SECOND ORDER SYMMETRIC TENSORS AND QUADRIC SURFACES IN GENERAL RELATIVITY

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(Received October 22, 1981)

The algebraic study of a (trace-free) symmetric tensor as a quadric surface in complex projective 3-space, leading to a quartic curve on the fundamental quadric surface determined by the metric tensor, is reconsidered. This approach, first given by Penrose and later by Cormack and Hall is simplified and more details are given. This enables a simple comparison with more conventional classification schemes. The geometrical aspects and interpretations are stressed throughout.

PACS numbers: 04.20.-q

1. Introduction

In a recent paper of Cormack and Hall [1] a discussion was presented of the classification of second order symmetric tensors in General Relativity using the spinor representation of such tensors and the methods of algebraic geometry. This paper followed up work by Penrose [2] and Ludwig and Scanlon [3] and showed that the approaches followed in these latter two papers were essentially the same. The aim of the present paper is firstly to present a simplified account of this work which uses tensor notation only and which gives more details of some of the points mentioned only briefly in the above references and secondly to present some further geometrical interpretations of the classification.

The notation will be the usual one, with M denoting a space-time manifold of signature $+2$. The Riemann, Ricci, Weyl and metric tensors and the Ricci scalar are connected by the relations

$$\text{(a)} \quad R_{abcd} = C_{abcd}+E_{abcd}+\tfrac{1}{6} R g_{a[c}g_{d]b},$$

$$\text{(b)} \quad E_{abcd} = \tilde{R}_{a[c}g_{d]b}+\tilde{R}_{b[d}g_{c]a},$$

$$\text{(c)} \quad R_{ab} = R^{c}_{acb}, \qquad R = R_{ab}g^{ab},$$

$$\text{(d)} \quad \tilde{R}_{ab} = R_{ab}-\tfrac{1}{4} R g_{ab} = E^{c}_{acb}, \tag{1.1}$$

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where $\tilde{R}_{ab}$ is the trace-free Ricci tensor and the tensor E_{abcd} is completely equivalent to it in a well defined sense [4] (see the Appendix). At a point $p \in M$ it is convenient to introduce a complex null tetrad of vectors $(l, n, m, \overline{m})$, where l and n are real null vectors and m is a complex null vector satisfying $l_a n^a = m_a \overline{m}^a = 1$ with all other inner products between tetrad members equal to zero. Also required at p are a real null tetrad of vectors (l, n, y, z) where $l_a n^a = y_a y^a = z_a z^a = 1$ are the only non-vanishing inner products and an orthonormal tetrad of vectors (t, x, y, z) where the only non-vanishing inner products are $-t_a t^a = x_a x^a = y_a y^a = z_a z^a = 1$. The following completeness relations then hold

$$g_{ab} = 2l_{(a}n_{b)} + 2m_{(a}\overline{m}_{b)} = 2l_{(a}n_{b)} + x_a x_b + y_a y_b$$
$$= -t_a t_b + x_a x_b + y_a y_b + z_a z_b. \tag{1.2}$$

Now any symmetric second order tensor at p (which, since this paper is only concerned with the algebraic properties of such tensors, may without loss in generality be taken to be the trace-free Ricci tensor) may be put into one of the following canonical forms by an appropriate choice of tetrad [5–7]

(a) $\tilde{R}_{ab} = -\varrho_0 t_a t_b + \varrho_1 x_a x_b + \varrho_2 y_a y_b + \varrho_3 z_a z_b,$

(b) $\tilde{R}_{ab} \doteq 2\varrho_1 l_{(a}n_{b)} \pm l_a l_b + \varrho_2 y_a y_b + \varrho_3 z_a z_b,$

(c) $\tilde{R}_{ab} = 2\varrho_1 l_{(a}n_{b)} + 2l_{(a}y_{b)} + \varrho_1 y_a y_b + \varrho_2 z_a z_b,$

(d) $\tilde{R}_{ab} = 2\varrho_1 l_{(a}n_{b)} + \varrho_2(l_a l_b - n_a n_b) + \varrho_3 y_a y_b + \varrho_4 z_a z_b, \quad (\varrho_2 \neq 0),$ (1.3)

where in each case the ϱ's satisfy the trace-free condition $\tilde{R}^a_a = 0$. The expressions in (1.3) correspond respectively to the Segré types $\{1, 1, 1, 1\}$, $\{2, 1, 1\}$, $\{3, 1\}$ and $\{z, \bar{z}, 1, 1\}$ for $\tilde{R}_{ab}$ where in the first type, the first digit in the Segré symbol corresponds to the timelike eigenvalue and where eigenvalue degeneracies are indicated by enclosing the appropriate digits inside round brackets. It will be of interest later to note that there are no real null eigendirections of $\tilde{R}_{ab}$ in case (a) above unless the timelike eigenvalue ϱ_0 is equal to one of the spacelike eigenvalues ϱ_1, ϱ_2 and ϱ_3, in which case there are at least two, that in the types (b) and (c) l spans the unique real null eigendirection of $\tilde{R}_{ab}$ and that in type (d) there are no real null eigendirections[1].

2. *The Ricci quadric*

The two tensors $\tilde{R}_{ab}$ and g_{ab} at p determine two quadric surfaces in complex projective 3-space $P^3(\mathbf{C})$ given by

$$(i)\ \tilde{R}_{ab}x^a x^b = 0, \qquad (ii)\ g_{ab}x^a x^b = 0. \tag{2.1}$$

Here, the x^a are a set of four complex numbers and are to be regarded as the homogeneous coordinates of a point in $P^3(\mathbf{C})$. The second quadric in (2.1) is called the *fundamental quadric*. It is a proper quadric consisting of all complex null directions at p and is denoted

[1] Further details of the classification may be found in a recent review article [7].

