ON GEODESIC NORMAL FLOW OF PERFECT AND VISCOUS FLUID

By V. I. Obozov

Orenburg Polytechnic Institute*

(Received December 11, 1981)

It is shown that flow-lines form time-like shear-free normal congruence of geodesics only in conformally flat gravitational fields of a perfect and viscous fluid. This is found to be a necessary and sufficient condition that the gravitational fields of a perfect and viscous fluid are conformal to a flat space-time.

PACS numbers: 04.20.–q

1. Introduction

In [4] we have established the following theorem: In nonstationary gravitational fields of an irrotational perfect fluid the acceleration of the fluid particles is zero, but this theorem is valid only for some nonstationary gravitational fields of an irrotational perfect fluid.

The purpose of the present paper is to show that the flow-lines are geodesics only in the conformally flat gravitational fields of a perfect and viscous fluid. In Sections 2, 3 we consider the gravitational fields of a perfect fluid. In Section 4 we investigate a viscous fluid.

In Section 2 it is found that the condition $t_k = 0$ gives the relations $\omega_{ik} = 0$, $\sigma_{ik} = 0$, where t_k, ω_{ik}, σ_{ik} represent the acceleration, vorticity and shear of the flow, respectively.

In Section 3 we have established a necessary and sufficient condition for the gravitational fields of a perfect fluid to be conformal to a flat space-time. For the realistic description of a perfect fluid we used the realistic equation of state in the form $\varrho = \varrho(p)$, where ϱ is density, p is pressure.

In Section 4 it is shown that a necessary and sufficient condition for the gravitational fields of a viscous fluid to be conformal to a flat space-time is equivalent to the same condition for a perfect fluid. Here we used the equation of the viscous fluid state in the form $\varrho = \varrho(p)$, $\zeta = \zeta(p)$, $\eta = \eta(p)$, where η and ζ denote the coefficients of shear and bulk

* Address: Orenburg Polytechnic Institute, Orenburg, 460018, USSR.

viscosity, respectively. We chose this equation of state because of the following consideration: The energy-momentum tensor of a viscous fluid can be written as

$$T_{ik} = (\varrho+\tilde{p})u_i u_k - \tilde{p} g_{ik} + 2\eta\sigma_{ik}, \tag{1.1}$$

where $\tilde{p} = p - \zeta\theta$, $\theta = u^n{}_{;n}$ is the volume expansion, u_i is the dynamical velocity and $c = 1$ (c is the light velocity). From (1.1) it follows that the flow of a viscous fluid is a thermal-free flow. Actually [1] the energy-momentum tensor for the thermal flows assumes a specific form. If the thermal flow in a viscous fluid is absent, the fluid motion is isothermal. Therefore, the equation of state in this case is $\varrho = \varrho(p)$, $\eta = \eta(p)$, $\zeta = \zeta(p)$.

2. *Velocity field of a perfect fluid with geodesic flow*

In the general case we have for the gravitational fields of a perfect fluid that

$$t_i = u_{i;n}u^n = \frac{1}{\varrho+p}(p_{,i} - p_{,n}u^n u_i), \tag{2.1}$$

where u_i is the velocity ($u_n u^n = 1$), $p_{,i} = \dfrac{\partial p}{\partial x^i}$, $c = 1$ and the semicolon denotes covariant differentiation.

If $t = 0$, it follows from (2.1) that

$$p_{,i} = p_{,n}u^n u_i. \tag{2.2}$$

The condition (2.2) implies that the corresponding family of streamlines is a normal congruence of geodesics. Actually we have from (2.2)

$$p_{,i;k} - p_{,k;i} = (p_{,n}u^n)_{,k}u_i - (p_{,n}u^n)_{,i}u_k + p_{,n}u^n(u_{i;k} - u_{k;i}) = 0. \tag{2.3}$$

