LAGRANGE FORMALISM IN INSTANTANEOUS PREDICTIVE RELATIVISTIC DYNAMICS

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A method of constructing examples of Predictive Relativistic Dynamics on the basis of Lagrange formalism is presented. Kerner's one dimensional example is obtained.

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In the instantaneous formalism of Predictive Relativistic Dynamics (PRD) [1, 2, 6, 7] the system of two point-like i.e. structureless particles is described by Newtonian-like differential equations:

$$\frac{d^2x_i^n}{dt^2} \equiv \ddot{x}_i^n = a_i^n(\vec{x}^n, \dot{\vec{x}}^1, \dot{\vec{x}}^2), \tag{1}$$

where $\vec{x}$ is the position of n-th particle, $\vec{x} = \vec{x}^1 - \vec{x}^2$ is the instantaneous interparticle distance, $\dot{\vec{x}}^n = \frac{d\vec{x}^n}{dt}$ is the particle velocity and a_i^n are the components of "force".

Currie and Hill [1, 2] have given the differential conditions which guarantee the Lorentz invariance of PRD. They form a set of nonlinear partial differential equations. In the case of the one dimensional motion of two particles, the Currie-Hill conditions are as follows:

$$(1-\dot{x}^1\dot{x}^1)\frac{\partial a^1}{\partial \dot{x}^1} + (1-\dot{x}^2\dot{x}^2 + xa^2)\frac{\partial a^1}{\partial \dot{x}^2} - \dot{x}^2 x\frac{\partial a^1}{\partial x} + 3\dot{x}^1 a^1 = 0,$$

$$(1-\dot{x}^2\dot{x}^2)\frac{\partial a^2}{\partial \dot{x}^2} + (1-\dot{x}^1\dot{x}^1 - xa^1)\frac{\partial a^2}{\partial \dot{x}^1} - \dot{x}^1 x\frac{\partial a^2}{\partial x} + 3\dot{x}^2 a^2 = 0. \tag{2}$$

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The nonlinearity makes their integration difficult. At present, five solutions of (2) are known in the explicit form. They all have the form

$$a^n = \frac{\phi_n(\dot{x}^1, \dot{x}^2)}{x}, \quad \phi_2(\dot{x}^1, \dot{x}^2) = -\phi_1(\dot{x}^2, \dot{x}^1), \tag{3}$$

and explicitly:

$$\phi_1 = \frac{(\dot{x}^1 - \dot{x}^2)^2}{2},$$

$$\phi_1 = \frac{1-\dot{x}^1\dot{x}^1}{\dot{x}^2}(\dot{x}^2 - \dot{x}^1), \quad \phi_1 = 2(1-\dot{x}^1)(1-\dot{x}^2),$$

$$\phi_1 = 2(1+\dot{x}^1)(1+\dot{x}^2), \quad \phi_1 = \frac{(1-\dot{x}^1\dot{x}^1)(\dot{x}^1-\dot{x}^2)}{1-\dot{x}^2}. \tag{4}$$

It is very hard to find any simple solution which falls off with interparticle distance x faster than $\frac{1}{x}$ [9]. The solutions (3) do not appear to be physically realistic.

In this paper, we present a method of constructing examples of PRD on the basis of Lagrange formalism. The question can be put as follows: is it possible to ensure the Lorentz invariance at the level of Lagrangian $L(x, \dot{x}^1, \dot{x}^2)$ and which conditions the Lagrangian should satisfy to guarantee the Lorentz invariance? In other words we look for the Lagrangian which, via the Euler-Lagrange equations, would provide us with the Lorentz invariant equations of motion (1).

