A CLASSICAL COLORED PARTICLE WITH SPIN IN AN EXTERNAL YANG-MILLS FIELD

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(Received January 6, 1982)

We consider the motion of a wave packet in the external SU (2) gauge field using the Foldy-Wouthuysen representation for the Dirac equation. We find that a satisfactory description of the motion of the wave packet in terms of a trajectory of a classical particle is possible only when the velocity of the wave packet is sufficiently large. We show that in addition to the vectors of classical spin $\vec{S}$ and color spin $\vec{I}$ of the particle, it is necessary to introduce a tensorial dynamical variable $[J^{ab}]$ describing a mixing of the spin and color spin. On the whole, it turns out that the classical particle has six independent internal dynamical variables, compactly described as an SO(3, 1) matrix, due to constraint relations between $\vec{I}$, $\vec{S}$ and $[J^{ab}]$.

PACS numbers: 11.10.Np, 03.50.-z

1. *Introduction*

This paper is devoted to the concept of a classical, colored particle, [1]. This concept has turned out to be very useful in investigations of several theoretical problems, [2], eventhough one should not expect to observe colored particles in any real experiment according to the color confinement concept. Furthermore, it has also provided a push to formal investigations of the classical mechanics of particles with internal degrees of freedom, see, e.g., [3].

We reconsider the derivation of equations of motion for the classical colored particle interacting with an external SU(2) gauge field from the Dirac equation presented in [1]. The derivation presented in [1] has two basic shortcomings described below. Also, the spin of the particle was not taken into account. We would like to present a more complete derivation which leads to a number of entirely new results.

Let us first sketch the derivation presented in [1]. The first step was to derive the Heisenberg equations of motion for the momentum and the color spin operators on the basis of the Dirac equation. Next, the replacement of the operators by c-number classical quantities led to equations which were interpreted as the equations of motion for the classical,

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color charged particle. The color charge of the particle was described by a 3-component color spin vector $\vec{I} = (I^a)$, I^a — real numbers. In the nonrelativistic limit, those equations were

$$m\ddot{\vec{x}} = gI^a\vec{E}^a + \frac{g}{c}\,\dot{\vec{x}}\times\vec{B}^a I^a, \tag{1}$$

$$\dot{I}^a = \frac{g}{\hbar}\,\varepsilon^{abc}\left(A_0^b - \vec{A}^b\,\frac{\dot{\vec{x}}}{c}\right)I^c, \tag{2}$$

where dots denote differentiations with respect to time. In these equations

$$E^{ai} = F^a_{0i}, \qquad B^{ak} = -\tfrac{1}{2}\,\varepsilon^{krt}F^a_{rt}, \tag{3}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \frac{g}{\hbar c}\,\varepsilon^{abc}A^b_\mu A^c_\nu \tag{4}$$

is the SU(2) field strength tensor.

The first shortcoming of the above described derivation is that it is based on the identification of the matrices $\vec{\alpha} = \gamma^0\vec{\gamma}$, present in the Hamiltonian for the Dirac equation, with the ordinary velocity $\dot{\vec{x}}$ of the classical particle. This identification, based on the Heisenberg picture equation

$$\frac{d\hat{\vec{x}}}{dt} = \frac{i}{\hbar}\,[H, \hat{\vec{x}}] = c\vec{\alpha},$$

is known to lead to difficulties in the physical interpretation of operators appearing in the theory, [4].

The known solution to these difficulties is to apply the Foldy-Wouthuysen transformation. In the Section 2 we do this for the Dirac particle in the external nonabelian gauge field. Then, we derive the Heisenberg picture equations of motion in the Foldy-Wouthuysen representation. In addition to the operators of momentum $\hat{\vec{p}}$ and color spin $\hat{\vec{T}}$ we also consider the operator $\hat{\vec{S}}$ of the spin of the particle. Next, we replace the quantum operators by the classical dynamical variables, thus obtaining classical equations for position, spin and color spin of the classical particle. When one neglects the spin of the particle these equations reduce to Wong's equations (1), (2). However, when the presence of the spin is taken into account, Wong's equation (2) gets modified by terms of order $\hbar^0$. Our approach applies only in the nonrelativistic limit and for weak external fields, because only in this limit can one construct a reliable Foldy-Wouthuysen transformation and define the positive energy sector of the theory, [4].

The second shortcoming of the approach presented in paper [1] is that it says nothing about the wave function of the colored particle. Therefore, in Section 3 we try to derive the classical equations of motion from the motion of a wave packet in the external gauge field. Our most interesting findings is that in the nonabelian case there are difficulties in obtaining gauge invariant equations, e.g., for the trajectory $\vec{x}(t)$ and for spin $\vec{S}(t)$ of the wave packet. It seems that only when the velocity of the wave packet is sufficiently large it is possible to describe the motion of the wave packet in terms of a classical, pointlike particle moving in the external gauge field. This would suggest that the concept of a classical, pointlike colored particle is of limited legitimacy from the point of view of Dirac's equation.

This fact does not contradict the knowledge about nonabelian gauge theories. Namely, the primary theory of color interactions is the quantized theory. The classical limit of this theory, if it exists at all, has to be calculated. This is in contrast to the abelian case, in which just the classical limit is primary — it has the form of the experimentally well-established Newton equations with Lorentz force, completed with classical Maxwell equations. This fundamental difference between abelian and nonabelian gauge theories is related to the fact that QCD is ultraviolet stable and QED is infrared stable.

In the region where the classical particle picture of the motion of the wave packet is applicable, we find that it is necessary to introduce many internal dynamical variables for the classical particle in order to obtain a closed set of equations of motion. In addition to the classical spin $\vec{S}$ and color spin $\vec{I}$, it is necessary to introduce a new classical quantity, namely a tensor $[J^{ab}]$, the expectation value of the product of the spin and color spin operators. Only in the particular case of states in which spin and color spin decouple, J^{ab} reduces to a function of $\vec{S}$ and $\vec{I}$, namely $J^{ab} = I^a S^b$. It is well-known, [5], that in nonabelian gauge theories it is even possible to generate spin from color spin, not to say about the simple mixing between them.

