EXACT RELATIVISTIC SOLUTIONS OF A PLANE-SYMMETRIC INTERACTING PERFECT FLUID AND ZERO-MASS SCALAR FIELDS

By B. Chatterjee and A. R. Roy

Department of Mathematics, Indian Institute of Technology, Kharagpur*

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Field equations corresponding to plane-symmetric interacting zero-mass scalar fields and a perfect fluid distribution have been solved exactly for the following physically important cases: (a) Disordered distribution of radiation ($D = 3p$); (b) Zeldovich fluid distribution ($D = p$); (c) Perfect fluid distribution ($D \neq p$); (d) Matter distribution in internebular space ($D = 3p/2$). For case (a) it has been observed that the only possible distribution that can exist is an unbounded plane-symmetric, asymptotically tending to plane-symmetric vacuum solution. Various physical behaviour that the solutions represent have been studied paying special attention to the pressure, density and energy-content per unit volume.

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1. *Introduction*

Tolman (1934) has observed that gravitation can never produce hydrostatic equilibrium in a finite relativistic fluid $D = 3p$ and that the only possible distribution that can exist is unbounded plane symmetric, asymptotically tending to the plane symmetric vacuum solution. Based on this result Teixeira et al. (1977) have obtained an exact solution corresponding to an equilibrium distribution of disordered radiation with planar symmetry in general relativity. They have observed that this equilibrium is due to the radiation. An analogous study with the source of gravitation as a linear combination of two interacting fields (viz., the zero-mass scalar fields and perfect fluid distributions) is worth attempting.

It may be mentioned that the study of interacting fields (one of the fields being a zero-mass scalar field) is basically an attempt to look into the yet unsolved problem of the unification of the gravitational and quantum theories. Considerable interest has been focussed on a set of field equations representing scalar fields coupled with the gravitational

* Address: Department of Mathematics, Indian Institute of Technology, Kharagpur — 721302, India.

field for the past two decades. Bergmann and Leipnik (1957), Bramhachary (1960), Das (1960, 1962), Stephenson (1962), Gautreau (1969), Roy, Rao and Tiwari (1972) are some of the authors who have studied various aspects of interacting fields in the framework of general relativity.

In this context we have solved the problem of interacting zero-mass scalar fields and perfect fluid distributions for the following cases:

(a) $D = 3p$ (Disordered distribution of radiation)
(b) $D = p$ (Zeldovich fluid distribution)
(c) $D \neq p$ (Perfect fluid distribution)
(d) $D = 3p/2$ (Matter distribution corresponding to internebular space).

2. *Field equations representing the interacting fields*

The Einstein field equations corresponding to an interacting zero-mass scalar field and perfect fluid distribution are given by

$$(R_{ij}-\tfrac{1}{2}Rg_{ij}) = -K(T_{ij(p)}+T_{ij(s)}), \tag{1}$$

where $T_{ij(p)}$ and $T_{ij(s)}$ are respectively the energy-momentum tensors corresponding to the perfect fluid and scalar fields, given by

$$T_{ij(p)} = (p+D)u_iu_j - g_{ij}p \tag{2}$$

and

$$T_{ij(s)} = v_iv_j - \tfrac{1}{2}g_{ij}v_kv^k, \tag{3}$$

where the scalar potential v satisfies the Klein-Gordon equation

$$g^{ij}v_{;ij} = 0. \tag{4}$$

As usual p is the pressure, D the density and u_i are the four velocity vectors satisfying $u^1 = u^2 = u^3 = 0$ and $u_4u^4 = 1$. A semicolon after an unknown function denotes covariant differentiation.

We now consider the plane-symmetric metric

$$ds^2 = e^{2\alpha}dt^2 - e^{2\beta}dx^2 - e^{\beta-\alpha}(dy^2+dz^2), \tag{5}$$

where α and β are functions of x and t only, and x^1, x^2, x^3, x^4 correspond respectively to x, y, z, t.