For two particles the infinitesimal Lorentz transformation of x, $\dot{x}^1$, $\dot{x}^2$ has the form [3]:

$$x' = x + \varepsilon(x^1\dot{x}^1 - x^2\dot{x}^2), \tag{5}$$

$$\dot{x}^{n'} = \dot{x}^n + \varepsilon(x^n\ddot{x}^n + \dot{x}^n\dot{x}^n - 1), \quad n = 1,2 \tag{6}$$

where ε denotes the infinitesimal parameter of the transformation. From Kerner's proof of "No-Interaction Theorem" [3, 4] it follows that the requirement for the Lorentz transformation to be canonical is too restrictive. In the Lagrange formalism this implies

$$L(x', \dot{x}^{1'}, \dot{x}^{2'}) = L(x, \dot{x}^1, \dot{x}^2) + \varepsilon\frac{dG(x, \dot{x}^1, \dot{x}^2)}{dt}, \tag{7}$$

which results in the Lagrangians of the form

$$L = c_1\sqrt{1-\dot{x}^1\dot{x}^1} + c_2\sqrt{1-\dot{x}^2\dot{x}^2} + c_3|x|. \tag{8}$$

They describe the free particle motion or the case of "forces" independent of x. Thus we abandon the condition (7) by putting:

$$L(x', \dot{x}^{1'}, \dot{x}^{2'}) = L(x, \dot{x}^1, \dot{x}^2) + \varepsilon\frac{dG(x, \dot{x}^1, \dot{x}^2)}{dt} + \varepsilon\Delta L(x, \dot{x}^1, \dot{x}^2), \tag{9}$$

requiring moreover that $L(x', \dot{x}^{1'}, \dot{x}^{2'})$ when expressed in non-primed variables gives the same equations of motion as those following from $L(x, \dot{x}^1, \dot{x}^2)$. This requirement will be automatically satisfied if

$$\hat{L}_m \Delta L \equiv \sum_{n=1}^{2} A_{mn}(x, \dot{x}^1, \dot{x}^2)\hat{L}_n L, \qquad m = 1,2 \tag{10}$$

where

$$\hat{L}_m \equiv \frac{d}{dt}\frac{\partial}{\partial \dot{x}^m} - \frac{\partial}{\partial x^m}, \tag{11}$$

and $A_{mn}(x, \dot{x}^1, \dot{x}^2)$ are arbitrary functions with det $A_{mn} \neq 0$.

A similar method has been proposed by Currie and Saletan [5] for the case of one degree of freedom.

The function $\Delta L(x, \dot{x}^1, \dot{x}^2)$ follows from the Lorentz transformations (5) and (6). Indeed, if we insert (5) and (6) into (9), perform the Taylor expansion in the parameter ε requiring that all terms linear in $\ddot{x}^n$ vanish we end up with the following set of equations:

$$\frac{\partial^2 L}{\partial \dot{x}^m \partial \dot{x}^n} = 0 \qquad \text{for } m \neq n.$$

They imply that

$$L = F_1(x, \dot{x}^1) + F_2(x, \dot{x}^2), \tag{12}$$

$$G = x^1 F_1 + x^2 F_2 + h(x^1, x^2), \tag{13}$$

where F_1, F_2 and h are arbitrary functions of their arguments. The remaining terms in (9) give:

$$\Delta L = \sum_{n=1}^{2}\left[(\dot{x}^n\dot{x}^n - 1)\frac{\partial F_n}{\partial \dot{x}^n} - \dot{x}^n F_n\right] + x\dot{x}^2\frac{\partial F_1}{\partial x} + x\dot{x}^1\frac{\partial F_2}{\partial x} + \frac{d}{dt}h(x^1, x^2). \tag{14}$$

Now we put (14) into (10) and thus we get

$$(\dot{x}^i\dot{x}^i - 1)\frac{\partial^3 F_i}{\partial \dot{x}^i \partial \dot{x}^i \partial \dot{x}^i} + (3\dot{x}^i - A_{ii})\frac{\partial^2 F_i}{\partial \dot{x}^i \partial \dot{x}^i} + x\dot{x}^k \frac{\partial^3 F_i^i}{\partial x \partial \dot{x}^i \partial \dot{x}^i} = 0, \tag{15}$$