by $\mathscr{B}$. Those members of $\mathscr{B}$ which in the above coordinates are complex multiples of a real null vector consitute the *reality section* $\mathscr{R}$ of $\mathscr{B}$. Now it is well known from algebraic geometry (see, for example, [8]) that the intersection of two quadrics in $P^3(\mathbf{C})$, one of which is proper, is a quartic curve μ in $P^3(\mathbf{C})$. The proper quadric $\mathscr{B}$ will be kept fixed throughout and each trace-free Ricci tensor will determine a quartic curve μ on $\mathscr{B}$ which is characteristic of $\tilde{R}_{ab}$ up to multiples. Consequently, a classification of the trace-free Ricci tensor can be achieved by classifying such quartic curves μ on $\mathscr{B}$ [1, 2]. In fact one learns from algebraic geometry that such a curve on $\mathscr{B}$ may be irreducible or may decompose into an appropriate number of irreducible components. These irreducible components will be either one of the reguli (generators) of the quadric $\mathscr{B}$, a proper conic (plane section[2]) of $\mathscr{B}$ which may be real or complex (that is either it may or it may not be possible to choose the coefficients in its equation to be real in the homogeneous coordinates used here) or a twisted cubic. One can also think of these components in the following way. A component of μ is called a (p, q) curve if it intersects the general member of one of the two families of generators of $\mathscr{B}$ p times and the other q times (counted properly). In this notation, an irreducible quadric is a (2, 2) curve, a twisted cubic is a (1,2) curve or a (2, 1) curve, a conic is a (1,1) curve and the generators themselves are (1,0) or (0,1) curves. The curve μ will then decompose into a certain number of irreducible components with the proviso, of course, since μ is a quartic curve, that the sum of the first integers and the sum of the second integers in the pairs (p, q) for the various members of the decomposition are each equal to two. However, the fact that the quadrics in (2.1) have simultaneously real coefficients severely restricts the decomposition of μ since, if a line (generator) appears in a decomposition of μ then so must the corresponding complex conjugate generator. Further, the twisted cubic is no longer a possibility since a decomposition containing such a cubic component would necessarily contain a single generator only. Finally, if a complex conic appears in such a decomposition then so must its complex conjugate conic. The irreducible components which remain as possibilities for μ are then in the notation of Penrose [2]

Q — irreducible quartic with real equation;
B — pair of complex conjugate irreducible conics;
C — irreducible real conic;
X — pair of complex conjugate lines.

The full list of possible decompositions for μ is thus Q, B, CC, C^2, CX, XX and X^2. Here, the symbol CC represents two distinct real conics whereas C^2 represents a repeated real conic. Similar comments apply to the cases XX and X^2. It will be seen later how this classification of μ is easily reconciled with the Segré classification of $\tilde{R}_{ab}$.

The classification of μ is then refined by firstly subdividing the cases Q and C according to the number of connected one dimensional pieces of the appropriate component which lie in the reality section $\mathscr{R}$ of $\mathscr{B}$. This number is then used as a suffix on the corresponding symbol and leads to types C and C_1 for proper real conics and Q, Q_1 and Q_2 for irreducible quartics where, following Penrose, the suffix zero has been dropped. It will be seen that

[2] More precisely, non-tangent plane sections of $\mathscr{B}$. A section of $\mathscr{B}$ with one of its tangent planes yields a pair of reguli (a degenerate conic).

no further refinement is achieved by applying the above analysis to components of type B or X or by considering isolated point intersections of the irreducible components of μ with $\mathscr{R}$.

The second refinement arises by considering the *real* multiple point structure of μ. The types of real multiple point which might occur, together with their symbols as given by Penrose, are

n — real node with two real branches (double point);
i — isolated real node with conjugate imaginary branches (double point);
nn — two real nodes;
ii — two isolated real nodes;
c — cusp (a double point with one branch and coincident tangents);
τ — tacnode (a double point with two branches, real or imaginary, and coincident tangents);
τ^{∞} — real curve of tacnodes (a repeated real curve);
t — triple point with one real and two conjugate imaginary tangents;
q — quadruple point with two repeated conjugate imaginary tangents.

It will be seen later that non-real multiple points other than double points are forbidden and that such double points add nothing to the refinement of the classification scheme already achieved. It is also noted that because of its quartic nature, if μ possesses a real triple or quadruple point or more than one double point (real or non-real) then it is necessarily reducible. This follows by a consideration of a plane through the triple or quadruple point of μ and two other points on μ, or a plane through the two double points of μ and another point on μ. A similar conclusion also holds if μ admits a tacnode.

3. *The general classification*

In order to see the connection between the way in which the quartic μ decomposes and the Segré type of $\tilde{R}_{ab}$ one can, either by means of a direct calculation using the canonical forms (1.3) or by following the arguments given by Ludwig and Scanlon [3], show that the (non-zero) trace-free Ricci tensor may always be written in the following form at p

$$2\tilde{R}_{ab} = r_{(a}s_{b)}+\bar{r}_{(a}\bar{s}_{b)}-\tfrac{1}{4}(r_c s^c+\bar{r}_c\bar{s}^c)g_{ab}, \tag{3.1}$$

where r and s are non-zero complex vectors at p. Ludwig and Scanlon use (3.1) to classify $\tilde{R}_{ab}$ according to the following scheme.

Type A. This occurs when r and s are real and proportional and gives two immediate subtypes according to the sign of the factor of proportionality. It is further subdivided according as r is timelike, spacelike or null.

Type B. This occurs when r and s are real but not proportional. It is subdivided according to the sign or zeros of $r_a r^a$, $s_a s^a$ and $r_a s^a$ and according as the 2-space spanned by r and s is timelike, spacelike or null.

Type C. This occurs when r and s are complex vectors (but not complex multiples of real vectors) when r is non-null and where $s = \pm\bar{r}$. Again, two immediate subtypes occur

according to the sign in the last equation. It is further subdivided according as the 2-space spanned by the real and imaginary parts of r is timelike, spacelike or null.
Type D. This comprises all the cases not covered by A, B or C. It is remarked that the non-null restriction on r in type C keeps this type disjoint from type B.

For the types A, B and C one has respectively

$$\text{(a)} \qquad A\pm;\ \tilde{R}_{ab} = \pm(r_a r_b - \tfrac{1}{4}\, r_c r^c g_{ab}),$$

$$\text{(b)} \qquad B;\ \tilde{R}_{ab} = r_{(a}s_{b)} - \tfrac{1}{4}\, r_c s^c g_{ab},$$

$$\text{(c)} \qquad C\pm;\ \tilde{R}_{ab} = \pm(r_{(a}\bar{r}_{b)} - \tfrac{1}{4}\, r_c \bar{r}^c g_{ab}). \tag{3.2}$$

In (3.2) (a), the real vector r is determined by $\tilde{R}_{ab}$ up to a sign. Similarly, in (3.2) (b), r and s are real non-parallel vectors determined by $\tilde{R}_{ab}$ up to the changes $r \to \kappa r$, $s \to \kappa^{-1}s$ $(0 \neq \kappa \in \mathbf{R})$ and $s \to r$, $r \to s$. In (3.2) (c), r is complex, non-null, not parallel to a real vector and determined by $\tilde{R}_{ab}$ up to the changes $r \to e^{i\theta}r$ $(\theta \in \mathbf{R})$ and $r \to \bar{r}$, $\bar{r} \to r$. The classes A, B, C and D are mutually disjoint and exhaustive of non-zero, trace-free Ricci tensors. The relationship between this classification and the classification by means of Segré types has been tabulated [1, 3]. For convenience, the details are repeated here (Table I).