Contracting Eq. (2.3) with u^i by virtue of $u_n u^n = 1$, $u_{n;k}u^n = 0$, $t_k = u_{k;n}u^n = 0$ gives

$$(p_{,n}u^n)_{,k} = (p_{,n}u^n)_{,m}u^m u_k,$$

then it follows from (2.3) that $u_{i;k} = u_{k;i}$. If $u_{i;k} = u_{i;k}$ the vorticity of the streamlines is zero, i.e. $\omega_{ik} = 0$. Consequently a family of streamlines with $t_i = 0$ is a normal congruence of geodesics. In this case the covariant derivative of the velocity field may be decomposed in the following way

$$u_{i;k} = \frac{\theta}{3}(g_{ik} - u_i u_k) + \sigma_{ik}, \tag{2.4}$$

where $\theta = u^n{}_{;n}$ is the expansion of volume, σ_{ik} is shear tensor. The shear tensor satisfies the relations

$$\sigma_{ik} = \sigma_{ki}, \quad \sigma_{nk}u^n = 0, \quad g^{nk}\sigma_{nk} = 0.$$

Further, we consider the gravitational fields of the perfect fluid satisfying (2.4). The field equations for a perfect fluid are

$$R_{ik}-\tfrac{1}{2}Rg_{ik} = -(\varrho+p)u_iu_k+pg_{ik}, \tag{2.5}$$

where units are chosen so that $8\pi G = c = 1$; the symbols have the usual significance.

The Bianchi identities, $R_{ik[lm;s]} = 0$, are equivalent to

$$R_{i[k;l]}-\tfrac{1}{6}g_{i[k}R_{,l]} = C^{n}{}_{ikl;n}, \tag{2.6}$$

where $C^{n}{}_{ikl}$ is the Weyl tensor, the square brackets denote antisymmetrization. Substituting R_{ik}, R from (2.5) and $u_{i;k}$ from (2.4) into (2.6), we obtain

$$\tfrac{1}{3}g_{i[k}\varrho_{,l]}-u_iu_{[k}\varrho_{,l]}-\frac{(\varrho+p)\theta}{3}g_{i[l}u_{k]}-(\varrho+p)\sigma_{i[l}u_{k]} = C^{n}{}_{ikl;n}. \tag{2.7}$$

Simultaneously we have from the conservation equations, $T^{n}{}_{k;n} = 0$, where $T^{n}{}_{k}$ is the energy-momentum tensor of a perfect fluid, that

$$\varrho_{,n}u^{n}+(\varrho+p)\theta = 0. \tag{2.8}$$

It follows from (2.2) and the equation of state $\varrho = \varrho(p)$ that

$$\varrho_{,i} = \varrho_{,n}u^{n}u_{i}.$$

With the aid of Eq. (2.8) we can rewrite the above relation in the following way

$$\varrho_{,i} = -(\varrho+p)\theta u_{i}.$$

Substituting this relation into (2.7) we obtain

$$(\varrho+p)\sigma_{i[k}u_{l]} = C^{n}{}_{ikl;n}. \tag{2.9}$$

Contraction of Eq. (2.9) with u^{i} by virtue of $u^{n}\sigma_{nk} = 0$ gives

$$u^{m}C^{n}{}_{mkl;n} = 0. \tag{2.10}$$

If $C^{n}{}_{mkl;n} \neq 0$, generally speaking, from (2.9) it follows that $\sigma_{ik} \neq 0$. Yet in [2] it is shown that the gravitational fields of a perfect fluid with (2.4), where $\sigma_{ik} \neq 0$, do not exist in General Relativity.

Therefore we now consider the condition $C^{n}{}_{ikl;n} = 0$ satisfying Eq. (2.10). In this case it follows from (2.9) that $\sigma_{i[k}u_{l]} = 0$. Contracting this with u^{l} and using $u^{n}\sigma_{in} = 0$, we obtain $\sigma_{ik} = 0$. Consequently in the permissible cases from (2.10) it follows that $\sigma_{ik} = 0$. Hence we have the following theorem:

Theorem 2.1. The velocity field of a perfect fluid with geodesic flow is defined by

$$u_{i;k} = \frac{\theta}{3}(g_{ik}-u_iu_k). \tag{2.11}$$

3. *Necessary and sufficient condition for* $C^n{}_{ikl} = 0$

In this Section we prove the following theorem:

Theorem 3.1. Eq. (2.11) is the necessary and sufficient condition for the gravitational field of a perfect fluid to be conformal to a flat space-time.