The non-vanishing components of the Ricci-tensor R_{ij} for the metric (5) are given by

$$R_{11} = (\beta_{11}-\tfrac{1}{2}\beta_1^2+\tfrac{3}{2}\alpha_1^2-\alpha_1\beta_1)-e^{(2\beta-2\alpha)}(\beta_{44}+2\beta_4^2-2\alpha_4\beta_4),$$

$$R_{22} = \tfrac{1}{2}e^{(-\beta-\alpha)}(\beta_{11}-\alpha_{11})-\tfrac{1}{2}e^{(\beta-3\alpha)}(\beta_{44}-\alpha_{44}+2\beta_4^2+2\alpha_4^2-4\alpha_4\beta_4),$$

$$R_{44} = -e^{(2\alpha-2\beta)}(\alpha_{11})+(2\beta_{44}-\alpha_{44}+\tfrac{3}{2}\beta_4^2+\tfrac{3}{2}\alpha_4^2-3\alpha_4\beta_4)$$

and

$$R_{14} = (\beta_{14}-\alpha_{14}-\tfrac{1}{2}\beta_1\beta_4+\tfrac{3}{2}\alpha_1\alpha_4-\tfrac{1}{2}\beta_1\alpha_4-\tfrac{1}{2}\beta_4\alpha_4). \tag{6}$$

With the help of (2), (3) and (6), the Einstein field equations (1) representing interacting zero-mass scalar field and perfect fluid distribution for the metric (5) assume the form

$$G_{11} = \left(-\frac{\beta_1^2}{4}+\tfrac{3}{4}\alpha_1^2-\tfrac{1}{2}\alpha_1\beta_1\right)+e^{(2\beta-2\alpha)}(\beta_{44}-\alpha_{44}+\tfrac{3}{4}\beta_4^2+\tfrac{7}{4}\alpha_4^2-\tfrac{5}{2}\alpha_4\beta_4)$$

$$= -\frac{K}{2}(v_1^2+e^{(2\beta-2\alpha)}v_4^2)-Ke^{2\beta}p, \tag{7}$$

$$G_{22}(=G_{33}) \equiv (-\beta_{11}-\alpha_{11}+\tfrac{1}{2}\beta_1^2-\tfrac{3}{2}\alpha_1^2+\alpha_1\beta_1)+e^{(2\beta-2\alpha)}(3\beta_{44}-\alpha_{44}+\tfrac{7}{2}\beta_4^2+\tfrac{3}{2}\alpha_4^2-5\alpha_4\beta_4)$$

$$= -K(-v_1^2+e^{(2\beta-2\alpha)}v_4^2)-2Ke^{2\beta}p, \tag{8}$$

$$G_{44} \equiv (\beta_{11}-\alpha_{11}-\tfrac{1}{4}\beta_1^2+\tfrac{3}{4}\alpha_1^2-\tfrac{1}{2}\alpha_1\beta_1)+e^{(2\beta-2\alpha)}\left(-\tfrac{5}{4}\beta_4^2-\frac{\alpha_4^2}{4}+\tfrac{3}{2}\alpha_4\beta_4\right)$$

$$= -\frac{K}{2}(v_1^2+e^{(2\beta-2\alpha)}v_4^2)-Ke^{2\beta}D \tag{9}$$

and

$$G_{14} \equiv (\beta_{14}-\alpha_{14}-\tfrac{1}{2}\beta_1\beta_4+\tfrac{3}{2}\alpha_1\alpha_4-\tfrac{1}{2}\beta_1\alpha_4-\tfrac{1}{2}\beta_4\alpha_1) = -\frac{K}{2}v_1v_4. \tag{10}$$

The Klein-Gordon equation (4) for the metric (5) reduces to the form

$$v_{11}-e^{(2\beta-2\alpha)}(v_{44}+2\beta_4v_4-2\alpha_4v_4) = 0. \tag{11}$$

The conservation equations given by the contracted Bianchi Identities $G^{ij}_{;j} = 0$, for the metric (5) are

$$p_1+(p+D)\alpha_1 = 0 \tag{12}$$

and

$$D_4+(p+D)(2\beta_4-\alpha_4) = 0. \tag{13}$$

Thus the set of field equations for the unknowns α, β, v, p and D is given by the over-determinate set of equations from (7) to (13). The question of over-determinacy of five unknowns in seven equations is settled finally by actual substitution of the solutions obtained in the field equations.