For the Ludwig-Scanlon type A, the equations (2.1) defining μ reduce to finding those complex null directions (members of $\mathscr{B}$) whose homogeneous coordinates x^a satisfy $(r_a x^a)^2 = 0$. This is a repeated real plane section of $\mathscr{B}$ and is thus a repeated proper real conic if r is spacelike or timelike (Penrose type C^2). If r is null, the resulting plane section is a tangent plane section of $\mathscr{B}$ and a repeated pair of generators results (Penrose type X^2). For Ludwig-Scanlon type B, equation (2.1) shows that μ consists of those complex null directions whose homogeneous coordinates satisfy $(r_a x^a)(s_a x^a) = 0$. Consequently, μ consists of two distinct real plane sections of $\mathscr{B}$ and is thus of one of the Penrose types CC, CX or XX, according as neither, exactly one of, or both of r and s are null. For Ludwig-Scanlon type C, a similar argument shows that μ consists of those complex null directions satisfying $(r_a x^a)(\bar{r}_a x^a) = 0$ and is thus composed of a conjugate pair of complex plane sections of $\mathscr{B}$. Since r is non-null, a conjugate pair of proper complex conics results and the resulting curve is of Penrose type B. Finally, for the Ludwig-Scanlon type D, the irreducible Penrose type Q for μ results.

The results of this section enable one to construct the basic connection between the Penrose classification and the Ludwig-Scanlon (and hence the Segré type) classification (Table I). The various refinements of this basic connection will be considered in the following sections.

4. The real multiple point structure of μ

Let l be a real null direction and construct any real null tetrad l, n, y, z. Then $\tilde{R}_{ab}$ may be written as

$$\tilde{R}_{ab} = 2\tilde{R}^1 l_{(a}n_{b)} + \tilde{R}^2 l_a l_b + \tilde{R}^3 n_a n_b + 2\tilde{R}^4 l_{(a}y_{b)} + 2\tilde{R}^5 l_{(a}z_{b)}$$
$$+2\tilde{R}^6 n_{(a}y_{b)} + 2\tilde{R}^7 n_{(a}z_{b)} + 2\tilde{R}^8 y_{(a}z_{b)} + \tilde{R}^9 y_a y_b + \tilde{R}^{10} z_a z_b, \tag{4.1}$$

TABLE I

Comparison of the Ludwig-Scanlon, Segré and Penrose types

Ludwig-Scanlon type	Sign of relevant invariant				Segré type	Penrose type
	$r_a r^a$	$s_a s^a$	$r_a s^a$	I		
$A_1 \pm$	$+$				$\{(1, 1, 1)1\}$	$C_1^2 \tau^\infty$
$A_2 \pm$	$-$				$\{1(1, 1, 1)\}$	C^2
$A_3 \pm$	0				$\{(2, 1, 1)\}$	$X^2 q$
B_{1a}	0	0	$-$	$+$	$\{(1, 1)(1, 1)\}$	$XXii$
B_{1b}	0	0	$+$	$+$	$\{(1, 1)(1, 1)\}$	$XXii$
B_{2a}	$+$	0	0	0	$\{(3, 1)\}$	$C_1 Xt$
B_{2b}	$+$	0	$-$	$+$	$\{2(1, 1)\}$	$C_1 Xi$
B_{2c}	$+$	0	$+$	$+$	$\{2(1, 1)\}$	$C_1 Xi$
B_{3a}	$+$	$+$	0	$-$	$\{(1, 1)1, 1\}$	$C_1 C_1 nn$
B_{3b}	$+$	$+$	$-$	$-$	$\{(1, 1)1, 1\}$	$C_1 C_1 nn$
B_{3c}	$+$	$+$	$+$	$-$	$\{(1, 1)1, 1\}$	$C_1 C_1 nn$
B_{3d}	$+$	$+$	$-$	0	$\{(2, 1)1\}$	$C_1 C_1 \tau$
B_{3e}	$+$	$+$	$+$	0	$\{(2, 1)1\}$	$C_1 C_1 \tau$
B_{3f}	$+$	$+$	$-$	$+$	$\{1, 1(1, 1)\}$	$C_1 C_1$
B_{3g}	$+$	$+$	$+$	$+$	$\{1, 1(1, 1)\}$	$C_1 C_1$
B_{4a}	0	$-$	$-$	$+$	$\{2(1, 1)\}$	CXi
B_{4b}	0	$-$	$+$	$+$	$\{2(1, 1)\}$	CXi
B_{5a}	$-$	$-$	$-$	$+$	$\{1, 1(1, 1)\}$	CC
B_{5b}	$-$	$-$	$+$	$+$	$\{1, 1(1, 1)\}$	CC
B_{6a}	$+$	$-$	0	$+$	$\{z, \bar{z}(1, 1)\}$	$C_1 C$
B_{6b}	$+$	$-$	$-$	$+$	$\{z, \bar{z}(1, 1)\}$	$C_1 C$
B_{6c}	$+$	$-$	$+$	$+$	$\{z, \bar{z}(1, 1)\}$	$C_1 C$
$C_1 \pm$				$+$	$\{1, 1(1, 1)\}$	B
$C_2 \pm$				0	$\{(1,2)1\}$	$B\tau$
$C_3 \pm$				$-$	$\{(1, 1)1, 1\}$	Bii
D_1	$\left\{\begin{matrix} \varrho_0 \gtrless \varrho_1, \varrho_2, \varrho_3 \\ \text{otherwise} \end{matrix}\right\}$				$\{1, 1, 1, 1\}$	$\left\{\begin{matrix} Q \\ Q_2 \end{matrix}\right\}$
D_2					$\{z, \bar{z}, 1, 1\}$	Q_1
D_3	$\left\{\begin{matrix} \varrho_2 \gtrless \varrho_1 \gtrless \varrho_3 \\ \varrho_1 \gtrless \varrho_2, \varrho_3 \text{ and either} \\ \varrho_1 > 0\,(+) \text{ or } \varrho_1 < 0\,(-). \\ \varrho_1 \gtrless \varrho_1, \varrho_3 \text{ and either} \\ \varrho_1 > 0\,(-) \text{ or } \varrho_1 < 0\,(+) \end{matrix}\right\}$				$\{2, 1, 1\}$	$\left\{\begin{matrix} Q_1 n \\ Q_1 i \\ Qi \end{matrix}\right\}$
D_4					$\{3, 1\}$	$Q_1 c$

The first five columns for types A, B and C and the first and third columns for type D give the Ludwig -Scanlon classification. Here, $I = (r_a s^a)^2 - (r_a r^a)(s_b s^b)$ and is positive, negative or zero according as the 2-space spanned by r and s is timelike, spacelike or null. The eigenvalues occurring in the type D case refer to equations (1.3) (a) and (b) and the notation $(+)$ and $(-)$ indicates the choice of sign in (1.3) (b). The sixth column (third column for type D) gives the Segré type, where the convention that all degeneracies are included inside round brackets and that in the diagonal case, the first digit corresponds to the timelike eigenvalue, is adopted. The final column gives the Penrose type.