The classical equations of motion describing the motion of the wave packet are different from Wong's equations (1), (2). Namely, the equation for the color spin $\vec{I}$ contains more terms than (2), the additional terms are of order $\hbar^0$. Moreover, in addition to (1) and to the modified (2) we have two more equations — an equation for the classical spin $\vec{S}(t)$ and for the tensor $J^{ab}(t)$. The new equations are of the first order.

In Section 4 we show that there are constraints relating $\vec{I}$, $\vec{S}$ and $[J^{ab}]$. They reduce the number of independent classical dynamical variables describing the internal motion of the particle to six. These variables cannot be regarded as the three components of $\vec{S}$ and the three components of $\vec{I}$. Furthermore, $\vec{I}$, $\vec{S}$, J^{ab} can be shown to form a matrix, closely related to an element of the SO(3, 1) group (the Lorentz group). Thus, the classical, SU(2) colored, spinning particle essentially possesses the SO(3, 1) internal structure.

In our approach, the spin and color spin degrees of freedom are treated in a rigorous manner. Only for translational degrees of freedom do we take the classical limit. Therefore, the presented approximation to the quantum mechanics of the Dirac particle is in fact semiclassical.

Further general remarks are presented in Section 5 of this paper.

2. *The Foldy-Wouthuysen transformation*

This transformation is constructed here in the form of an expansion in powers of $1/mc$, in strict analogy to the abelian case considered in [4]. In fact, the calculations and the resulting Hamiltonian are almost identical as in the abelian case. The only difference is that various components of the gauge potential $\hat{A}_\mu$ (in the matrix notation) do not commute, with the result that the nonabelian field strength tensor $\hat{F}_{\mu\nu}$ replaces $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ from the abelian case. Therefore, we present here only the most important formulae for the sake of completeness of our paper and the final result.

The Dirac equation

$$\gamma^\mu \left(i\hbar\partial_\mu - \frac{g}{c} \hat{A}_\mu \right) \psi - mc\psi = 0, \tag{5}$$

where $\hat{A}_\mu = A_\mu^a \hat{T}^a$, $a = 1, 2, 3$, $\hat{T}^a$ are generators of the fundamental representation of SU(2), can be written in the Schrödinger form

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi \tag{6}$$

with the following Hamiltonian

$$H = c\vec{\alpha} \left(\vec{p} - \frac{g}{c} \hat{\vec{A}} \right) + g\hat{A}_0 + mc^2\beta, \tag{7}$$

where $\alpha^i = \gamma^0\gamma^i$, $\beta = \gamma^0$. We choose γ^μ to be hermitean matrices, β diagonal, e.g.,

$$\beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix},$$

where σ^i are Pauli matrices, σ^0 is a 2×2 unit matrix.

The Foldy-Wouthuysen transformation has the form

$$\psi' = \exp(iM)\psi, \tag{8}$$

$$H' = \exp(iM)H\exp(-iM) - i\hbar \exp(iM) \frac{\partial}{\partial t} \exp(-iM). \tag{8}$$

Then,

$$i\hbar \frac{\partial}{\partial t} \psi' = H'\psi'. \tag{9}$$

The transformation (8) is performed in order to remove odd powers of α matrices from the Hamiltonian (7). Thus, H' may contain only products of an even number of α's. The advantage of the Foldy-Wouthuysen representation is that in this representation the effects due to the "Zitterbewegung" of the Dirac particle, [6], are explicitly exhibited.

In the simple case when $\hat{A}_\mu$ is gauge equivalent to zero, H' can be calculated exactly,

$$H' = c \cdot \beta \cdot \sqrt{m^2c^2 + \left(\vec{p} - \frac{g}{c} \hat{\vec{A}} \right)^2} + g\hat{A}_0,$$

$$\hat{A}_\mu = \frac{i\hbar c}{g} \partial_\mu \omega \omega^{-1}, \quad \omega \in SU(2).$$

In the general case H' can be found only in the form of the expansion in powers of $1/mc$ $\left(\text{in fact, the dimensionless expansion parameters are } \frac{\hbar}{mc}\nabla, \frac{\hbar}{mc^2}\frac{\partial}{\partial t}\right)$. Performing three successive transformations of the type (8) with

$$M_1 = -\frac{i}{2mc}\beta\vec{\alpha}\left(\vec{p} - \frac{g}{c}\hat{\vec{A}}\right),$$

$$M_2 = \frac{g\hbar}{4m^2c^3}\vec{\alpha}\hat{\vec{E}},$$

$$M_3 = \frac{i}{4m^3c^3}\beta\left\{\frac{g\hbar^2}{2c^2}\vec{\alpha}\left(\frac{\partial\hat{\vec{E}}}{\partial t} + \frac{ig}{\hbar}[\hat{A}_0, \hat{\vec{E}}]\right)\right.$$
$$\left. + \left[\left(\vec{p} - \frac{g}{c}\hat{\vec{A}}\right)^2 + \frac{\hbar g}{2c}\varepsilon^{iks}S^s\hat{F}_{ik}\right]\vec{\alpha}\left(\vec{p} - \frac{g}{c}\hat{\vec{A}}\right)\right\},$$

where

$$\hat{E}^i = \hat{F}_{0i}, \quad \hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu + \frac{ig}{\hbar c}[\hat{A}_\mu, \hat{A}_\nu],$$

and

$$\hat{S}^b = \frac{1}{2}\begin{pmatrix}\sigma^b & 0\\ 0 & \sigma^b\end{pmatrix}$$