3.1. The proof of necessity

Let a gravitational field of a perfect fluid be conformally flat. In this case we have $C^n{}_{ikl} = 0$. From Eq. (2.6) with the aid of Eq. (2.5) and $C^n{}_{ikl} = 0$ we find

$$(\varrho+p)u_i u_{[k;l]} + (\varrho+p)u_{i[;l}u_{k]} - \tfrac{1}{3} g_{i[k}\varrho_{,l]} - u_i u_{[l}(\varrho+p)_{,k]} = 0. \tag{3.1}$$

Moreover the metric of conformally flat gravitational fields may be written as

$$ds^2 = \phi^2[(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2].$$

Hence we have that the vorticity in this case is zero, i.e. $\omega_{ik} = 0$. It the vorticity is zero, the tensor $u_{i;k}$ may be split up as follows

$$u_{i;k} = t_i u_k + \frac{\theta}{3}(g_{ik}-u_i u_k)+\sigma_{ik}.$$

Hence $u_{[i;k]} = t_{[i}u_{k]}$, where the tensor t_i satisfies Eq. (2.1). With the aid of $u_{[i;k]} = t_i u_k$ and (2.1) and some simplifications from (3.1) we have

$$(\varrho+p)u_{i[;l}u_{k]} - \tfrac{1}{3} g_{i[k}\varrho_{,l]} - u_i u_{[l}\varrho_{,k]} = 0. \tag{3.2}$$

The contraction of Eq. (3.2) with u^i gives

$$\varrho_{,k}u_l = \varrho_{,l}u_k. \tag{3.3}$$

The field equations are given for the equation of state for which $\varrho = \varrho(p)$. Therefore from (3.3) and $\varrho = \varrho(p)$ it follows that

$$p_{,k}u_l = p_{,l}u_k.$$

In this case by virtue of (2.1) we have that $t_i = 0$. The contraction (of (3.2) with u^k, by virtue of $t_i = 0$ and (3.3), gives

$$u_{i;l} = -\frac{(\varrho_{,n}u^n g_{il} - \varrho_{,l}u_i)}{3(\varrho+p)}. \tag{3.4}$$

From (3.3) we have $\varrho_{,l} = \varrho_{,n}u^n u_l$. Therefore Eq. (3.4) is equivalent to

$$u_{i;l} = -\frac{\varrho_{,n}u^n}{3(\varrho+p)}(g_{il}-u_i u_l).$$

With the aid of this and (2.8) we obtain (2.11).

Consequently, the necessity of condition (2.11) for $C^n{}_{ikl} = 0$ is proved.

3.2. The proof of sufficiency

Let the velocity field of a perfect fluid satisfies (2.11). Then we prove that the corresponding gravitational field of a perfect fluid is conformally flat.

The integrability conditions of (2.11) are equivalent to

$$u_{i[;k;l]} = \tfrac{1}{2}\, u_n R^n{}_{ikl},$$

where $R^n{}_{ikl}$ is the Riemann tensor of space-time. This, by virtue of (2.11), can be written as follows

$$\tfrac{1}{3}\, g_{i[k}\theta_{,l]} - \frac{u_i}{3}\, u_{[k}\theta_{,l]} - \frac{\theta^2}{9}\, g_{i[l}u_{k]} = \tfrac{1}{2}\, u_n R^n{}_{ikl}. \tag{3.5}$$

In the general case $C^n{}_{ikl} \neq 0$ and we have

$$R_{nikl} = C_{nikl} + R_{i[k}g_{l]n} + R_{n[l}g_{k]i} - \tfrac{1}{3}\, R g_{i[k}g_{l]n}. \tag{3.6}$$

With the aid of (3.6), (2.5) and (3.5) we find

$$\tfrac{1}{3}\, g_{i[k}\theta_{,l]} - \frac{u_i}{3}\, u_{[k}\theta_{,l]} - \frac{\theta^2}{9}\, g_{i[l}u_{k]} - \frac{\varrho+3p}{6}\, g_{i[l}u_{k]} = \tfrac{1}{2}\, u_n C^n{}_{ikl}. \tag{3.7}$$

The contraction of Eq. (3.7) on i and l gives

$$\tfrac{1}{3}\,\theta_{,n}u^n u_k + \tfrac{2}{3}\,\theta_{,k} + \frac{\theta^2}{3}\, u_k + \frac{\varrho+3p}{2}\, u_k = 0. \tag{3.8}$$