where the tracefree relation $2\tilde{R}^1+\tilde{R}^9+\tilde{R}^{10} = 0$ holds. One now considers a neighbourhood of l in $\mathscr{B}$ in the usual topology, the members of which may be written with homogeneous coordinates

$$x^a = l^a - \tfrac{1}{2}(\alpha^2+\beta^2)n^a + \alpha y_a + \beta z_a \qquad (\alpha, \beta \in \mathbf{C}). \tag{4.2}$$

Equations (4.1) and (4.2) together with (2.1) lead to the following equation for μ

$$F(\alpha, \beta) \equiv -(\alpha^2+\beta^2)\tilde{R}^1 + \tfrac{1}{4}(\alpha^2+\beta^2)^2\tilde{R}^2 + \tilde{R}^3 - \alpha(\alpha^2+\beta^2)\tilde{R}^4 - \beta(\alpha^2+\beta^2)\tilde{R}^5$$
$$+2\alpha\tilde{R}^6 + 2\beta\tilde{R}^7 + 2\alpha\beta\tilde{R}^8 + \alpha^2\tilde{R}^9 + \beta^2\tilde{R}^{10} = 0, \tag{4.3}$$

where one is, in effect, using a one-to-one projection of the neighbourhood of l in $\mathscr{B}$ onto an open subset of the tangent plane to $\mathscr{B}$ at l, the latter open subset being considered as an open subset of $\mathbf{C}^2$ and the image point being represented by the pair of complex numbers (α, β). The necessary and sufficient condition that l be a real multiple point of μ is then (see, for example, [9])

$$F(0,0) = 0; \quad F_\alpha \equiv \left(\frac{\partial F}{\partial \alpha}\right)_{\alpha=\beta=0} = 0; \quad F_\beta = \left(\frac{\partial F}{\partial \beta}\right)_{\alpha=\beta=0} = 0. \tag{4.4}$$

But the equations (4.4) are equivalent to $\tilde{R}^3 = \tilde{R}^6 = \tilde{R}^7 = 0$ in (4.1) which, in turn, are equivalent to l being a real null eigendirection of $\tilde{R}_{ab}$. (This can also be deduced by a direct algebraic-geometric argument.) Thus the real multiple points of μ correspond precisely to the real null eigendirections of $\tilde{R}_{ab}$. As a consequence μ will admit a unique real multiple point if and only if the associated $\tilde{R}_{ab}$ has Segré type $\{2, 1, 1\}$ or $\{3, 1\}$ or a degeneracy of one of these types, more than one real multiple point if and only if the associated $\tilde{R}_{ab}$ has Segré type $\{(1, 1)1, 1\}$ or some degeneracy of this type and no real multiple points in all other cases.

To examine the real multiple point structure in more detail, one first notes that one can, by an appropriate choice of null tetrad based on l, ensure that in addition to the coefficients $\tilde{R}^3$, $\tilde{R}^6$ and $\tilde{R}^7$, the coefficients $\tilde{R}^5$ and $\tilde{R}^8$ in (4.1) are also zero. This will be assumed done. One then recalls that l is a multiple point of order r if all the k-th partial derivatives of F vanish at l when $k < r$, but that the r-th partial derivatives do not [9]. Now at the real multiple point l, one has $F = F_\alpha = F_\beta = 0$ and in an obvious notation

$$F_{\alpha\alpha} = 2(\tilde{R}^9 - \tilde{R}^1), \quad F_{\alpha\beta} = 0, \quad F_{\beta\beta} = 2(\tilde{R}^{10} - \tilde{R}^1),$$
$$F_{\alpha\alpha\alpha} = -6\tilde{R}^4, \quad F_{\alpha\alpha\beta} = 0, \quad F_{\alpha\beta\beta} = -2\tilde{R}^4, \quad F_{\beta\beta\beta} = 0,$$
$$F_{\alpha\alpha\alpha\alpha} = F_{\beta\beta\beta\beta} = 6\tilde{R}^2, \quad F_{\alpha\alpha\beta\beta} = 2\tilde{R}^2, \quad F_{\alpha\alpha\alpha\beta} = F_{\alpha\beta\beta\beta} = 0. \tag{4.5}$$

Thus l will be a real double point (a multiple point of order 2) if and only if $\tilde{R}^3 = \tilde{R}^5 = \tilde{R}^6 = \tilde{R}^7 = \tilde{R}^8 = 0$ and either $\tilde{R}^1 \neq \tilde{R}^9$ or $\tilde{R}^1 \neq \tilde{R}^{10}$ or both. Now the equation of the tangents to the curve μ at the point l is given by

$$F_{\alpha\alpha}\alpha^2 + 2F_{\alpha\beta}\alpha\beta + F_{\beta\beta}\beta^2 = 0 \tag{4.6}$$