we obtain up to the order $\left(\frac{1}{mc}\right)^2$

$$H' \equiv H_2 = mc^2\beta + \frac{\beta}{2m}\left(\vec{p} - \frac{g}{c}\hat{\vec{A}}\right)^2 + g\hat{A}_0$$
$$+ \frac{g\hbar}{2mc}\varepsilon^{iks}\hat{S}^s\hat{F}_{ik} - \frac{g\hbar^2}{8m^2c^2}D_i\hat{E}^i$$
$$- \frac{g\hbar}{4m^2c^2}\varepsilon^{iks}\hat{S}^s\left[\hat{E}^i\left(p^k - \frac{g}{c}\hat{A}^k\right) + \left(p^k - \frac{g}{c}\hat{A}^k\right)\hat{E}^i\right], \tag{10}$$

where

$$D_k\hat{E}^i = \partial_k\hat{E}^i + \frac{ig}{\hbar c}[\hat{A}_k, \hat{E}^i] = \left(\partial_k E^{ia} - \frac{g}{\hbar c}\varepsilon^{abc}A_k^b E^{ic}\right)\hat{T}^a$$

is the covariant derivative of the color electric field.

The Hamiltonian (10) represents the first three terms in the expansion of the unknown exact Hamiltonian H' in powers of $\frac{1}{mc}$. The operator $\hat{\vec{S}}$ is essentially the operator of the spin of the particle in the Foldy-Wouthuysen representation; precisely the spin operator is $\hbar\hat{\vec{S}}$. In the following, we adopt as the spin operator the operator $\hat{\vec{S}}$. Because $\hat{T}^a = \frac{\sigma^a}{2}$ for SU(2) group, we have the relation

$$\hat{T}^a\hat{T}^b = \tfrac{1}{4}\,\delta^{ab}+\tfrac{1}{2}\,i\varepsilon^{abc}\hat{T}^c. \tag{11}$$

Therefore, H_2 is a linear function of $\hat{\vec{T}}$.

From the Hamiltonian H_2 we obtain the following Heisenberg equations of motion. We introduce the mechanical momenta

$$\Pi^r = p^r - \frac{g}{c}\hat{A}^r. \tag{12}$$

Then

$$\frac{dx^i}{dt} = \frac{i}{\hbar}[H_2, x^i] = \frac{\beta}{m}\Pi^i - \frac{g\hbar}{2m^2c^2}\varepsilon^{tis}\hat{S}^s\hat{E}^t, \tag{13}$$

and

$$\frac{d\Pi^r}{dt} = -\frac{g}{2c}\left(\hat{F}_{ri}\frac{dx^i}{dt}+\frac{dx^i}{dt}\hat{F}_{ri}\right)+g\hat{F}_{0r}-\frac{g\hbar}{2mc}\varepsilon^{iks}\hat{S}^s(D_r\hat{F}_{ik})$$

$$+\frac{g\hbar}{4mc^2}\beta\varepsilon^{iks}\hat{S}^s\left[(D_r\hat{E}^i)\frac{dx^k}{dt}+\frac{dx^k}{dt}(D_r\hat{E})\right]+\frac{g\hbar^2}{8m^2c^2}D_r(D_i\hat{E}^i). \tag{14}$$

For the spin operator $\hat{\vec{S}}$ we obtain

$$\frac{d\hat{S}^t}{dt} = -\frac{g}{mc}\hat{F}_{tk}\hat{S}^k$$

$$+\tfrac{1}{4}\frac{g}{mc^2}\beta\hat{S}^p\left(\hat{E}^t\frac{dx^p}{dt}-\hat{E}^p\frac{dx^t}{dt}+\frac{dx^p}{dt}\hat{E}^t-\frac{dx^t}{dt}\hat{E}^p\right), \tag{15}$$

and for the color spin operator $\hat{\vec{T}}$

$$\frac{d\hat{T}^a}{dt} = \frac{g}{\hbar}\left[\frac{1}{2c}\left(\frac{dx^i}{dt}A^{bi}+A^{bi}\frac{dx^i}{dt}\right)-A_0^b\right]\varepsilon^{bac}\hat{T}^c$$

$$-\frac{g}{2mc}\varepsilon^{its}\hat{S}^sF^b_{it}\varepsilon^{bac}\hat{T}^c+\frac{g\hbar}{8m^2c^2}(D_i\hat{E})^b\varepsilon^{bac}\hat{T}^c$$

$$+\frac{g}{4mc^2}\beta\varepsilon^{ips}\hat{S}^s\left(E^{bi}\frac{dx^p}{dt}+\frac{dx^p}{dt}E^{bi}\right)\varepsilon^{bac}\hat{T}^c. \tag{16}$$

One could also consider "composite" operators like $\hat{J}^{ab} = \hat{T}^a\hat{S}^b$. Because

$$\frac{d\hat{J}^{ab}}{dt} = \hat{T}^a\frac{d\hat{S}^b}{dt}+\frac{d\hat{T}^a}{dt}\hat{S}^b, \tag{17}$$

the Heisenberg equations of motion for $\hat{J}^{ab}$ can be easily obtained by substituting in (17) the equations (15), (16). Of course, the operators $\hat{J}^{ab}$ are not new independent quantum observables for the Dirac particle. The reasons for considering these operators will be given in the Sections 3 and 4.

In order to avoid antiparticles on the classical mechanics level, we project (13)–(16) on the subspace of positive energies. This amounts to replacing β by the unit 2×2 matrix and to replacing $\hat{\hat{S}}$ by $\frac{1}{2}\vec{\sigma}$, and it is equivalent to considering only the states ψ of the form

$$\psi = \begin{pmatrix} u \\ v = 0 \end{pmatrix}, \tag{18}$$

where u is a 2-component spinor.