From (3.8) it follows that

$$u_{[l}\theta_{,k]} = 0, \qquad \theta_{,k} = \theta_{,n}u^n u_k. \tag{3.9}$$

Using these relations in (3.8) we obtain

$$\theta_{,k} = -\left(\frac{\theta^2}{3} + \frac{\varrho+3p}{2}\right) u_k. \tag{3.10}$$

Substituting Eqs (3.9), (3.10) into (3.7) we find $u_n C^n{}_{ikl} = 0$. By [3], if $u_n u^n \neq 0$ and the dimensionality of space-time $n = 4$, the condition $u_n C^n{}_{ikl} = 0$ implies that $C^n{}_{ikl} = 0$. $C^n{}_{ikl} = 0$ means that the corresponding gravitational field of a perfect fluid is conformally flat.

Consequently theorem 3.1 is proved.

4. *The viscous fluid with geodesic flow*

The field equations for the energy-momentum tensor (1.1) are the following

$$R_{ik} - \tfrac{1}{2}\, R g_{ik} = -(\varrho + \tilde{p})u_i u_k + \tilde{p} g_{ik} - 2\eta\sigma_{ik}, \tag{4.1}$$

where units are chosen so that $8\pi G = c = 1$. If the fields satisfying (4.1) are conformally flat, it follows from Bianchi identities (2.6) that

$$(\varrho + \tilde{p})t_k = \tfrac{2}{3}\,(\varrho_{,k} - \varrho_{,n}u^n u_k) + \tilde{p}_{,k} - \tilde{p}_{,n}u^n u_k + 2\eta t^n \sigma_{nk}, \tag{4.2}$$

where $t_k = u_{k;n}u^n$ is the acceleration. If $t_k = 0$, Eq. (4.2) can be written as follows

$$\tfrac{2}{3}(\varrho_{,k}-\varrho_{,n}u^n u_k)+\tilde{p}_{,k}-\tilde{p}_{,n}u^n u_k = 0. \tag{4.3}$$

It can be shown after the detailed calculations which are similar to the calculations in Sec. 2, that Eq. (4.3) implies $\omega_{ik} = 0$. The condition $\omega_{ik} = 0$ defines the normal congruence of geodesics. If $\omega_{ik} = 0$ and $t_k = 0$, the tensor $u_{i;k}$ may be split as follows

$$u_{i;k} = \frac{\theta}{3}(g_{ik}-u_i u_k)+\sigma_{ik}. \tag{4.4}$$

Now we consider the case $\sigma_{ik} = 0$. If $\sigma_{ik} = 0$, Eq. (4.4) is reduced to

$$u_{i;k} = \frac{\theta}{3}(g_{ik}-u_i u_k) \tag{4.5}$$

and the field equations are equivalent to

$$R_{ik}-\tfrac{1}{2}Rg_{ik} = -(\varrho+\tilde{p})u_i u_k+\tilde{p}g_{ik}. \tag{4.6}$$

Eq. (4.6) is similar to Eq. (2.5). Therefore, it can be shown with the aid of calculations which are similar to the calculations in Sec. 3 that Eq. (4.5) is a sufficient condition for the gravitational field (4.1) to be conformal to a flat space-time.

Now we prove that Eq. (4.5) is a necessary condition for the gravitational field of a viscous fluid to be conformal to a flat space-time.

Let the gravitational field of a viscous fluid be conformally flat, i.e. $C^n{}_{ikl} = 0$. In this case the Bianchi identities, $R_{ik[lm;n]} = 0$, are equivalent to

$$R_{i[k;l]}-\tfrac{1}{6}g_{i[k}R_{,l]} = 0.$$

With the aid of Eq. (4.1) we can write the above relation as follows

$$\begin{aligned}&\tfrac{1}{3}g_{i[k}\varrho_{,l]}-u_i u_{[k}(\varrho+\tilde{p})_{,l]}-(\varrho+\tilde{p})u_{i[;l}u_{k]}\\ &-(\varrho+\tilde{p})u_i u_{[k;l]}-2\sigma_{i[k}\eta_{,l]}-2\eta\sigma_{i[k;l]} = 0.\end{aligned} \tag{4.7}$$