and represents two lines in the tangent space to $\mathscr{B}$ at l which may be coincident. Now the real double point l will be a cusp or a tacnode when these tangents are coincident and this occurs precisely when either $\tilde{R}^1 = \tilde{R}^9$ or $\tilde{R}^1 = \tilde{R}^{10}$ but not both. Thus $\tilde{R}_{ab}$ will have Segré type $\{(1, 1, 1)1\}$, $\{(2, 1)1\}$ or $\{3, 1\}$ where the indicated degeneracies are the only ones permitted. These correspond respectively to equation (1.3) (a) with ϱ_0 equal to two of ϱ_1, ϱ_2 and ϱ_3 but distinct from the third, to equation (1.3) (b) with $\varrho_1 = \varrho_2$ or $\varrho_1 = \varrho_3$ but not both and to equation (1.3) (c) with $\varrho_1 \neq \varrho_2$. For l to be an isolated node, the tangents at l must be distinct and l should be the only real point on them. This occurs precisely when $\tilde{R}^1 \lessgtr \tilde{R}^9, \tilde{R}^{10}$, and corresponds to $\tilde{R}_{ab}$ having Segré type $\{(1, 1)1, 1\}$ or its degeneracy $\{(1, 1)(1, 1)\}$ (equation (1.3) (a) with ϱ_0 equal to one of ϱ_1, ϱ_2 and ϱ_3 but distinct from the other two and, if $\varrho_0 = \varrho_1$ say, $\varrho_0 \lessgtr \varrho_2, \varrho_3$) or Segré type $\{2, 1, 1\}$ or its degeneracy $\{2(1, 1)\}$ (equation (1.3) (b) with $\varrho_1 \lessgtr \varrho_2, \varrho_3$). Again no further degeneracies are allowed. For l to be a node, the tangents at l must be distinct and the real solution of (4.6) must consist of two distinct straight lines. This occurs precisely when $\tilde{R}^9 \lessgtr \tilde{R}^1 \lessgtr \tilde{R}^{10}$ and corresponds to $\tilde{R}_{ab}$ having Segré type $\{(1, 1)1, 1\}$ (equation (1.3) (a) with ϱ_0 equal to one of ϱ_1, ϱ_2 and ϱ_3 and lying strictly between the other two) or Segré type $\{2, 1, 1\}$ (equation (1.3) (b) with $\varrho_2 \lessgtr \varrho_1 \lessgtr \varrho_3$). In those cases of Segré type $\{(1, 1)1, 1\}$ or $\{(1, 1)(1, 1)\}$ where a node or isolated node appears, the real null direction which shares an eigenvalue with l is also respectively a node or isolated node. For l to be a triple point of μ, equations (4.4) and (4.5) and the choice of null tetrad show that $\tilde{R}^3 = \tilde{R}^5 = \tilde{R}^6 = \tilde{R}^7 = \tilde{R}^8 = 0$ and $\tilde{R}^1 = \tilde{R}^9 = \tilde{R}^{10}$ but that $\tilde{R}^4 \neq 0$. The trace free condition $\tilde{R}^a_a = 0$ then gives $\tilde{R}^1 = \tilde{R}^9 = \tilde{R}^{10} = 0$ and so $\tilde{R}_{cb}$ has Segré type $\{(3, 1)\}$, being easily rewritten in the form of (1.3) (c) with $\varrho_1 = \varrho_2 = 0$. Finally, for l to be a quadruple point, equations (4.4) and (4.5), the choice of null tetrad, and the trace free condition on $\tilde{R}_{ab}$ show that $\tilde{R}^3 = \tilde{R}^5 = \tilde{R}^6 = \tilde{R}^7 = \tilde{R}^8 = \tilde{R}^1 = \tilde{R}^9 = \tilde{R}^{10} = \tilde{R}^4 = 0$ but that $\tilde{R}^2 \neq 0$. Thus $\tilde{R}_{ab}$ has Segré type $\{(2, 1, 1)\}$ (equation (1.3) (b) with $\varrho_1 = \varrho_2 = \varrho_3 = 0$). This completes the discussion of the real multiple point structure of μ, the remainder of the detail given in Table I and in the list of multiple point types given at the end of section 2 being easily checked.

5. *The real part of* μ

In this section, the intersection of the curve μ with the reality section $\mathscr{R}$ will be investigated. This is the final refinement of the Penrose classification scheme. It turns out that the curve μ may have no intersection with $\mathscr{R}$ or that it may intersect $\mathscr{R}$ in a number of one dimensional connected pieces or a number of isolated points or both. A brief discussion of this feature will be given here where it will be shown that in order to refine the classification already achieved, one need only consider the one dimensional connected pieces in which μ intersects $\mathscr{R}$ and then only when μ is of Penrose type Q or C.

A point in which μ intersects $\mathscr{R}$ corresponds to a *real* null direction k such that $\tilde{R}_{ab}k^a k^b = 0$. So if μ possesses a component of type X one seeks real solutions for k of the equation $l_a k^a = 0$ where l is a real null direction. Consequently a component of μ of type X intersects $\mathscr{R}$ in a single isolated point and no further consideration of this case need be given. For a component of type C one seeks solutions for k of the equation $r_a k^a = 0$ where r is a real non-null direction. The result is that when r is timelike the intersection is empty (case C) and when r is spacelike the intersection is easily shown to be a one dimensional connected piece of μ (case C_1). These results together with the numbering system given by Ludwig and Scanlon [3] (see Table I) show that corresponding to the Ludwig-Scanlon types $A_1, A_2, B_2, B_3, B_4, B_5$ and B_6 one has the respective Penrose types $C_1{}^2, C^2, C_1X, C_1C_1, CX, CC$ and C_1C. For a component of μ of Penrose type B one considers real solutions for k of the equation $r_a k^a = 0$ where r is complex and non-null. Such solutions correspond to real null directions k orthogonal to the 2-space spanned by the real and imaginary parts of r. Consequently, there is a discrete set of solutions, the set comprising none, one or two members according as this 2-space is timelike, null or spacelike. (This result can be seen geometrically by noting that if a member of R lies on a complex irreducible conic component then it necessarily lies on its conjugate component and since these conics lie in different planes this part must also lie on the line of intersection of these planes. Since a line and a proper conic lying in the same plane intersect in two parts (properly counted) the number of members of $\mathscr{R}$ lying on these conics is either none, one or two.) For these numbers of intersections, the corresponding Ludwig-Scanlon types are C_1, C_2 and C_3 and from Table I it is seen that these types have already been distinguished between by their real multiple point structure. Hence, curves of Penrose type B have no one dimensional intersections with R and a study of the number of discrete intersections with R provides no refinement of the classification already achieved.

Finally, let μ be an irreducible quartic curve (Penrose type Q). Thus $\tilde{R}_{ab}$ is of Ludwig-Scanlon type D and has no degeneracies among its eigenvalues. To study the intersection of the curve μ with $\mathscr{R}$ one constructs an orthonormal tetrad (t, x, y, z) and writes the real null direction k in the form

$$k_a = t_a + \alpha x_a + \beta y_a + \gamma z_a \quad (\alpha, \beta, \gamma \in \boldsymbol{R}, \alpha^2+\beta^2+\gamma^2-1 = 0) \tag{5.1}$$

and studies the solutions for k of the equation $\tilde{R}_{ab}k^a k^b = 0$. First consider the case when $\tilde{R}_{ab}$ has Segré type $\{1, 1, 1, 1\}$ and is given by equation (1.3) (a) where ϱ_0, ϱ_1, ϱ_2 and ϱ_3 are distinct. Then equations (5.1), (1.3) (a) and the condition $\tilde{R}_{ab}k^a k^b = 0$, with γ eliminated by means of the bracketed equation in (5.1), gives

$$\left(\frac{\varrho_1-\varrho_3}{\varrho_0-\varrho_3}\right)\alpha^2 + \left(\frac{\varrho_2-\varrho_3}{\varrho_0-\varrho_3}\right)\beta^2 = 1. \tag{5.2}$$

One now seeks the real solutions for α and β of (5.2) which satisfy $\alpha^2+\beta^2 \leqslant 1$, this latter condition being necessary to ensure corresponding real solutions for γ and hence for k. Now without loss in generality, one may suppose that $\varrho_1 < \varrho_2 < \varrho_3$ and so there are four cases to consider;

(*i*) $\varrho_1 < \varrho_2 < \varrho_3 < \varrho_0$. In this case, no real solutions of (5.2) exist.

(*ii*) $\varrho_1 < \varrho_2 < \varrho_0 < \varrho_3$. In this case, the real solutions of (5.2) represent an ellipse lying entirely inside the circle $\alpha^2+\beta^2 = 1$ and so each solution pair (α, β) gives rise to two real solutions for γ. Two distinct real one-dimensional connected pieces for μ result.