From equations (13)–(16) it is possible to obtain certain classical equations of motion by the replacement of the Heisenberg picture operators by the c-number classical quantities. Such a procedure is of course highly ambiguous. For example, any term present in the equations could be zeroed by multiplying it by the unit operator

$$\mathbf{1} = \frac{i}{3\hbar}[\Pi^k, x^k], \tag{19}$$

which vanishes when Π^k, x^k become the c-number quantities. Therefore, this procedure has to be accompanied with the limit $\hbar \to 0$, because only in this limit can the r.h.s. of (19) be equal to one when Π^k and x^k tend to the c-numbers. Another source of ambiguities is the relation

$$\hat{T}^c = -\frac{i}{2}\varepsilon^{abc}[\hat{T}^a, \hat{T}^b], \tag{20}$$

and the same for $\hat{S}^b$. When $\hat{T}^a$ and $\hat{S}^b$ become the c-numers, the r.h.s. of (20) vanishes. Thus, we have to decide whether $\hat{T}^a$, $\hat{S}^b$ become zero or I^a, S^b. In order to obtain nontrivial classical equations we choose the last possibility.

Taking the above remarks into account we obtain from (13)–(16) the following classical equations

$$m\frac{d^2x^r}{dt^2} = -\frac{g}{c}F^a_{ri}\frac{dx^i}{dt}I^a + gF^a_{0r}I^a, \tag{21}$$

$$\frac{dI^a}{dt} = \frac{g}{\hbar c}\varepsilon^{bac}A^{bi}I^c\frac{dx^i}{dt} - \frac{1}{\hbar}A^b_0 I^c\varepsilon^{bac} - \frac{g}{2mc}\varepsilon^{its}\varepsilon^{bac}S^s I^c F^b_{it}$$

$$+\frac{g}{2mc^2}\varepsilon^{ips}\varepsilon^{bac}S^s I^c E^{bi}\frac{dx^p}{dt}, \tag{22}$$

$$\frac{dS^t}{dt} = -\frac{g}{mc}F^a_{tk}S^k I^a + \frac{g}{2mc^2}S^p\left(E^{at}I^a\frac{dx^p}{dt} - E^{ap}I^a\frac{dx^t}{dt}\right). \tag{23}$$

Equations (21)–(23) form a closed set of classical equations. There is no need to introduce here the tensor $[J^{ab}]$ mentioned in the Introduction. From (22) and (23) it follows that $\vec{I}^2$ and $\vec{S}^2$ are constants of motion. Equation (22) reduces to the Wong's equation (2) only when $\vec{S} = 0$. Notice also that these equations are gauge invariant.

Of course, the classical equations (21)–(23) can not be called "derived" from the Heisenberg equations of motion because the above procedure for obtaining them is ambiguous. Moreover, there is no quantitative relation between the c-numbers I^a, S^b, J^{ab} and the quantum operators $\hat{T}^a$, $\hat{S}^b$, $\hat{J}^{ab}$. Therefore, equations (21–23) should be regarded merely as a particular possibility for a closed set of equations for $\vec{I}$, $\vec{S}$, $\vec{x}(t)$, having a loose relation to the Dirac equation.

In the next Section we shall present a more precise approach to the classical limit of the Dirac equation, consisting of considerations of the expectation values of quantum observables, in analogy to Ehrenfest's approach to the classical limit of nonrelativistic quantum mechanics of a spinless particle [7].

3. The motion of a wave packet in the external nonabelian gauge field

We will investigate the motion of a localized wave packet in the external nonabelian gauge field in order to find out the most adequate description of this motion in terms of a classical particle. In other words, the classical particle will be regarded here as an idealization of the wave packet. We work in the Foldy-Wouthuysen representation in the positive energy sector. We assume that the gauge potentials are slowly changing with $\vec{x}$. This will allow us to regard $\hat{A}_\mu$ as constant over the region occupied by the wave packet.

We assume the following form for the wave packet

$$\Psi^{\alpha\eta}(\vec{x}, t) = u^{\alpha\eta}(\vec{x}, t)\varphi(\vec{x}-\vec{x}(t)), \tag{24}$$

where $\vec{x}(t)$ is the trajectory of the corresponding classical particle. Here $\varphi(\vec{x}-\vec{x}(t))$ is a c-number valued wave packet, localized at $\vec{x}(t)$ with the average momentum

$$\langle\varphi|\vec{p}|\varphi\rangle = m\dot{\vec{x}}(t). \tag{25}$$

These two requirements, together with the requirement of minimal uncertainty, $\langle(\Delta x_i)^2\rangle \langle(\Delta p_i)^2\rangle \approx \hbar^2$, essentially determine the form of the wave packet. We shall assume that φ is normalized to one. The spinor $u^{\alpha\eta}(\vec{x}, t)$ takes into account the spin and color degrees of freedom.

Because the wave packet φ is localized at $\vec{x}(t)$, it would be natural to consider u as a function of t only,

$$u(\vec{x}, t) \rightarrow u(\vec{x}(t), t).$$

However, it turns out to be very convenient to assume that in the vicinity of the point $\vec{x}(t)$ (for fixed t)

$$u(\vec{x}, t) = \left(1+\frac{ig}{\hbar c}\hat{A}^i(\vec{x}(t), t)\Delta x^i\right)u_0(t), \tag{26}$$

where $\Delta x^i = x^i - x^i(t)$. We assume that $u_0(t)$ is normalized to one.