3. *Solutions of the field equations*

The field equations have been solved for the following two cases:

(*i*) Static case: When the metric parameters α, β and the unknowns v, p and D are functions of x only.

(*ii*) Non-static case: When the unknowns are functions of t only.

Static solutions

(a) when $D = 3p$

From (7), (8) and (9) we have the two equations

$$(\beta_{11}+\alpha_{11})-(\beta_1-\alpha_1)(\beta_1+3\alpha_1) = -2Kv_1^2 \tag{14}$$

and

$$2(\beta_{11}-\alpha_{11})+(\beta_1-\alpha_1)(\beta_1+3\alpha_1) = 2Kv_1^2. \tag{15}$$

Equation (11) yields

$$v = ax+b. \tag{16}$$

Here and in what follows, small letters are used to represent constants of integration. Using the value of v, we gave from (14) and (15)

$$3\beta = \alpha-cx-d. \tag{17}$$

In order that the solution be analytic at the plane of symmetry, we take $g_{tt} = -g_{xx} = -g_{yy} = 1$, on the plane $x = 0$. This condition for the metric (5) implies $\alpha = \beta = 0$ when $x = 0$. With this, (17) gives

$$3\beta = \alpha-cx, \text{ as } d = 0. \tag{18}$$

Substituting for β and v in (14), we get

$$12\alpha_{11}+(2\alpha_1+c)(10\alpha_1-c) = -18Ka^2,$$

which has a solution

$$\alpha = -\left(\frac{c}{5}+q\right)x+\tfrac{3}{5}\log\left(m+ne^{\frac{10}{3}qx}\right), \tag{19}$$

where

$$q = \pm\tfrac{3}{10}(c^2-10Ka^2)^{1/2}.$$

By imposing the boundary conditions $\alpha(0) = \alpha_1(0) = 0$, the constants m and n are then obtained as

$$m = \left(\tfrac{1}{2}+\frac{c}{10q}\right)$$

and

$$n = \left(\tfrac{1}{2}-\frac{c}{10q}\right). \tag{20}$$

The boundary conditions imposed imply that the system represents mirror symmetry w.r.t. the plane $x = 0$.

From (7), the pressure p can be easily obtained as

$$p = \left(\frac{5q^2}{9K} - \frac{c^2}{45K}\right) e^{-4\alpha}. \tag{21}$$

Denoting the pressure at $x = 0$ by p_0, our results, then, are given by

$$ds^2 = f^3 e^{-\xi} dt^2 - f e^{-\xi} dx^2 - f^{-1}(dy^2 + dz^2)$$

and

$$p = p_0 f^{-6} e^{2\xi},$$

where

$$f(\xi) = (m + n e^{\frac{10q}{3c}\xi})^{2/5} \exp\left(\left(\tfrac{1}{5} - \frac{2q}{3c}\right)\xi\right),$$

$$\xi(X) = s(Kp_0 X^2)^{1/2} \geqslant 0, \quad \text{with} \quad s = \tfrac{1}{3}\left(\frac{5q^2}{c^2} - \tfrac{1}{5}\right)^{-1/2}. \tag{22}$$

In order to study the asymptotic behaviour of the solution in regions far from the central plane $x = 0$ we introduce a new coordinate system as given by

$$\frac{T}{t} = \left(\frac{1}{n}\right)^{3/5}, \quad \frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \left(\frac{1}{n}\right)^{1/5} \tag{23}$$

in terms of which the exact solution takes the form

$$ds^2 = h^3 e^{-\eta} dT^2 - h e^{-\eta} dX^2 - h^{-1}(dY^2 + dZ^2)$$

and

$$p = l h^{-6} e^{2\eta},$$

where

$$h(\eta) = [e^{\left(\frac{1}{5} - \frac{2q}{3c}\right)\eta}]\left[e^{\frac{10q}{3c}\eta} + \left(\frac{m}{n}\right)^{2/5}\right],$$

$$\eta(X) = r(KqX^2)^{1/2} \geqslant 0, \quad l = \left(\frac{1}{n}\right)^{12/5} p_0, \quad r = \tfrac{1}{3}\left(\frac{5q^2}{c^2} - \tfrac{1}{5}\right)^{1/2}. \tag{24}$$