(*iii*) $\varrho_1 < \varrho_0 < \varrho_2 < \varrho_3$. In this case, the real solutions of (5.2) represent an ellipse with major axis greater than unity and minor axes less than unity. Again, two distinct real one-dimensional connected pieces for μ result.

(*iv*) $\varrho_0 < \varrho_1 < \varrho_2 < \varrho_3$. In this case, the real solutions of (5.2) represent an ellipse lying entirely outside the circle $\alpha^2+\beta^2 = 1$. Consequently, μ does not intersect $\mathscr{R}$. The conclusion is that, in this case, the curve μ is either of Penrose type Q or Q_2 depending on the inequalities satisfied by the eigenvalues (see Table I). A similar but tedious analysis can be carried out for the other Segré types using (5.1) and the appropriate equation in (1.3) and where the null tetrad used in (1.3) (b), (c) and (d) is readily connected to the orthonormal tetrad in (5.1). The results are that for the Segré types $\{3, 1\}$ and $\{z, \bar{z}, 1, 1\}$ a single real one-dimensional connected piece for μ results, whereas for the Segré type $\{2, 1, 1\}$ either μ has a single real one-dimensional connected piece and this can occur with or without a separate real discrete point, or μ has only a single real discrete point. As in the case of Segré type $\{1, 1, 1, 1\}$, the various possibilities occurring for the Segré type $\{2, 1, 1\}$ are distinguished by certain eigenvalue inequalities (see Table I). From the geometrical viewpoint, however, these possibilities are also distinguished by the existence or non-existence of the real one-dimensional connected piece for μ and by the nature of its (unique) real multiple point.

This concludes the discussion of the intersection of the curve μ with the reality section $\mathscr{R}$ and also shows why only the real one-dimensional connected pieces of μ need be considered in order to refine the classification already achieved.

6. *The complex multiple point structure of* μ

The complex multiple point structure of μ turns out to be much more restricted and less complicated than the real multiple point structure as might be expected. (In this section the word "complex" will always be understood to mean non-real.) The following is a summary of the relevant features, the proof of which will be given later.

(*i*) Suppose that the complex null direction m lies on the curve μ. Then m is a complex multiple point of μ if and only if m is a complex null eigendirection of $\tilde{R}_{ab}$. Consequently, the complex multiple points of μ occur in complex conjugate pairs.

(*ii*) Any such curve possessing a complex multiple point is reducible.

(*iii*) Any complex multiple point of μ is necessarily a double point, higher order points being forbidden.

(*iv*) The number n of complex multiple points possessed by μ is restricted to $n = 0$, $n = 2$ or $n = \infty$.

(*v*) For a curve with exactly two complex multiple points, the tangents at any of these points are distinct. A curve with infinitely many complex multiple points however has coincident tangents at each such point and is necessarily confined to a single plane section of $\mathscr{B}$. A curve of one of the repeated Penrose types C^2, ${C_1}^2$ or X^2 results and each point on it is a tacnode.

(*vi*) Finally, the following very simple result is noted. If m is a complex null eigenvector of $\tilde{R}_{ab}$ with eigenvalue λ, then λ is real and the real and imaginary parts of m are real orthogonal spacelike eigenvectors of $\tilde{R}_{ab}$ each with eigenvalue λ. Conversely, if $\tilde{R}_{ab}$ has two spacelike eigenvectors with equal (real) eigenvalue λ then $\tilde{R}_{ab}$ admits a complex null eigenvector with eigenvalue λ.

To prove these statements, suppose firstly that m is a complex multiple point of μ. Then one can construct a complex null tetrad $(l, n, m, \bar{m})$ based on m and proceed with an argument similar to that given in section 4 and based on equation (4.3). This method also reveals that only double points are possible. An alternative approach to this latter result starts by letting m and $\bar{m}$ be a pair of complex multiple points of μ one of which has order of multiplicity at least three. Then any plane containing m and $\bar{m}$ contains a component of μ and this implies that the line $m\bar{m}$ is a generator of $\mathscr{B}$. This is a contradiction since no generator of $\mathscr{B}$ can contain a complex point and its conjugate. Next, the fact that such a curve μ which possesses a complex multiple point (and hence at least two such points) is necessarily reducible follows from the comments at the end of section 2. Now suppose that m is a complex null eigenvector of $\tilde{R}_{ab}$. Then one writes $m^a = u^a + iv^a$ where u^a and v^a are real orthogonal spacelike vectors of equal norm and

$$\tilde{R}_{ab}(u^b + iv^b) = (\lambda + i\sigma)(u_a + iv_a)\,(\lambda, \sigma \in \mathbf{R}). \tag{6.1}$$

The real and imaginary parts of (6.1) together with the symmetry condition $\tilde{R}_{ab}u^a v^b = \tilde{R}_{ab}v^a u^b$ shows that $\sigma = 0$ and that u^a and v^a are eigenvectors of $\tilde{R}_{ab}$ with equal (real) eigenvalue λ. The converse is similar. It now follows that if more than two complex multiple points are admitted, then infinitely many are admitted for if m_1 and m_2 ($m_2 \neq \bar{m}_1$) are complex multiple points, they determine distinct spacelike eigen 2-spaces of $\tilde{R}_{bb}$ whose eigenvalues are necessarily equal (eigenvectors of $\tilde{R}_{ab}$ with distinct eigenvalues are orthogonal). Then infinitely many complex null eigenvectors of $\tilde{R}_{ab}$ are generated by the resulting infinitely many pairs of orthogonal spacelike eigenvectors with equal norms and eigenvalues. It follows that if exactly two complex multiple points are admitted, the Segré type of $\tilde{R}_{ab}$ is $\{1, 1, (1, 1)\}$, $\{(1, 1)(1, 1)\}$, $\{2(1, 1)\}$ or $\{z, \bar{z}(1, 1)\}$ with no further degeneracies allowed. In each case, the appropriate canonical form in (1.3) and the methods of section 4 show that the tangents at these multiple points are distinct. Similarly, if infinitely many complex multiple points are admitted, the Segré type of $\tilde{R}_{ab}$ is $\{1, 1, 1)1\}$, $\{1(1, 1, 1)\}$ or $\{(2, 1, 1)\}$ and coincident tangents occur at each complex multiple point. In these cases, a repeated Penrose curve of type C^2, C_1^2 or X^2 results. It is now easily seen that no further refinement of the classification scheme already obtained is achieved by considering the complex multiple point structure.