The Ansatz (26) has to be completed with certain assumptions. The reason is that the term

$$\frac{ig}{\hbar c}\hat{A}^i \Delta x^i$$

gives a gauge noninvariant contribution to the expectation values, which does not vanish in the classical limit. In order to illustrate this problem let us consider the expectation value of $\hat{S}^b$. Because $\langle\varphi|\Delta x^i|\varphi\rangle \approx 0$, we obtain

$$\int \Psi^{\dagger}\hat{S}^b\Psi d^3x = \langle u_0|\hat{S}^b|u_0\rangle$$

$$+\frac{g^2}{\hbar^2 c^2}[\langle u_0|\hat{A}^{\parallel}\hat{A}^{\parallel}\hat{S}^b|u_0\rangle\,\langle(\Delta\vec{x}^{\parallel})^2\rangle + \langle u_0|\hat{A}^{\perp}\hat{A}^{\perp}\hat{S}^b|u_0\rangle\,\langle(\Delta\vec{x}^{\perp})^2\rangle], \qquad (27)$$

where $\|$ $(\perp)$ denotes the component of $\hat{\vec{A}}$ parallel (perpendicular) to the velocity $\dot{\vec{x}}$, and $\hat{\vec{A}}$ is taken at the point $\vec{x}(t)$. The trouble is that the terms quadratic in $\hat{\vec{A}}$ do not form a gauge invariant function of the external gauge potential $\vec{A}_\mu$. In effect, these terms would lead to difficulty in obtaining a gauge invariant equation for the classical spin S^b. Similarly, there are difficulties of this same kind in obtaining a gauge invariant equation for the trajectory $\vec{x}(t)$. Therefore, those terms have to be removed somehow.

Unfortunately, the troublesome terms in (27) do not vanish in the classical limit $\hbar \to 0$, because they are of the order $\hbar^0$, as follows from the facts that

$$\langle(\Delta x_i)^2\rangle \approx \frac{\hbar^2}{\langle(\Delta p_i)^2\rangle},$$

and that $\langle(\Delta p_i)^2\rangle$ cannot be taken arbitrarily large without spoiling the assumed picture of the positive energy wave packet moving along the given trajectory $\vec{x}(t)$ with definite velocity $\dot{\vec{x}}(t)$.

Our solution to this problem is the following. We adopt the point of view that all our investigations of the wave packet are carried out in the particular gauge. Namely, for the fixed in (24) classical trajectory $\vec{x}(t)$ we adjust the gauge in such a way that

$$\hat{A}_{\perp}(\vec{x}(t), t) = 0. \qquad (28)$$

It is easy to convince oneself that such a gauge exists for a wide class of trajectories $\vec{x}(t)$. In fact, the requirement $\hat{A}_{\perp} = 0$ is a rather weak one, because it concerns only a single line in three-dimensional space. Unfortunately, in general it is not possible to improve the gauge further in order to also have $\hat{A}^{\parallel}(\vec{x}(t), t) = 0$. Thus, in order to eliminate the term containing $\hat{A}^{\parallel}$ we have to assume that $\langle(\Delta\vec{x}^{\parallel})^2\rangle$ is small, i.e., that $\langle(\Delta p^{\parallel})^2\rangle$ is large. This assumption is not in contradiction to the fact that the wave packet $\varphi(\vec{x}-\vec{x}(t))$ moves with definite velocity only if the velocity is large enough. Taking $\langle(\Delta\vec{p}^{\parallel})^2\rangle = \alpha^2 m^2|\dot{\vec{x}}|^2$, where α is a small number we obtain the following condition under which we can neglect the troublesome term in the expectation value (27)

$$\frac{g^2}{4\alpha^2 m^2 c^2} A^{a\|} A^{a\|} \ll |\dot{\vec{x}}|^2. \tag{29}$$

Of course, in spite of the fact that we work in the particular gauge (28), we require that all physical characteristics of the particle be given by gauge invariant quantities formed from the external gauge potential $\hat{A}_\mu$ and from the dynamical variables of the particle. In particular, the trajectory $\vec{x}(t)$ should be calculated from a gauge invariant equation. As a matter of fact, the role of (28), (29) is to remove the undesired consequences of the form of the Ansatz (24), (26) for the wave packet. The shortcoming of the form of this Ansatz is that it is not gauge covariant in the sense that the gauge transformation of $\hat{A}_\mu$ is not equivalent to a gauge transformation of Ψ.

It is easy to check that condition (29) is not necessary in the case of an abelian gauge group because in this case one can always have $A^{\|} = 0$ also.

It is natural to ask why we do not simply abandon the troublesome factor in (26). The answer is that without this factor the expectation values (33), (34) below would be different, and that they would lead to problems with the gaue invariance of the classical equations. In order to solve these problems, we would be forced again to assume (29) and the particular gauge (28).

Now we are ready to investigate the motion of the wave packet. From (24), (25), (28), (29) we obtain

$$\langle\Psi|\vec{p}|\Psi\rangle = m\dot{\vec{x}}(t)+\vec{A}^a(\vec{x}(t), t)I^a, \tag{30}$$

where

$$I^a(t) = \langle\Psi|\hat{T}^a|\Psi\rangle \approx \langle u_0|\hat{T}^a|u_0\rangle \equiv u_0^\dagger \hat{T}^a u_0, \tag{31}$$

because $\langle\varphi|\Delta x^i|\varphi\rangle = 0$. Equation (30) can be rewritten as

$$\langle\Psi|\vec{\Pi}|\Psi\rangle = m\dot{\vec{x}}(t), \tag{32}$$

where Π^r is the mechanical momentum (12).

It is just due to the Ansatz (26) and the assumption (29) that we obtain the following simple formulae:

$$\tfrac{1}{2}\langle\Psi|[\hat{T}^a, \Pi^r]_+|\Psi\rangle = mI^a(t)\dot{x}^r(t), \tag{33}$$

and

$$\tfrac{1}{2}\langle\Psi|\hat{S}^b[\hat{T}^a, \Pi^r]_+|\Psi\rangle = m\dot{x}^r\langle u_0|\hat{S}^b\hat{T}^a|u_0\rangle. \tag{34}$$

In other words, with the Ansatz (26), (28), (29) the translational degrees of freedom are decoupled from the internal degrees of freedom.

