CONNECTION BETWEEN THE BEHAVIOUR OF ELASTIC AND TOTAL CROSS SECTIONS AT FINITE ENERGIES

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The rigorous lower bound for the elastic-to-total cross-section ratio is derived at finite energies in analytical form. The obtained bound allows one to make numerical estimates of total cross-section which are sufficiently close to corresponding experimental data.

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One of the most important directly observed consequences of the general principles of the quantum field theory is the existence of the connection between the values of elastic and total cross sections.

The first bound on the ratio of the elastic cross section to the total one was obtained by Martin [1], who proved that at asymptotic energies

$$\frac{\sigma_{el}(s)}{\sigma_{tot}(s)} > C\,\frac{\sigma_{tot}(s)}{\ln^2(s/s_0)}. \tag{1}$$

The strongest result of this type was obtained by Singh and Roy [2] and by Łukaszuk [3]:

$$\frac{\sigma_{el}}{\sigma_{tot}} \geqslant \frac{m_\pi^2 \sigma_{tot}}{\pi}\,\frac{1}{\ln^2(s/c\sigma_{tot})}. \tag{2}$$

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The bounds (1) and (2) show, that classical Froissart-Martin bound [4, 5] can be improved by the factor σ_{el}/σ_{tot} (approximately).

In the present paper we shall show that the analogous bound is valid for finite energies as well. On the one hand, this circumstance allows to improve essentially the Froissart--Martin bound at finite energies and, on the other hand, it gives the opportunity to obtain the lower bound of the value of elastic cross section when the total cross section is known.

Let us begin from the simplest case of $\pi\pi$-scattering and consider $\pi^0\pi^0$-scattering for exactness. As in the problem of derivation of ordinary Froissart-Martin bound at finite energies [6–12], we shall use an information about the t-channel D-wave scattering length a_2^t, which, according to Froissart-Gribov formulae, can be expressed through the integral of the imaginary part of elastic amplitude for unphysical value $t = 4m_\pi^2$:

$$a_2^t = \frac{8}{15\pi}\int_4^\infty \frac{dsA(s,4)}{s^{5/2}\sqrt{s-4}}, \qquad m_\pi^2 = 1. \tag{3}$$

Our task is to obtain the bound on the total cross section within some interval of energies (s_1, s_2) if the ratio $\sigma_{el}/\sigma_{tot} \equiv b(s)$ is fixed. This bound is valid in some "middle" point of given interval.

At first, it should be noted that:

$$a_2^t > \frac{8}{15\pi}\int_{s_1}^{s_2} \frac{dsA(s,4)}{s^3} \equiv \alpha_2(s_1, s_2) \tag{3'}$$

since $A(s, t) > 0$ for $t > 0$. Representing $A(s, 4)$ as $A(s, 4) \equiv s\varphi(s, 4)$ and using the middle value theorem,

$$A(\bar{s}, 4) = \frac{15\pi}{8}\alpha_2(s_1, s_2)\frac{s_1 s_2}{s_2 - s_1}\bar{s}, \qquad \bar{s} \in (s_1, s_2) \tag{4}$$

is obtained.

For solving the problem of maximizing $A(\bar{s}, 0)$ if $A(\bar{s}, 4)$ and $b(\bar{s})$ are given, we shall use the generalized method of Lagrange multipliers [13]. For our case the Lagrange function is written as

$$\begin{aligned}\mathscr{L} = {} & \sum_{l=0}^{\infty}(2l+1)a_l(\bar{s}) + D\Big[A(\bar{s},4) - \sum_{l=0}^{\infty}(2l+1)a_l(\bar{s})P_l(z)\Big] \\ & + C\Big\{b(\bar{s})\sum_{l=0}^{\infty}(2l+1)a_l(\bar{s}) - \sum_{l=0}^{\infty}(2l+1)\left[a_l^2(\bar{s}) + r_l^2(\bar{s})\right]\Big\} \\ & + \sum_{l=0}^{\infty}(2l+1)\lambda_l\left[a_l(\bar{s}) - a_l^2(\bar{s}) - r_l^2(\bar{s})\right].\end{aligned} \tag{5}$$

Here $z \equiv 1+8/(\bar{s}-4)$; D, C, λ_l are indefinite Lagrange multipliers, $a_l(\bar{s})$ and $r_l(\bar{s})$ are imaginary and real parts of partial wave amplitude respectively; l are even.