7. *Further comments on the geometry of $\mathscr{B}$. Invariant 2-spaces of $\tilde{R}_{ab}$*

Let l be a real null direction in $\mathscr{B}$ and let L and $\bar{L}$ be the conjugate pair of generators of $\mathscr{B}$ through l. The members of L are contained in the tangent space to $\mathscr{B}$ at l and are hence orthogonal to l as well as being null. Now if m is any complex (non-real) member of the generator L, let L' be the other generator of $\mathscr{B}$ through m. The point m uniquely determines the conjugate point $\bar{m}$ which lies in both $\bar{L}$ and $\bar{L}'$. Hence m uniquely determines another real null direction n distinct from l and lying at the point of intersection of L' and $\bar{L}'$. Thus after an appropriate scaling, a comlex null tetrad $(l, n, m, \bar{m})$ is determined up to the ambiguities $m \to e^{i\theta}m$, $l \to Al$, $n \to A^{-1}n$ ($\theta, A \in \mathbf{R}$, $A \neq 0$). Now the members of L can be represented formally by the 2-parameter family of directions $\{m + \alpha l\colon \alpha \in \mathbf{C}\}$

together with the direction l. Consequently, with l fixed, a change of m within L results in a change of null tetrad (with the above mentioned ambiguities taken into account) which may be written in the component form for the usual null rotations about l (cf. [10])

$$l'^a = Al^a,\ m'^a = e^{i\theta}(m^a - A\bar{B}l^a),\ n'^a = -A^{-1}n^a + Bm^a + \overline{Bm^a} - AB\bar{B}l^a, \tag{7.1}$$

where $A, \theta \in \mathbf{R}, B \in \mathbf{C}$ and where A has been taken positive so that the direction l may always be considered, say, future pointing. The null rotations (7.1) preserve the set of future pointing timelike and null vectors and also the orientation of a triad of spacelike vectors. This latter orientation would not be preserved if, with l fixed, one changed from m in L to some member of $\bar{L}$.

Each point in $\mathscr{B}\backslash\mathscr{R}$ has real and imaginary parts corresponding to two real orthogonal spacelike vectors of equal norm determined up to a rotation within the 2-space defined by them and there is a one-to-one correspondence between such points and spacelike 2-spaces at $p \in M$. In particular, such points which lie in L correspond to the spacelike 2-spaces orthogonal to l, the "wave surfaces" of l. These comments enable one to give a simple geometrical interpretation of two recent results of Cormack and Hall [11] and in particular of the concept of an invariant 2-space of $\tilde{R}_{ab}$ at $p \in M$. A two-dimensional subspace V of the tangent space $T_p(M)$ to M at p is an *invariant 2-space* of $\tilde{R}_{ab}$ if, in components, whenever $v^a \in V$, $R^a_b v^b \in V$. The following results are known [4, 6, 7, 12]:

(*i*) for any tracefree Ricci tensor an invariant 2-space always exists;

(*ii*) a null 2-space is an invariant 2-space of $\tilde{R}_{ab}$ if and only if its unique null direction is an eigendirection of $\tilde{R}_{ab}$;

(*iii*) a 2-space V is an invariant 2-space of $\tilde{R}_{ab}$ if and only if its orthogonal complement $\overset{*}{V}$ is also.

It is now clear that a null invariant 2-space (in fact an orthogonal pair of null invariant 2-spaces) is then interpreted in terms of the geometry of $\mathscr{B}$ as a real multiple point of μ. Next, let G be the four-dimensional (Grassman) manifold of all 2-spaces at $p \in M$ and $\bar{G}$ the (four-dimensional) open submanifold of G consisting of all non-null 2-spaces at p. One can now construct a real-valued differentiable map ϕ_p on $\bar{G}$ where if F is a non-null 2-space at p and F_{ab} any (necessarily non-null) representative simple bivector for F, then[3]

$$\phi_p(F) = \frac{E_{abcd}F^{ab}F^{cd}}{2g_{a[c}g_{d]b}F^{ab}F^{cd}}, \tag{7.2}$$

where E_{abcd} is given by (1.1). The value of $\phi_p(F)$ is clearly independent of the representative bivector chosen for F. Now if F is a spacelike 2-space at p, its othogonal complement $\overset{*}{F}$ is a timelike 2-space at p and it follows from (7.2) that $\phi_p(F) = -\phi_p(\overset{*}{F})$ [7, 11]. Thus ϕ_p is determined by its values on spacelike 2-spaces at p. From the remarks above one can, therefore, consider ϕ_p as a real-valued differentiable function on $\mathscr{B}\backslash\mathscr{R}$, where if $m \in \mathscr{B}\backslash\mathscr{R}$ corresponds to the spacelike 2-space F, then with an abuse of notation

$$\phi_p(m) = \phi_p(F) = \frac{E_{abcd}F^{ab}F^{cd}}{2g_{a[c}g_{d]b}F^{ab}F^{cd}} = \frac{\tilde{R}_{ab}\bar{m}^{(a}m^{b)}}{g_{ab}m^{(a}\bar{m}^{b)}}, \tag{7.3}$$

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[3] This should be compared with the Riemannian curvature (sectional curvature) function.

where m^a is a representative complex vector for m and where the last equality follows from (1.1) (b) and the choice $iF_{ab} = 2m_{[a}\overline{m}_{b]}$. The following results can now be established from (7.3).

(a) if L is a generator of $\mathscr{B}$ and l is the unique member of $\mathscr{R}$ contained in L, then the value of $\phi_p(m)$, $m \in L\backslash\{l\}$ is independent of m if and only if l is a (real) null eigendirection of $\tilde{R}_{ab}$.

(b) if one examines the values of ϕ_p in a neighbourhood in $\mathscr{B}\backslash\mathscr{R}$ of the point m, that is at points of the form $m'^a = m^a+\alpha l^a+\beta n^a-\alpha\beta\overline{m}^a$ ($\alpha, \beta \in$ C) where the complex null tetrad $(l, n, m, \overline{m})$ is constructed as above, then the critical points of ϕ_p (in the obvious sense) are precisely those points in $\mathscr{B}\backslash\mathscr{R}$ which represent spacelike invariant 2-spaces of $\tilde{R}_{ab}$.

Thus all invariant 2-spaces of $\tilde{R}_{ab}$ are determined by the behaviour of ϕ_p on $\mathscr{B}\backslash\mathscr{R}$. The proofs of these two results, given in a different form in [11] are readily gathered from (7.3) by considering an expansion of $\tilde{R}_{ab}$ in terms of the complex null tetrad $(l, n, m, \overline{m})$ similar to that given in terms of a real null tetrad in (4.1) and referring to the theorem in section 2 of reference [11].

This completes the discussion of the geometrical interpretation of certain of the important properties of $\tilde{R}_{ab}$ by means of quartic curves on $\mathscr{B}$. Some physical applications of the classification scheme have been discussed elsewhere [13, 14] (see also [7] for further references).