Now, we shall derive the set of classical equations describing the motion of the wave packet. In all our considerations below we shall neglect the spreading out of the wave packet. The spreading out gives a contribution of order $\hbar$ to the time derivative $\frac{d\Psi}{dt}$. Therefore, in order to be consistent, we also neglect all terms of this same or higher order in $\hbar$.

The equation for the trajectory $\vec{x}(t)$ can be obtained by considering the time dependence of the expectation value of $\vec{\Pi}$ in the state $|\psi\rangle$. We work in the Schrödinger picture because we have assumed the form of the time-dependent wave function in (24). Because

$$\frac{d}{dt}\langle\Psi|\vec{\Pi}|\Psi\rangle = \frac{i}{\hbar}\langle\Psi|\,[H_2, \vec{\Pi}]\,|\Psi\rangle+\langle\Psi|\frac{\partial\vec{\Pi}}{\partial t}|\Psi\rangle,$$

we obtain from (32) and (10) that

$$m\ddot{x}^r(t) = -\frac{g}{c}\dot{x}^i F^a_{ri}(\vec{x}(t), t)I^a(t)+gF^a_{0r}(\vec{x}(t), t)I^a, \tag{35}$$

where we have neglected the terms of order $\hbar$. This is again Wong's equation (1).

In order to have a closed set of equations we have to add an equation for $\vec{I}$. From (31), (10), (33), (34), neglecting terms of order $\hbar$, we obtain

$$\frac{dI^a}{dt} = \frac{g}{\hbar}\left(\frac{1}{c}A^{bi}\dot{x}^i - A^b_0\right)\varepsilon^{bac}I^c$$

$$+\frac{g}{2mc}\left[-F^c_{it}+\frac{1}{2c}(E^{ci}\dot{x}^t-E^{ct}\dot{x}^i)\right]\varepsilon^{itb}\varepsilon^{cad}J^{db}(t), \tag{36}$$

where

$$J^{cs}(t) = \langle u_0|\hat{T}^c\hat{S}^s|u_0\rangle. \tag{37}$$

Because $\hat{T}^c$, $\hat{S}^s$ are hermitean operators, J^{cs} are real numbers. Observe now that $\vec{I}^2$ is not a constant of motion in general.

Thus, we also need an equation for $J^{cs}(t)$. From (37), (10), (33), (34) we obtain the equation

$$\frac{dJ^{ab}}{dt} = \frac{g}{\hbar}\left(\frac{1}{c}A^{ci}\dot{x}^i - A^c_0\right)\varepsilon^{cad}J^{db}$$

$$+\frac{g}{8mc}\varepsilon^{ipb}\varepsilon^{cad}\left(-F^c_{ip}+\frac{x^p}{c}E^{ci}\right)I^d$$

$$+\frac{g}{4mc}\left[-F^a_{bp}+\frac{1}{2c}(E^{ab}\dot{x}^p-E^{ap}\dot{x}^b)\right]S^p, \tag{38}$$

where

$$S^p = \langle u_0|\hat{S}^p|u_0\rangle. \tag{39}$$

From (38) it follows that we have to add an equation for $S^p(t)$. From (39), (10), (33), (34) we obtain

$$\frac{dS^t}{dt} = \frac{g}{mc}\left[-F_{tp}+\frac{1}{2c}(E^{at}\dot{x}^p-E^{ap}\dot{x}^t)\right]J^{ap}(t). \tag{40}$$

Observe that from (40) it does not follow in general that $\vec{S}^2(t) = \text{const}$.

The four multicomponent equations (35), (36), (38), (40) form the closed set of classical equations for the expectation values. These equations can be regarded as the equations of motion for the classical particle with spin and color spin. These equations are more general than the equations (21)–(23) of the previous Section because they take into account the possibility of a mixing between spin and color spin.

J^{ab} is the new dynamical variable for the classical particle, independent of I^a, S^b. Observe that if at certain instant t_0

$$u_0^{\alpha\eta}(t_0) = \xi^\alpha(t_0)\chi^\eta(t_0), \tag{41}$$

then

$$J^{ab}(t_0) = I^a(t_0)S^b(t_0). \tag{42}$$

However, it is easy to check from (36), (38), (40) that the quantity $Q^{ab} \equiv J^{ab} - I^a S^b$ is not a constant of motion. Therefore, in general (41), (42) do not hold for $t \neq t_0$, and therefore J^{ab} does not cease to be the independent dynamical variable.

Equations (35), (36), (38), (40) have to be completed with constraint equations. The reason is that the fifteen numbers I^a, S^b, J^{ab} are expectation values in the single state u_0. Therefore, these expectation values depend on 6 independent, real numbers forming u_0 (t) (because u_0 is normalized to 1 and because the overall phase factor of u_0 does not change the expectation values). Thus, the constraints are necessary if the classical mechanics based on the equations (35), (36), (38), (40) is to be related to the quantum mechanical Dirac particle. We find these constraint equations in the next Section. We also shall show how to calculate $u_0(t)$ from known $I^a(t), S^b(t), J^{ab}(t)$, and we shall find the time evolution equation for $u_0(t)$.

4. *The constraint equations and the determination of the wave function $u_0(t)$*

First, let us show that knowledge of the classical quantities $\vec{I}$, $\vec{S}$, $[J^{ab}]$ determines the wave function $u_0(t)$ up to an arbitrary time dependent phase factor. This fact we shall regard as proof that the above set of classical dynamical variables describing the internal motion of the particle is complete, in the sense that any other classical, internal dynamical variable, i.e., the expectation value of an operator in the state u_0, is a function of $\vec{I}$, $\vec{S}$, $[J^{ab}]$.