Thus in regions far from the plane $X = 0$, we have the approximate (asymptotic) solution given by

$$ds^2 = e^{\frac{\eta}{5}} dT^2 - e^{-\frac{3\eta}{5}} dX^2 - e^{-\frac{2\eta}{5}}(dY^2 + dZ^2),$$

$$p = l e^{-2\eta/5}, \quad \eta \gg 1.$$

This solution corresponds to the exact Levi-Civita (1918) static vacuum solution with planar symmetry viz.,

$$ds^2 = e^{\frac{2\varphi}{c^2}} dT^2 - e^{-\frac{6\varphi}{c^2}} dX^2 - e^{-\frac{4\varphi}{c^2}}(dY^2 + dZ^2),$$

where $\varphi(X) = \left(\dfrac{Kc^4\sigma}{4}\right)|X|$ and σ is the surface mass density.

To evaluate the energy content per unit area on the plane $X = 0$, we start with the formula [Tolman (1934)] for the energy content of a volume element viz.,

$$d^3E = (-g)^{1/2}(2T_4^4 - T)dX\,dY\,dZ,$$

where $g = \det g_{\mu\nu}$. For our system

$$\frac{d^3E}{dY\,dZ} = 6lh^{-5}e^{\eta}dX.$$

Integrating this differential $d\varepsilon'$ of the surface energy density between two planes $X = \pm$ constant, we get

$$\varepsilon'(\eta) = \frac{18c}{5rq}\left(\frac{l}{K}\right)^{1/2}\left(\tfrac{1}{2} + \frac{c}{10q}\right)\left(e^{\frac{10}{3c}q\eta} - 1\right)\left(e^{\frac{10}{3}q\eta} + \frac{m}{n}\right)^{-1}.$$

When $|X|$, $\eta \to \infty$, we obtain a surface energy density having a finite value

$$\varepsilon' = \frac{18c}{5rq}\left(\frac{l}{K}\right)^{1/2}\left(\tfrac{1}{2} + \frac{c}{10q}\right).$$

In the absence of zero-mass scalar fields, the expression for the density coincides with the Levi-Civita surface-energy-density

$$c^2\sigma = 2\left(\frac{l}{K}\right)^{1/2}.$$

The mass-density, of the plane-symmetric fluid considered, is maximum and finite on the central plane $X = 0$ and decreases monotonically to zero in both directions normal to the plane.

(b) when $p = D$ (Zeldovich fluid)

Proceeding similarly as above with the same boundary conditions, the solution for this case is obtained as

$$\alpha = \beta - ax,$$

$$\beta = \left(\tfrac{3}{4}a + \frac{Kc^2}{2a}\right)x + \frac{f}{2a}(e^{2ax} - 1),$$

$$v = cx + d,$$

$$p(\equiv D) = \frac{fa}{K}e^{-2\alpha},$$

with the relation

$$\left(\frac{a}{2} - \frac{Kc^2}{2a^2}\right) = 2f. \tag{25}$$

For $x \to 0$, we obtain the surface energy density having a finite value $\varepsilon' = \dfrac{2f}{K}$.

The density of the fluid is maximum and finite at the central plane $x = 0$ and decreases monotonically to zero in both directions normal to the plane.

(c) $p \neq D$

Because of the highly nonlinear character of the field equations a general solution is almost impossible for this case. One is naturally therefore tempted to consider some relations either between the metric parameter α and β or between one of the metric parameters and the scalar potential v, which may possibly lead to some exact solutions. Also a relation between the pressure and the density may simplify the field equations for a possible solution.

To start with, we have first taken the relations (as is apparent from the field equations): (*i*) $\beta_1 = \alpha_1$, (*ii*) $\beta_1 = -3\alpha_1$, (*iii*) $\beta_1 = \alpha_1 + A$ and (*iv*) $\beta_1 = -3\alpha_1 + A$. In all these cases, though we get an exact solution, the solutions are found to be non-physical in the sense that either the pressure or the density or both turn out to be negative.