The presence of the last term in (5) is connected with the necessity to take into account the unitarity condition:

$$a_l(s) \geqslant a_l^2(s)+r_l^2(s).$$

The variations of L with respect to a_l and r_l lead to the following partial-wave regime that realizes the maximum of $A(\bar{s}, 0)$:

$$r_l(\bar{s}) = 0$$

$$a_l(\bar{s}) = \begin{cases} 1, & \tilde{a}_l > 1 \\ \tilde{a}_l, & 0 \leqslant \tilde{a}_l \leqslant 1 \\ 0, & \tilde{a}_l < 0 \end{cases}$$

$$\tilde{a}_l \equiv \frac{1+Cb(\bar{s})-DP_l(z)}{2C} \equiv d-\frac{DP_l(z)}{2C}. \tag{6}$$

It is obvious that $\tilde{a}_l$ decreases monotonically with the increasing of l. Distracting from the fact that l are integer, we can define such L that $a_L = 0$. Later on we shall make our bound only worse taking the nearest upper even integer instead of L. Then we obtain

$$a_l = d\left[1-\frac{P_l(z)}{P_L(z)}\right].$$

It can be expected that "$a_l = 1$ regime" will not be realized for sufficiently small values of $b(s)$. For this purpose it is enough that the inequality $d \leqslant 1$ is valid.

More specifically the conditions under which the "$a_l = 1$ regime" is not realized we shall consider later, and now let us assume that they are fulfilled, and

$$a_l = \begin{cases} d\left[1-\dfrac{P_l(z)}{P_L(z)}\right], & l \leqslant L \\ 0 \qquad\qquad\quad , & l > L. \end{cases} \tag{6'}$$

Inserting (6′) into the partial-wave expansion and summing we obtain

$$A(\bar{s}, 0) = d\left[L(L+1)-2z\frac{P_L'(z)}{P_L(z)}\right],$$

$$A(\bar{s}, 4) = dP_L(z)\left\{-L(L+1)+\frac{(z^2+1)}{z}\frac{P_L'(z)}{P_L(z)}+(z^2-1)\frac{[P_L'(z)]^2}{[P_L(z)]^2}\right\},$$

$$b(\bar{s}) = d\,\frac{\left\{2L(L+1)-\dfrac{(3z^2+1)}{z}\dfrac{P_L'(z)}{P_L(z)}-(z^2-1)\dfrac{[P_L'(z)]^2}{[P_L(z)]^2}\right\}}{\left[L(L+1)-2z\dfrac{P_L'(z)}{P_L(z)}\right]}. \tag{7}$$

For simplification of equations (7) we shall use the following estimations of Legendre polynomials and their derivatives [10–12]:

$$\frac{e^{\gamma}}{\sqrt{2\pi\gamma}}\left(1+\frac{1}{8\gamma}\right) \leqslant P_n(\operatorname{ch}\alpha) \leqslant \frac{e^{\gamma}}{\sqrt{2\pi\gamma}}\left(1+\frac{1}{8\gamma}+\frac{2}{15\gamma^2}\right),$$

$$\frac{1}{\alpha\cdot\operatorname{sh}\alpha}\cdot\frac{e^{\gamma}\sqrt{\gamma}}{\sqrt{2\pi}}\left(1-\frac{3}{8\gamma}-\frac{1}{4\gamma^2}\right) \leqslant P_n'(\operatorname{ch}\alpha) \leqslant \frac{1}{\alpha\cdot\operatorname{sh}\alpha}\cdot\frac{e^{\gamma}\sqrt{\gamma}}{\sqrt{2\pi}}\left(1-\frac{3}{8\gamma}\right),$$

$$\gamma \equiv (n+\tfrac{1}{2})\alpha, \tag{8}$$

which are valid for $\gamma > 1$, $\alpha \ll 1$. In our case $\operatorname{ch}\alpha = 1+8/(\bar{s}-4)$, $\alpha < 4/\sqrt{\bar{s}-4}$, i.e. the condition $\alpha \ll 1$ is practically fulfilled if $s \gtrsim 10$ GeV^2. Using inequalities (8) we can derive the following bound on the total cross section for $\gamma > 1$:

$$\sigma_{\text{tot}}(\bar{s}) < b(\bar{s})\gamma^2\left(1-\frac{1}{\gamma}+\frac{1}{\gamma^2}\right) < b(\bar{s})\gamma^2. \tag{9}$$