If $C^n{}_{ikl} = 0$, the metric of space-time may be written in a special form

$$ds^2 = f^2[(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2].$$

The above expression implies that $\omega_{ik} = 0$. In this case the velocity field of a viscous fluid may be decomposed as

$$u_{i;k} = t_i u_k+\frac{\theta}{3}(g_{ik}-u_i u_k)+\sigma_{ik}. \tag{4.8}$$

Substituting (4.8) into Ricci's identities

$$u_{i[;k;l]} = \tfrac{1}{2}u_n R^n{}_{ikl}$$

we obtain

$$\tfrac{1}{2}\, u_n R^n{}_{ikl} = t_{i[;l}u_{k]} + t_i t_{[k}u_{l]} + \tfrac{1}{3}\, g_{i[k}\theta_{,l]}$$

$$-\frac{u_i}{3}\, u_{[k}\theta_{,l]} - \frac{\theta^2}{9}\, g_{i[l}u_{k]} - \frac{\theta}{3}\,\sigma_{i[l}u_{k]} - \frac{\theta}{3}\, u_i t_{[k}u_{l]} + \sigma_{i[k;l]}. \tag{4.9}$$

Now with the aid of Eq. (4.2), (4.7), (4.8) we obtain

$$\tfrac{1}{3}\, h_{i[k}\varrho_{,l]} + \frac{(\varrho+\tilde{p})\theta}{3}\, h_{i[k}u_{l]} - (\varrho+\tilde{p})\sigma_{i[l}u_{k]} - 2\eta u_i t^n \sigma_{n[k}u_{l]}$$

$$-2\sigma_{i[k}\eta_{,l]} - 2\eta\sigma_{i[k;l]} = 0, \tag{4.10}$$

where $h_{ik} = g_{ik} - u_i u_k$. Further from (4.10) we find $\sigma_{i[k;l]}$ and use this in (4.9). As a result we obtain

$$\eta u_n R^n{}_{ikl} = 2\eta \left\{ t_{i[;l}u_{k]} + t_i t_{[k}u_{l]} + \tfrac{1}{3}\, g_{i[k}\theta_{,l]} - \frac{u_i}{3}\, u_{[k}\theta_{,l]} \right.$$

$$\left. - \frac{\theta^2}{9}\, g_{i[l}u_{k]} - \frac{\theta}{3}\,\sigma_{i[l}u_{k]} - \frac{\theta}{3}\, u_i t_{[k}u_{l]} \right\} + \tfrac{1}{3}\, h_{i[k}\varrho_{,l]}$$

$$+\tfrac{1}{3}(\varrho+\tilde{p})\theta h_{i[k}u_{l]} - (\varrho+\tilde{p})\sigma_{i[l}u_{k]} - 2\eta u_i t^n \sigma_{n[k}u_{l]} - 2\sigma_{i[k}\eta_{,l]}. \tag{4.11}$$

Contraction of (4.11) with u^l by virtue of $t_n u^n = 0$, $\sigma_{in}u^n = 0$, $h_{in}u^n = 0$, gives

$$2\eta t_{i;k} = 2\eta t_{i;n}u^n u_k - 2\eta u^n u^m R_{nikm} + 2\eta t_i t_k$$

$$+ \frac{2\eta}{3}\,\theta_{,n}u^n h_{ik} + \frac{2\eta\theta^2}{9}\, h_{ik} + \frac{2\eta\theta}{3}\,\sigma_{ik} - \frac{2\eta\theta}{3}\, u_i t_k + (\varrho+\tilde{p})\sigma_{ik}$$

$$+\tfrac{1}{3}\,[\varrho_{,n}u^n + (\varrho+\tilde{p})\theta] h_{ik} - 2\eta_{,n}u^n\sigma_{ik} - 2\eta u_i t^n \sigma_{nk}. \tag{4.12}$$