Next considering a possible functional relationship between the metric parameter β, the scalar potential function V and the pressure p in the form

$$\beta = \beta(v, p), \tag{26}$$

from field equations (7), (8) and (9) we get

$$\beta_{vv}v_1^2 + \beta_{pp}p_1^2 + \beta_p p_{11} + \tfrac{5}{2}\, e^{2\beta}(D - 5p) = 0.$$

Now on physical grounds $g^{ij}p_i p_j \neq 0$. We, therefore, assume

$$\beta_{vv} = 0, \qquad \beta_{pp} = 0, \qquad \beta_p = 0, \qquad D = 5p.$$

The above equations lead to the possible relation

$$\beta = mv, \qquad D = 5p. \tag{27}$$

With these relations, the solution of the field equations is obtained after a straightforward calculation:

$$\alpha = \frac{ma}{3}\, x + \tfrac{1}{3} \log\left(\cosh(\theta)/\cosh(d)\right), \qquad \beta = mv, \qquad v = ax + b,$$

$$p = \frac{3c^2}{4K}\, e^{-2\beta}\,\mathrm{sech}^2(\theta), \qquad D = \frac{15c^2}{4K}\, e^{-2\beta}\,\mathrm{sech}^2(\theta), \tag{28}$$

where $\theta = (3cx + d)$ and $c = \pm \dfrac{a}{3}\,(6K - m^2/4)^{1/2}$. Here it may be noted that the integration constants are determined by imposing the same boundary conditions as in case 3(a).

Here we may mention that when we consider the metric coefficient g_{22} (or g_{33}) to be a function of the scalar potential function v and the pressure p, we arrive at a solution (on the same grounds as already argued) which is the same as obtained in 3(b) for the case $D = p$.

Finally, with the equation of state $D = np$, we get a solution of the field equations given by:

$$\alpha = \frac{(n+3)}{(5-n)}\beta + cx,$$

$$\beta = -\frac{(n+1)}{l}cx + \frac{2}{l}\log(\cosh(\theta)/\cosh(r)),$$

$$p = \frac{lm^2}{K(5-n)}e^{-2\beta}\operatorname{sech}^2(\theta), \qquad v = ax+b, \tag{29}$$

where $\theta = \left(\frac{ml}{2}x+r\right)$, $l = \frac{(n-1)(n+7)}{(5-n)}$ and $m^2 = \frac{(5-n)^2}{(1-n)(n+7)} \times \left[\frac{3}{4}c^2 + \frac{Ka^2}{2} + \frac{(n+1)^2a^2}{(1-n)(n+7)}\right]$. The constant r is given by $\tanh(r) = \frac{2c(5-n)}{lm(n+3)}$, using the boundary conditions. However, on physical grounds, we observe that the above solution is valid for all values of $n > 1$, barring $n = 5$.

This solution automatically leads to a solution for matter distribution in internebular space by putting $n = 3/2$.

For both two sets of solutions viz., (28) and (29), the behaviour of the density is found to be the same as in the previous cases. The energy content per unit area is again found to have a finite value at $x = 0$.

Non-static solutions

(b) when $p = D$

We have from (11)

$$v_4 = ae^2(\alpha-\beta). \tag{30}$$

Now we assume a relationship between the metric parameters α and β viz., $\beta = n\alpha$. Then from (7) and (8) we have

$$\alpha = \frac{l}{2(n-1)}\log[2(n-1)t+b]. \tag{31}$$

Equation (13) yields the pressure as given by

$$p = p_0 e^{2(1-2n)\alpha}, \tag{32}$$

where

$$p_0 = \frac{(n-1)(5n-1)+2K^2a^2}{4K}.$$

With these values of α and β the scalar potential v is obtained as

$$v = \frac{a}{2(n-1)}\log[b+(n-1)t], \qquad n \neq 1. \tag{33}$$

Hence, the solution is represented by (31), (32) and (33). As negative values of n lead to unphysical results, n is restricted only to positive values barring $n = 1$.

