-Baker-Willey function $F^{[1]}$ [7]. So we see that at high temperature the effective lagrangian is equal to:

$$\mathscr{L}(B, T) = T^4\Omega(T^2/m^2, \alpha) - \frac{\alpha}{2\alpha_0}B^2, \quad T^2 \gg m^2 \gg eB, \tag{6}$$

and the magnetization of the virtual electron-positron pairs gas is proportional to the magnetic field B.

To make a close analogy between QED at small distances, strong magnetic fields and high temperatures let me consider the effective lagrangian in the one-loop approximation taking into account only that part which is proportional to the lagrangian of the Maxwell theory. In the k-th order of perturbation theory[2] a diagram which contributes to that part of the effective lagrangian has $k-1$ photon and $2k-2$ fermion lines. There are also k integrations over virtual photon momenta. The Feynman fermion and photon

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[2] I confine my consideration to $k \geqslant 2$ since the case of $k = 1$ was studied in [2].

free propagators that enter into such a diagram have the following form in the high-temperature limit:

$$\tilde{S}_F(\vec{p}, T, N) = iT^{-1} \int_0^\infty ds(m_B T^{-1} - \vec{\gamma} \cdot \vec{p} T^{-1} \tag{7}$$

$$+(2N+1)i\pi\gamma_0) \exp [-is(m_B^2 T^{-2} - i\varepsilon + \vec{p}^2 T^{-2})$$

$$-is\pi^2(2N+1)^2] \xrightarrow[m_B T^{-1} \to 0]{} iT^{-1} \tilde{g}(\vec{p} T^{-1}, N),$$

$$\tilde{D}_{F\mu\nu}(\vec{k}, T, N) = -g_{\mu\nu} iT^{-2} \int_0^\infty d\tau \exp [-i\tau \vec{k}^2 T^{-2}$$

$$-4i\tau\pi^2 N^2] = -g_{\mu\nu} iT^{-2} \tilde{d}(\vec{k}^2 T^{-2}, N), \tag{8}$$

where m_B is the electron bare mass and $N = 0, \pm 1, \pm 2, \ldots$ Taking advantage of these expressions I find that in the k-th order of perturbation theory the proportional to B^2 part of the effective lagrangian at high temperature takes the form:

$$\mathscr{L}_M^{(k)}(eB, T, \alpha_B, n) = -\frac{(eB)^2}{8\pi} \alpha_B^{k-1} T^{k(n-4)} L_M^{(k)}(n), \tag{9}$$

where α_B is the bare coupling constant, n is the parameter of the dimensional regularization and $L_M^{(k)}$ is the calculable function of n.

Repeating the calculations performed in [8] I find that in the one-loop approximation the effective lagrangian is of the form:

$$\mathscr{L}_{eff}^{[1]}(B, T, \alpha, m) = T^4 \Omega^{[1]}(T/m, \alpha) - \frac{B^2}{2} \eta^{[1]}(T/m, \alpha), \qquad T^2 \gg m^2 \gg eB, \tag{10}$$

where

$$\eta^{[1]}(T/m, \alpha) = Q_T^{[1]}(\alpha) + F^{[1]}(\alpha) \ln (T/m), \tag{11}$$

and $F^{[1]}(\alpha)$ is the JBW function, i.e., the effective lagrangian at high temperature defines the same function $F^{[1]}(\alpha)$ as the renormalized photon propagator at small distances [3] and the effective lagrangian at strong magnetic field [8].

Up till now I have studied the high-temperature corrections to QED. But it is well-known that there exist objects of extremely high densities. So, it is interesting to investigate the high-density corrections to QED, and in particular, to the effective lagrangian. I confine my considerations to that part of the effective lagrangian which is proportional to the lagrangian of the Maxwell theory. Summing diagrams that give the second power of the magnetic field I obtain:

$$-\frac{B^2}{2}(1 - \tfrac{2}{3} e^2 j(2; \mu/m, \mu/T)), \tag{12}$$

where μ is the chemical potential and

$$j(v; x, y) = 2(4\pi)^{-v} y^{-2v+4} \int_0^{\infty e^{i\delta}} dz z^{v-3} \exp(-x^2 z)$$

$$\times \left(1 - 2y^{-1}(2kK(k)z)^{1/2} \exp\left\{z + \frac{\pi}{2K(k)}\left[\int_0^{4y^{-1}K(k)z} du Z(u)\right.\right.\right.$$

$$\left.\left.\left. + \ln \mathrm{cn}\,(4y^{-1}K(k)z)\right]\right\}\right), \quad 0 < \delta \leqslant \pi/2. \tag{13}$$

Here, $K(k)$, $K'(k)$ and $Z(u)$, cn (u) are the elliptic integrals and functions respectively [9], and k, as the function of z and y, is defined by the equation:

$$\frac{K'(k)}{K(k)} = 4\pi y^{-2} z. \tag{14}$$

To determine theasymptotic behaviour of $j(2; \mu/m, \mu/T)$ one can use the Mellin transform. The result is:

$$j(2; \mu/m, \mu/T) = \frac{1}{4\pi^2} \ln\left(\frac{a_3 \mu}{m}\right), \quad \mu \gg m \gg T, \tag{15}$$

where a_3 is a constant of order 1. So we see that the factor multiplying the free electromagnetic lagrangian $-B^2/2$ has the same logarithmic behavior as those we have encountered in the cases of strong magnetic field and high temperature. I conjecture also that the effective lagrangian at high density determines, as in the previous cases, the Johnson-Baker--Willey function $F^{[1]}$, which plays a very important role in the finite QED as the function which zeros are equal to the asymptotic coupling α_0 and/or the observable coupling constant $\alpha \approx 1/137$ [3].

To recapitulate, all these properties of QED are the consequences of both the renormalizability of this theory at the external electromagnetic field, finite temperature and density, and the fact that the quantities $eF_{\mu\nu}$, T and μ are the renormalization invariants [10].

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