Substituting $2\eta t_{i;k}$ and $2\eta t_{i;l}$ from (4.12) into (4.11) we find

$$\eta(u_n R^n{}_{ikl} + u^n u^m R_{nilm}u_k - u^n u^m R_{nikm}u_l)$$

$$= \frac{2\eta}{3}\, g_{i[k}\theta_{,l]} - \frac{2\eta}{3}\, u_i u_{[k}\theta_{,l]} + \frac{2\eta}{3}\,\theta_{,n}u^n g_{i[l}u_{k]} + \tfrac{1}{3}\, h_{i[k}\varrho_{,l]}$$

$$+\tfrac{1}{3}\,\varrho_{,n}u^n h_{i[l}u_{k]} - 2\sigma_{i[k}\eta_{,l]} - 2\eta_{,n}u^n\sigma_{i[l}u_{k]}. \tag{4.13}$$

If $C^n{}_{ikl} = 0$, we have that

$$R_{nikl} = R_{i[k}g_{l]n} + R_{n[l}g_{k]i} - \frac{R}{3}\, g_{n[l}g_{k]i}.$$

Using this relation and Eq. (4.1) we can rewrite Eq. (4.13) as follows

$$\frac{2\eta}{3}\, h_{i[k}\theta_{,l]} - \frac{2\eta}{3}\,\theta_{,n}u^n h_{i[k}u_{l]} + \tfrac{1}{3}\, h_{i[k}\varrho_{,l]}$$

$$-\tfrac{1}{3}\,\varrho_{,n}u^n h_{i[k}u_{l]} - 2\eta_{,n}u^n\sigma_{i[l}u_{k]} - 2\sigma_{i[k}\eta_{,l]} = 0. \tag{4.14}$$

Contraction of (4.14) with g^{ik} gives

$$\frac{2\eta}{3}(\theta_{,l}-\theta_{,n}u^n u_l)+\tfrac{1}{3}(\varrho_{,l}-\varrho_{,n}u^n u_l)+\eta^n\sigma_{rl} = 0, \tag{4.15}$$

where $\eta^n = g^{nm}\eta_{,m}$. Substituting (4.15) into (4.14) we find

$$\eta^n\sigma_{n[k}h_{l]i}-2\eta_{,n}u^n\sigma_{i[l}u_{k]}-2\sigma_{i[k}\eta_{,l]} = 0.$$

This relation is equivalent to

$$\eta^n a_{nikl} = 0, \tag{4.16}$$

where

$$a_{nikl} = \sigma_{n[k}h_{l]i}+2\sigma_{i[l}h_{k]n}.$$

By virtue of $u^n\sigma_{nk} = 0$, $u^n h_{nk} = 0$ we have

$$u^n a_{nikl} = 0. \tag{4.17}$$

It is easily seen that both equations (4.16) and (4.17) are satisfied by

$$a_{nikl} = 0$$

or

$$\eta_{,k} = \eta_{,n}u^n u_k \tag{4.18}$$

(the solution $\eta_{,k} = 0$ does not satisfy Eq. (4.16) and (4.1) by virtue of Eqs $\eta = \eta(p)$, $\zeta = \zeta(p)$, $\varrho = \varrho(p)$).

If $a_{nikl} = 0$, the contraction of $a_{nikl} = 0$, on i and l gives $\sigma_{nk} = 0$. By [5], if $\sigma_{nk} = 0$, $C^n{}_{ikl} = 0$, we have that $t_k = 0$. Consequently in this case the condition (4.8) is transformed into the condition (4.5). Therefore in this case the necessity of condition (4.5) for $C^n{}_{ikl} = 0$ is proved.

Let $a_{nikl} \neq 0$, but Eq. (4.18) is valid. From (4.18) and the equations of state $\varrho = \varrho(p)$, $\zeta = \zeta(p)$, $\eta = \eta(p)$ it follows that

$$\varrho_{,k} = \varrho_{,n}u^n u_k, \quad \zeta_{,k} = \zeta_{,n}u^n u_k, \quad p_{,k} = p_{,n}u^n u_k. \tag{4.19}$$

With the aid of (4.19), (4.18), (4.15) we have

$$\theta_{,k} = \theta_{,n}u^n u_k. \tag{4.20}$$

Now using Eq. (4.20), (4.19) in (4.2) we obtain

$$(\varrho+\tilde{p})t_k = 2\eta t^n\sigma_{nk}.$$

By virtue of $t_n u^n = 0$ the above relation is equivalent to

$$t^n b_{nk} = 0, \quad (\varrho+\tilde{p})h_{nk}-2\eta\sigma_{nk} = b_{nk}.$$

It is easily seen that the equation

$$u^n b_{nk} = 0$$

is valid. From $t^n b_{nk} = 0$, $u^n b_{nk} = 0$ by virtue of $t_n u^n = 0$ it follows that $t_k = 0$ or $b_{nk} = 0$.