(c) when $D \neq p$

By assuming that the derivatives of the metric parameters α and β (w.r.tot) differ by an arbitrary constant viz.,

$$\beta_4 = \alpha_4 + a,$$

a rigorous solution of the corresponding field equations is finally obtained:

$$\alpha = c - \frac{a}{2} t - \frac{1}{2a} e^{b-2at},$$

$$\beta = d + \frac{a}{2} t - \frac{1}{2a} e^{b-2at},$$

$$D = \frac{e^{-2\alpha}}{K} \left[a e^{b-2at} + \frac{3a^2}{4} - \frac{K}{2} e^{2n-4at} \right],$$

$$p = \frac{e^{-2\alpha}}{K} \left[a e^{b-2at} - \frac{5a^2}{4} - \frac{K}{2} e^{2n-4at} \right],$$

$$v = -\frac{1}{2a} e^{n-2at}. \tag{34}$$

4. *Some general features*

(a) Behaviour of a test particle:

The motion of a test particle in the model described by the metric (5) is governed by the equations of the geodesic given by

$$\frac{d^2 x^\mu}{ds^2} + \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix} \frac{dx^\alpha}{ds} \cdot \frac{dx^\beta}{ds} = 0, \tag{35}$$

which becomes

$$\frac{d^2 x}{ds^2} + \beta_1 \left(\frac{dx}{ds} \right)^2 + e^{(2\alpha - 2\beta)} \alpha_1 \left(\frac{dt}{ds} \right)^2 + 2\beta_4 \frac{dx}{ds} \frac{dt}{ds} = 0$$

and

$$\frac{d^2 t}{ds^2} + e^{(2\beta - 2\alpha)} \beta_4 \left(\frac{dx}{ds} \right)^2 + \alpha_4 \left(\frac{dt}{ds} \right)^2 + 2\alpha_1 \frac{dx}{ds} \frac{dt}{ds} = 0. \tag{36}$$

If the particle is initially at rest, then the path is given by (for the static case)

$$s = b \int e^{\beta + a} dx + c \tag{37}$$

and (for the non-static case)

$$s = n \int e^{\alpha + l} dt + m. \tag{38}$$

(b) The matter distribution in the model

The scalar of expansion Θ which is defined as

$$\Theta = U^i_{;i}, \tag{39}$$

for our system reduces to

$$\Theta = e^{-\alpha}(2\beta_4 - \alpha_4), \tag{40}$$

where U^i is the flow vector of the distribution.

For all the solutions it has been observed that $U^i_{;j}\, U^j = 0$ [Synge (1964)] implying thereby that the lines of flow are geodesics. The condition $W_{ij} \equiv U_{i;j} - U_{j;i}$ for our system yields $W_{ij} = 0$ i.e., the fluid filling the universe is non-rotational.

The components of the shear stress are defined by

$$\sigma_{ij} = U_{i;j} - [g_{ij} - U_i U_j]. \tag{41}$$

For our system the non-vanishing components of σ_{ij} are given by

$$\sigma_{11} = e^{(2\beta-\alpha)}(\beta_4 - \alpha_4)$$

and

$$\sigma_{22}(\equiv \sigma_{33}) = \tfrac{1}{2}\, e^{(\beta-2\alpha)}(3\beta_4 - \alpha_4). \tag{42}$$

(c) The reality conditions

The non-vanishing components of the energy-momentum tensor T^i_j are given by

$$T^1_1 = -(p + \tfrac{1}{2}\, e^{-2\beta} v_1^2 + \tfrac{1}{2}\, e^{-2\alpha} v_4^2),$$

$$T^2_2(= T^3_3) = (-p + \tfrac{1}{2}\, e^{-2\beta} v_1^2 - \tfrac{1}{2}\, e^{-2\alpha} v_4^2),$$

$$T^4_4 = (D + \tfrac{1}{2}\, e^{-2\beta} v_1^2 + \tfrac{1}{2}\, e^{-2\alpha} v_4^2). \tag{43}$$

The expression for T^4_4, which represents the energy, is positive for all the solutions obtained.

Further the condition of Hawking and Penrose (1970) which is given by the expression

$$(T_{\alpha\beta} - \tfrac{1}{2}\, T g_{\alpha\beta}) u^\alpha u^\beta \geqslant 0 \tag{44}$$

is also found to be satisfied for the solutions.

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