APPENDIX

The equivalence of the tensors $\tilde{R}_{ab}$ and E_{abcd} has been utilised to obtain a classification scheme for $\tilde{R}_{ab}$ and hence for R_{ab} by an analysis of the tensor E_{abcd} [4, 7]. In these references, however, the analysis concerned the tensor $\overset{+}{E}_{abcd} = E_{abcd}+iE^*_{abcd}$, where * denotes the usual duality operator. In this appendix, a brief discussion is given of a direct analysis of the algebraic structure of E_{abcd}. It is convenient to use the block index replacement where capital Latin letters A, B take the values 1–6 and replace a skew pair of small Latin indices according to the scheme $1 \leftrightarrow 23$, $2 \leftrightarrow 31$, $3 \leftrightarrow 12$, $4 \leftrightarrow 10$, $5 \leftrightarrow 20$, $6 \leftrightarrow 30$ (see for example [15]). With this notation, the components E_{abcd} at p are replaced by the 6×6 matrix E_{AB} and a bivector F_{ab} is replaced by a 6-vector F_A. Block indices are raised and lowered in the usual way and for the duality operator, one notes the result $^*E_{AB} = -E^*_{AB}$ which together with the tracefree condition $E^A{}_A = 0$ will prove useful in what is to follow.

The following three results enable the classification to be given. They concern the study of the eigenvectors of E_{AB}, that is 6-vectors F^A satisfying $E_{AB}F^B = \lambda F_A$ where F_A and λ may be complex. Such eigenvectors are intimately connected with the invariant 2-spaces of R_{ab} [4, 7].

(*A1*) E_{AB} has an even number of independent real eigenvectors

Proof. If E_{AB} has a real eigenvector F_A then $\overset{*}{F}_A$ is also an eigenvector and is independent of F_A. If E_{AB} admits another independent real eigenvector G_A then again $\overset{*}{G}_A$ is an eigenvector and it is easily shown that F_A, $\overset{*}{F}_A$, G_A and $\overset{*}{G}_A$ are independent. A similar argument holds if a fifth independent real eigenvector H_A is admitted, for then F_A, $\overset{*}{F}_A$, G_A, $\overset{*}{G}_A$, H_A and $\overset{*}{H}_A$ are six independent real eigenvectors.

(*A2*) If a non-simple elementary divisor of order $\geqslant 3$ exists for E_{AB} and if E_{AB} has only real eigenvalues then its Segré type is necessarily $\{3, 3\}$

Proof. The existence of a non-simple elementary divisor of order $\geqslant 3$ implies the existence of three independent real vectors F_A, G_A and H_A satisfying

$$E_{AB}F^B = \alpha F_A,\ E_{AB}G^B = \alpha G_A + F_A,\ E_{AB}H^B = \alpha H_A + G_A.$$

The duals of these equations can be written as

$$E_{AB}(-\overset{*}{F}{}^B) = -\alpha(-\overset{*}{F}_A) \qquad E_{AB}\overset{*}{G}{}^B = -\alpha\overset{*}{G}_A + (-\overset{*}{F}_A)$$

$$E_{AB}(-\overset{*}{H}{}^B) = -\alpha(-\overset{*}{H}_A) + \overset{*}{G}_A.$$

Now it can be shown that the vectors F_A, G_A, H_A, $-\overset{*}{F}_A$, $\overset{*}{G}_A$ and $-\overset{*}{H}_A$ are independent and so they constitute a basis of vectors from which the Segré type of E_{AB} is clearly seen to be $\{3, 3\}$.

(*A3*) If E_{AB} admits a complex eigenvalue then, in an obvious notation, it has Segré type $\{z, \bar{z}, -z, -\bar{z}, 1, 1\}$

Proof. If $W_A = F_A + iG_A$ (F_A and G_A real) satisfies $E_{AB}W^B = \omega W_A$, where $\omega = \alpha + i\beta$, $\alpha, \beta \in \mathbf{R}$, then necessarily $\overline{W}_A$, $\overset{*}{W}_A$ and $\overline{\overset{*}{W}}_A$ are eigenvectors of E_{AB} with eigenvalues $\bar{\omega}$, $-\omega$ and $-\bar{\omega}$ respectively. It can then be shown that W_A, $\overline{W}_A$, $\overset{*}{W}_A$ and $\overline{\overset{*}{W}}_A$ constitute an independent set of eigenvectors of E_{AB}. One then shows that the elementary divisors corresponding to any complex eigenvalue is necessarily simple. In this step the tracefree condition is useful. One is then left with the two possibilities $\{z, \bar{z}, -z, -\bar{z}, \omega, \bar{\omega}\}$ and $\{z, \bar{z}, -z, -\bar{z}, 1, 1\}$ (the $z, \bar{z}, -z, -\bar{z}, 2\}$ case being ruled out by result (*A1*) above). It is then straightforward to show that the former is impossible and the result follows.

The classification is now completed by noting that the only possible Segré types for a 6×6 real matrix when all its eigenvalues are real are

(*i*) $\{1, 1, 1, 1, 1, 1\}$	(*ii*) $\{2, 1, 1, 1, 1\}$	(*iii*) $\{2, 2, 1, 1\}$
(*iv*) $\{2, 2, 2\}$	(*v*) $\{3, 1, 1, 1\}$	(*vi*) $\{3, 2, 1\}$
(*vii*) $\{3, 3\}$	(*viii*) $\{4, 1, 1\}$	(*ix*) $\{4, 2\}$
(*x*) $\{5, 1\}$	(*xi*) $\{6\}$.	

To these must be added those cases where complex eigenvalues occur. Now the result (*A1*) eliminates the cases (*ii*), (*iv*), (*vi*), (*viii*) and (*xi*) above whilst the result (*A2*) eliminates, in addition, (*v*), (*ix*) and (*x*). Finally, result (*A3*) reduces the possibilities when complex eigenvalues occur to a single case. The only Segré types which can occur are therefore

$$\{1, 1, 1, 1, 1, 1\}, \quad \{2, 2, 1, 1\}, \quad \{3, 3\}, \quad \{z, \bar{z}, -z, -\bar{z}, 1, 1\}$$

which are easily shown to correspond to the canonical forms for $\tilde{R}_{ab}$ given by (1.3) (a), (b), (c), (d) respectively.

One of the authors (G.S.H.) wishes to acknowledge valuable discussions with and the hospitality of Professor A. Trautman, Professor A. Zajtz, Professor J. Łopuszański, Dr Z. Oziewicz, Dr G. Sobczyk, Professor W. Roter and Professor A. Derdziński during his recent visits to The Institute of Theoretical Physics in Warsaw, the Institute of Mathematics at the Jagellonian University in Kraków and The Institute of Theoretical Physics and The Institute of Mathematics in the University of Wrocław. He also gratefully acknowledges the financial support of The Polish Academy of Sciences and The Royal Society. The other author gratefully acknowledges the award of a Science Research Council Studentship. Both authors wish to thank Dr W. J. Cormack for many valuable discussions.

Editorial note. This article was proofread by the editors only, not by the authors.

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