If $b_{nk} = 0$, the analysis gives $\varrho+\tilde{p} = 0$ and $\sigma_{nk} = 0$. In this case Eq. (4.8) is reduced to (4.5) [5].

Now let $b_{nk} \neq 0$ and $t_k = 0$. By virtue of $t_k = 0$ we can write Eq. (4.8) as

$$u_{i;k} = \frac{\theta}{3}(g_{ik}-u_iu_k)+\sigma_{ik}. \tag{4.21}$$

If Eq. (4.21) is valid we can use a co-moving coordinate system (CCS). In CCS we have the following relations

$$u_0 = u^0 = 1, \quad u_\alpha = u^\alpha = 0, \quad g_{00} = g^{00} = 1,$$

$$g_{0\alpha} = 0, \quad -g_{\alpha\beta} = \gamma_{\alpha\beta}, \quad \alpha, \beta \ldots = 1, 2, 3,$$

$$ds^2 = (dx^0)^2-\gamma_{\alpha\beta}dx^\alpha dx^\beta. \tag{4.22}$$

In this case it follows from (4.20) and (4.21) that

$$\theta = u^n{}_{;n} = \tfrac{1}{2}\gamma^{\alpha\beta}\frac{\partial\gamma_{\alpha\beta}}{\partial x^0} = f(x^0). \tag{4.23}$$

It can be shown that the condition (4.23) and $C^n{}_{ikl} = 0$ are satisfied, if the metric form (4.22) is reduced to

$$ds^2 = (dx^0)^2-F(x^0)c_{\alpha\beta}(x^\sigma)dx^\alpha dx^\beta. \tag{4.24}$$

In CCS Eq. (4.21) can be rewritten as

$$\kappa_{\alpha\beta} = \tfrac{1}{3}\kappa^\sigma_\sigma\gamma_{\alpha\beta}-2\sigma_{\alpha\beta}, \tag{4.25}$$

where

$$\kappa_{\alpha\beta} = \frac{\partial\gamma_{\alpha\beta}}{\partial x^0}, \quad \kappa^\sigma_\sigma = \gamma^{\alpha\beta}\kappa_{\alpha\beta}.$$

The substitution of $\gamma_{\alpha\beta} = F(x^0)c_{\alpha\beta}(x^\sigma)$ from (4.24) into (4.25) gives $\sigma_{\alpha\beta} = 0$. Simultaneously by virtue of $u^n\sigma_{nk} = 0$ and (4.22) we have $\sigma_{00} = 0$, $\sigma_{0\alpha} = 0$. The relations $\sigma_{00} = 0$, $\sigma_{0\alpha} = 0$, $\sigma_{\alpha\beta} = 0$ are equivalent to $\sigma_{ik} = 0$. Consequently if $t_k = 0$, $b_{nk} \neq 0$. Eq. (4.8) is transformed into Eq. (4.5).

We have considered all the cases. In all the cases, if $C^n{}_{ikl} = 0$, the velocity field of a viscous fluid is defined by (4.5). Therefore the necessity of the condition (4.5) for $C^n{}_{ikl} = 0$ is proved. Consequently we have proved the following theorem:

Theorem 4.1. The relation (4.5) is a necessary and sufficient condition for the gravitational field of a viscous fluid to be conformal to a flat space-time.

Editorial note. This article was proofread by the editors only, not by the author.

REFERENCES

[1] G. W. Misner, K. S. Thorne, I. A. Wheeler, *Gravitation*, W. H. Freeman and Company, San Francisco 1973, vol. 2.
[2] V. I. Obozov, Dep. in VINITI, reg. Dep. N 4781-80 (1980).
[3] H. W. Brinkmann, *Ann. Math.* **91**, 269 (1924).
[4] V. I. Obozov, *Acta Phys. Pol.* **B13**, 17 (1982).
[5] V. I. Obozov, *Izv. Vusov MV SSO SSSR, ser. fiz.* N 1, 36 (1981).