$$x\frac{\partial}{\partial x}\left(\frac{\partial F_i}{\partial \dot{x}^i} + \frac{\partial F_k}{\partial \dot{x}^k}\right) = A_{ik}\frac{\partial^2 F_k}{\partial \dot{x}^k \partial \dot{x}^k}, \tag{16}$$

$$\begin{aligned}(\dot{x}^1 - \dot{x}^2)&\left[(\dot{x}^i\dot{x}^i - 1)\frac{\partial^3 F_i}{\partial x \partial \dot{x}^i \partial \dot{x}^i} + x\dot{x}^k\frac{\partial^3 F_i}{\partial x \partial x \partial \dot{x}^i}\right] + (-1)^i\left\{x\dot{x}^k\frac{\partial^2 L}{\partial x \partial x}\right. \\ &\left. - \sum_{n=1}^{2}[1 - \dot{x}^k\dot{x}^k + (\dot{x}^k - \dot{x}^i)A_{in}]\frac{\partial^2 F_n}{\partial x \partial \dot{x}^n} + (A_{ik} - A_{ii})\frac{\partial L}{\partial x}\right\} = 0,\ i, k = 1,2, \quad i \neq k. \end{aligned} \tag{17}$$

Equations (15, 16) come from the comparison of terms linear in accelerations $\ddot{x}^n$ which enter Eq. (10). It turns out that Eqs. (15)–(17) imply that the forces deduced from L satisfy the Currie-Hill conditions. The inverse statement is more difficult to prove, because it is not clear whether each "force" a^n satisfying the Currie-Hill equations (2) is compatible with the existence of a Lagrangian satisfying equations (13), (15)–(17).

At the first look, the obtained system of equations is more complicated than the Currie-Hill conditions, since A_{mn} evaluated from (15), (16) and inserted into (17) give a system of nonlinear equations even more untractable than (2). We hope, however, that these equations could determine some class of Lagrangians thus providing us with the whole class of dynamical models.

Let us consider the particularly simple cases when:

$$A_{11} = A_{22} = 0, \quad A_{12} \neq 0, \quad A_{21} \neq 0$$

$$A_{12} = A_{21} = 0, \quad A_{11} \neq 0, \quad A_{22} \neq 0$$

Both of them lead to Lagrangian (8).

Note that in equations (17) almost all coefficients are polynomials in velocities. This suggests to look for Lagrangians which via equations (15), (16) would give the functions A_{mn} being polynomials or ratios of two polynomials in velocities. The simplest realization of this is to put:

$$L = \frac{c_1(x)}{2}\dot{x}^1\dot{x}^1 + \frac{c_2(x)}{2}\dot{x}^2\dot{x}^2 + f(x)(\dot{x}^1+\dot{x}^2) + d(x), \tag{18}$$

where c_1, c_2, d and f are functions to be determined. Inserting (18) into (15)–(17) one finds that the only Lagrangian of the form (18) is equal to

$$L = \alpha\frac{x}{2}(\dot{x}^1\dot{x}^1 - \dot{x}^2\dot{x}^2), \tag{19}$$

with

$$A_{mn} = \begin{pmatrix} 3\dot{x}^1+\dot{x}^2, & \dot{x}^2-\dot{x}^1 \\ \dot{x}^1-\dot{x}^2, & \dot{x}^1+3\dot{x}^2 \end{pmatrix}. \tag{20}$$

This leads to the "forces" found by Kerner [10], namely

$$a^1 = -a^2 = \frac{(\dot{x}^1-\dot{x}^2)^2}{2x}.$$

Other examples, including three dimensional ones, are under investigation.

Note that within this approach constants of motion such as $P = \frac{\partial L}{\partial \dot{x}^1} + \frac{\partial L}{\partial \dot{x}^2}$ or $E = \sum_{n=1}^{2} \dot{x}^n \frac{\partial L}{\partial \dot{x}^n} - L$ do not necessarily represent physical momentum and energy with appropriate transformation properties. Here the Lagrangian L represents a mathematical object, only which enables us to determine the forces and therefore we do not fall in contradiction with the theorem proved by Jordan [11, 2].

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