To this end we shall regard the spinor $[u_0^{\alpha\eta}]$ as 2×2 matrix u_0. Then, the normalization condition $u_0^{*\alpha\eta} u_0^{\alpha\eta} = 1$ takes the form

$$\mathrm{Tr}\,(\hat{u}_0^\dagger \hat{u}_0) = 1. \tag{43}$$

Furthermore,

$$I^a \equiv \langle u_0 | \hat{T}^a | u_0 \rangle = \tfrac{1}{2}\,\mathrm{Tr}(\hat{u}_0^* \sigma^a \hat{u}_0^{\mathrm{T}}) = \tfrac{1}{2}\,\mathrm{Tr}(\hat{u}_0 \sigma^{a\mathrm{T}} \hat{u}_0^\dagger), \tag{44}$$

$$S^t = \tfrac{1}{2}\,\mathrm{Tr}(\hat{u}_0^\dagger \sigma^t \hat{u}_0), \tag{45}$$

$$J^{ab} = \tfrac{1}{4}\,\mathrm{Tr}(\hat{u}_0^\dagger \sigma^b \hat{u}_0 \sigma^{a\mathrm{T}}), \tag{46}$$

where the star denotes the complex conjugation, and Tr denotes the transposition of the matrix. It is clear that we cannot determine the overall phase factor of $\hat{u}_0$.

From (43)–(45) it follows that

$$\hat{u}_0\hat{u}_0^\dagger = \tfrac{1}{2}\sigma^0+\vec{S}\vec{\sigma}, \tag{47}$$

$$\hat{u}_0^\dagger\hat{u}_0 = \tfrac{1}{2}\sigma^0+\vec{I}\vec{\sigma}^{\mathrm{T}}. \tag{48}$$

Equations (47), (48) imply that

$$|\det\hat{u}|^2 = \tfrac{1}{4}-\vec{I}^2 = \tfrac{1}{4}-\vec{S}^2. \tag{49}$$

Thus, we see that for SU(2)-colored particle

$$\vec{I}^2 = \vec{S}^2. \tag{50}$$

It is easy to see that this fact is consistent with the equations (40), (36) only if

$$\varepsilon^{bac}I^aJ^{cs} = \varepsilon^{skr}S^kJ^{br}. \tag{50'}$$

Utilising (44)–(48) it is easy to prove that the condition (50′) is satisfied. Relations (50), (50′) are examples of the constraint equations.

From (49) it follows that $\hat{u}_0$ is a singular matrix only when $\vec{S}^2 = \vec{I}^2 = 1/4$. It is easy to prove that $\det\hat{u}_0 = 0$ is equivalent to

$$u_0^{\alpha\eta} = \xi^\alpha\chi^\eta, \tag{51}$$

i.e., in this case the spin and color spin decouple. In this degenerate case knowledge of $\vec{I}$ and $\vec{S}$, together with the normalization conditions

$$\xi^\dagger\xi = 1,\ \chi^\dagger\chi = 1,$$

determines ξ, χ up to the arbitrary time-dependent phase factor. For example, when $I^3 \neq -1/2$,

$$\chi = \exp\left[i\alpha(t)\right](\tfrac{1}{2}+I^3)^{-1/2}\begin{pmatrix}\tfrac{1}{2}+I^3\\ I^1+iI^2\end{pmatrix}, \tag{52}$$

and if $I^3 = -1/2$

$$\chi = \exp\left[i\alpha(t)\right]\begin{pmatrix}0\\1\end{pmatrix}.$$

Analogous formulae hold for ξ.

Let us remark here that the above relation between $\vec{I}(\vec{S})$ and the 2-spinor $\chi(\xi)$ can be refined by utilising the coherent states for the SU (2) group, [8].

In the degenerate case (51) we have $J^{ab} = I^aS^b$. The constraint (50′) becomes trivialized to $0 = 0$. However, J^{ab} does not cease to be an independent dynamical variable for the particle, as we have argued in the previous Section. This means that the relation $\det\hat{u}_0 = 0$ is not conserved in time.

Now, let us consider the general case, which also includes $\det\hat{u}_0 = 0$. In order to determine $\hat{u}_0$ we recall that any 2×2 matrix can be written in the form

$$\hat{u}_0 = HV, \tag{53}$$

where

$$H = H^{\dagger} = \sqrt{\hat{u}_0 \hat{u}_0^{\dagger}} \tag{54}$$

is a positive definite, hermitean matrix, and V is a unitary matrix determined from (53). If $\hat{u}_0$ is not singular, the matrix V is determined uniquely, namely

$$V = H^{-1}\hat{u}_0. \tag{55}$$

From (47), (54) we obtain

$$H = \frac{1}{\sqrt{2\lambda}}(\lambda\sigma^0 + \vec{S}\vec{\sigma}), \tag{56}$$

where

$$\lambda = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-4\vec{S}^2}, \tag{57}$$

and σ^0 is the 2×2, unit matrix, Furthermore, because

$$\vec{\sigma}^{\mathrm{T}} = \tilde{\vec{\sigma}} = \begin{pmatrix} \sigma^1 \\ -\sigma^2 \\ \sigma^3 \end{pmatrix},$$

we obtain from (47), (48), (56) that

$$V^{\dagger}\vec{S}\vec{\sigma}V = \tilde{\vec{I}}\vec{\sigma}, \tag{58}$$

i.e., V represents a rotation which rotates $\vec{S}$ in to $\tilde{\vec{I}}$, where

$$\tilde{\vec{I}} = \begin{pmatrix} I^1 \\ -I^2 \\ I^3 \end{pmatrix}.$$

For instance, when $\vec{S} \neq -\tilde{\vec{I}}$ we may take

$$V_0 = i\vec{\sigma}\frac{\vec{S}+\tilde{\vec{I}}}{|\vec{S}+\tilde{\vec{I}}|}. \tag{59}$$

Obviously, (58) does not determine V completely. Namely, we can take

$$V = V_0 C$$

where C is any unitary matrix commuting with $\tilde{\vec{I}}\vec{\sigma}$. Any such C has the form

$$C = \exp\,[i\beta(t)]\exp\left[i\frac{\gamma(t)}{2}\frac{\tilde{\vec{I}}\vec{\sigma}}{|\vec{I}|}\right]. \tag{60}$$

Thus, we see that in the general case $\hat{u}_0$ is not determined by the knowledge of $\vec{I}(t)$ and $\vec{S}(t)$ — apart from the unessential phase factor $\exp\,[i\beta(t)]$ we do not know the function $\gamma(t)$.