The value of γ is determined from condition (for $\gamma > 2$):

$$A(\bar{s}, 4) > b(\bar{s})\frac{\sqrt{2\gamma}\, e^{\gamma}}{\sqrt{\pi}\,\alpha^2}\, g(\gamma), \tag{10}$$

where

$$g(\gamma) \equiv 1-\frac{1}{4\gamma}+\frac{1}{\gamma^2}-\frac{2}{\gamma^3}.$$

It is easily seen that the function $g(\gamma)$ increases monotonically for $\gamma > 2$ and that $\lim_{\gamma\to\infty} g(\gamma) = 1$. Since $g(\gamma) > 0.85$ for $\gamma > 2$ and $g(\gamma) > 0.95$ for $\gamma > 2.5$, the influence of this function is very weak. Solving (10) as in Ref. [11] and conserving the main terms in the expression for γ, we come to the following final inequality for the total cross section:

$$\sigma_{\text{tot}}(\bar{s}) < \pi\left(\frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}}\right)\ln^2\frac{\chi}{\sqrt{e\cdot\ln\chi}}, \tag{11}$$

where

$$\chi \equiv \frac{15\pi^{3/2}\alpha_2(s_1, s_2)}{\sqrt{2}\, b(\bar{s})}\cdot\frac{s_1 s_2}{(s_2-s_1)}.$$

The dependence of our bound on s will be more obvious if we use another way of averaging $A(s, 4)$ in (3′). Representing $A(s, 4)$ ad $A(s, 4) \equiv s^2\varphi(s, 4)$, we obtain

$$A(\bar{s}, 4) = \frac{15\pi}{8}\alpha_2(s_1, s_2)\frac{\bar{s}^2}{\ln(s_2/s_1)}$$

instead of (4), and consequently χ in (11) is replaced by

$$\chi = \frac{15\pi^{3/2}\alpha_2(s_1, s_2)}{\sqrt{2}\, b(\bar{s})\ln(s_2/s_1)}\cdot\bar{s}.$$

We are still to make clear under what condition the supposed partial wave regime is valid. This problem can be solved by using (7) and (8):

$$d(\bar{s}) < b(\bar{s})\left(1+\frac{1}{\gamma}+\frac{3}{\gamma^2}\right). \tag{12}$$

In the case of $\pi^0\pi^0$-scattering γ is proved to be greater than 2 even for the energies higher than several GeV. Consequently, the necessary condition $d(\bar{s}) \leqslant 1$ is fulfilled if $b(\bar{s}) \leqslant 0.44$. Since γ monotonically increases with energy, the condition, imposed on $b(\bar{s})$, becomes more and more liberal with the growth of energy.

Let us now consider the problem of πp-scattering. In this case we must take spin into account. For the exactness we shall consider the crossing-symmetric process of pion-proton scattering, i.e. we shall look for an upper bound on the quantity

$$\sigma_{\text{tot}}^{(+)} = \tfrac{1}{2}\left(\sigma_{\text{tot}}^{\pi^+\text{p}}+\sigma_{\text{tot}}^{\pi^-\text{p}}\right) = \tfrac{1}{3}\left(\sigma_{\text{tot}}^{(1/2)}+2\sigma_{\text{tot}}^{(3/2)}\right).$$

For taking spin into account we shall utilize the helicity formalism [14]. Then

$$A(s,t) = \sum_{l=0}^{\infty}\left[(l+1)B_l\left(1+\frac{t}{2q^2}\right)a_{l+}(s)+lB_{l-1}\left(1+\frac{t}{2q^2}\right)a_{l-}(s)\right], \tag{13}$$

where

$$B_l(x) = \frac{P'_{l+1}(x)-P'_l(x)}{l+1},$$

q is the c.m. momentum.

As it has been shown by Yndurain and Common [14], the Froissart-Gribov formulae in this case lead to the following bound on $A\,(s, 4\mu^2)$:

$$\int_{(M+\mu)^2}^{\infty}\frac{1+e^{-|\varphi'|}}{(s-M^2+\mu^2)^3}\cdot\frac{\sqrt{s}}{q}A(s,4\mu^2)ds \leqslant \frac{\hat{a}}{4\pi}, \tag{14}$$

where

$$\hat{a} = \frac{15\pi\alpha_2^{(+)}}{2(M^2-\mu^2)}-\frac{G^2}{\mu^4},$$

$$\alpha_2^{(+)} = \lim_{t\to 4\mu^2}\frac{h_2^{(+)}(t)}{4\mu^2-t}.$$

Here $h_2^{(+)}(t)$ is the t-channel D-wave, M and μ are the masses of nucleon and π-meson respectively, G^2 is the πN$\overline{\text{N}}$ coupling constant,

$$\sin\varphi' = -\frac{2M\sqrt{t[su-(M^2-\mu^2)^2]}}{\sqrt{\lambda(s,M^2,\mu^2)\cdot\lambda(u,M^2,\mu^2)}},$$

$$\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2yz-2xz.$$

Since

$$B_L(z) = P_L(z) + \frac{(z-1)}{(L+1)} P'_L(z) = P_L(z)\left[1 + \frac{\alpha}{2} + 0(\alpha^2)\right], \quad \alpha \sim \frac{\mu}{\sqrt{s}},$$

the decomposition of $A(s, t)$ in equation (13) coincides with an ordinary decomposition in Legendre polynomials in the spinless case with the precision of terms, having the order $\mu/\sqrt{s}$. Therefore the inequality (11) remains valid.

Let us now consider the numerical calculations of our bound on the total cross sections for different intervals (s_1, s_2). For this purpose we shall evaluate the integrals in the left side of (14) between the limits $(M+\mu)^2$ and s_1, using the experimental data on the total and elastic cross sections of πp-scattering (s_1 being the initial point of the interval within which we are looking for the bound). For evaluation of low-energy part of integral below (2.3 GeV)^2 the calculations that are based on phase analysis [15, 16] were carried out. By inverting the considered problem we can get, using the same method, the lower bound on $A(s, 4\mu^2)$ if the total cross section is given. And then we can evaluate our integral between limits (2.3 Gev)^2 and s_1.

According to calculations of Yndurain and Common [14], the value of $\hat{a}$ in (14) is equal to 0.177 μ^{-4}. Our calculations show that integral of $A(s, 4\mu^2)$ from the beginning of the cut to (2.3 GeV)^2 is equal to 0.139 μ^{-4}.

Here are the final evaluations of quantity

$$\hat{a}(s_1) \equiv \hat{a} - 4\pi \int\limits_{(M+\mu)^2}^{s_1} ds \frac{1+e^{-|\varphi'|}}{(s-M^2+\mu^2)^3} \cdot \frac{\sqrt{s}}{q} A(s, 4\mu^2):$$

$s_1 = (2.3\ \text{GeV})^2 \qquad \hat{a}(s_1) = 3.8 \times 10^{-2}\ \mu^{-4}$

$s_1 = (4.0\ \text{GeV})^2 \qquad \hat{a}(s_1) = 1.8 \times 10^{-2}\ \mu^{-4}$

$s_1 = (6.9\ \text{GeV})^2 \qquad \hat{a}(s_1) = 1.2 \times 10^{-2}\ \mu^{-4}$

$s_1 = (12.0\ \text{GeV})^2 \qquad \hat{a}(s_1) = 8.7 \times 10^{-3}\ \mu^{-4}$

The final bounds on total cross section are given in the Table.

TABLE

Upper bounds on total cross sections of crossing-symmetric πp-scattering

Interval of lab. momenta (GeV/c)	Interval of c.m. energies (GeV)	$\sigma_{tot}^{(+)}(s)$ (mb)
2.3÷25	2.3÷6.9	37.3
8.0÷76	4.0÷12.0	44.8
25÷230	6.9÷20.8	58.0
76÷690	12.0÷36.0	77.5

Thus, the obtained bounds are sufficiently close to saturation by experimental data and considerably tighter than analogous bounds that do not take into account the ratio of elastic cross section to the total one. A comparatively slow growth of bounds with increasing of energy should be noted, which is connected with the decreasing of ratio of elastic cross section to the total one with the growth of energy.

It must be noted in conclusion that the inequality (11) can be used for obtaining the evaluations of elastic cross section for the energies at which the experimental data exist only for the total cross section.

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