From (56), (59), (60) we obtain

$$\hat{u}_0 = \exp[i\beta(t)] \frac{1}{\sqrt{2\lambda}|\vec{S}+\tilde{\vec{I}}|}\left\{[\tfrac{1}{2}|\vec{S}+\tilde{\vec{I}}|^2+(\vec{S}\times\tilde{\vec{I}})\vec{\sigma}]\left(\cos\frac{\gamma}{2}+i\frac{\lambda}{I}\sin\frac{\gamma}{2}\right)\right.$$

$$\left.+\lambda(\vec{S}\vec{\sigma}+\tilde{\vec{I}}\vec{\sigma})\left(\cos\frac{\gamma}{2}+i\frac{I}{\lambda}\sin\frac{\gamma}{2}\right)\right\}. \tag{61}$$

In the degenerate case $\lambda = I = S = 1/2$, and therefore in this case all dependence on $\gamma(t)$ takes the form of the undeterminable phase factor $\exp[i\gamma(t)/2]$. It can be shown that in this case (61) can be written in the form (51) with ξ, χ given by (52).

In the general case however, the function $\gamma(t)$ does not appear in the form of a phase falctor, so it has to be determined. This determination is possible if in addition to $\vec{I}, \vec{S}$ we aso know the matrix $[J^{ab}]$. From (61) we obtain

$$J^{ab} = \frac{1}{4}\frac{I^a S^b}{\vec{I}^2} + \sin\gamma A^{ab} + \cos\gamma B^{ab}, \tag{62}$$

where

$$A^{ab} = \frac{\lambda - 2\vec{S}^2}{4\lambda|\vec{I}|}\left[\varepsilon^{ad\bar{b}} I^d - 2\varepsilon^{adc}\frac{I^d S^{\bar{c}}(S^b+I^{\bar{b}})}{|\vec{S}+\tilde{\vec{I}}|^2}\right], \tag{63}$$

$$B^{ab} = -\frac{I^a S^b}{4\vec{I}^2} + \frac{\lambda-2\vec{S}^2}{2\lambda|\vec{S}+\tilde{\vec{I}}|^2}(S^a+I^{\bar{a}})(S^b+I^{\bar{b}}) + \frac{1}{4\lambda}[2S^b I^a - (\lambda - 2\vec{S}^2)\delta^{\bar{a}b}],$$

In these formulae the barred indices $\bar{a}$, $\bar{b}$, etc., denote the change of sign of the vector or the tensor when the value of the index equals two, e.g.,

$$(\delta^{\bar{a}b}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The presence of the barred indices is due to the fact that $\sigma^{aT} = \sigma^{\bar{a}}$. From (62) we can determine γ if know $\vec{I}, \vec{S}, [J^{ab}]$.

Thus, we have proved that $\vec{I}, \vec{S}, [J^{ab}]$ form the complete set of classical dynamical variables for the internal motion of the particle. Now, we shall find the constraints. For 15 real number valued quantities I^a, S^b, J^{ab} we have to find 9 independent equations, in order to be left with 6 independent quantities.

In order to find the constraints, we consider the matrix $M^{\nu}{}_{\mu}$, defined by

$$M^{\nu}{}_{\mu} = \tfrac{1}{2}\,\mathrm{Tr}\,(u_0^{\dagger}\sigma^{\nu}u_0\sigma^{\mu}). \tag{64}$$

Comparing (64) with definitions of I^a, S^b, J^{ab}, we see that

$$M^0{}_0 = \tfrac{1}{2},$$

$$M^b{}_0 = S^b, \quad M^0{}_a = I^a, \quad M^b{}_a = 2J^{\bar{a}b}. \tag{65}$$

The matrix $M^{\nu}{}_{\mu}$ obeys the relation

$$M^{\nu}{}_{\mu} g_{\nu\varrho} M^{\varrho}{}_{\sigma} = |\det \hat{u}_0|^2 g_{\mu\sigma}, \tag{66}$$

where $(g_{\mu\nu}) = (1, -1, -1, -1)$ is the Minkowski space-time metrics. In the degenerate case this relation can be easily verified by a direct calculation. In the nondegenerate case, $\det \hat{u}_0 \neq 0$, this identity comes simply from the fact that $L^{\nu}{}_{\mu} = M^{\nu}{}_{\mu}/|\det \hat{u}_0|$ is a Lorentz transformation, because the matrix $\hat{u}_0/|\det \hat{u}_0|^{1/2}$ is an element of the SL (2, C) group. Here we use the well-known relation between SL (2, C) and the proper, ortochronous Lorentz group, [9]. The Lorentz transformations obey the relation

$$L^{\nu}{}_{\mu} g_{\nu\varrho} L^{\varrho}{}_{\sigma} = g_{\mu\sigma}$$

This relation leads to (66).

Let us recall that $|\det \hat{u}_0|$ is determined by $\vec{I}$ or $\vec{S}$ via (49).

From (65), (66) we obtain that

$$\tfrac{1}{4} - \vec{S}^2 = |\det \hat{u}_0|^2, \tag{67}$$

$$\tfrac{1}{4} I^a - J^{ab} S^b = 0, \tag{68}$$

$$4J^{ab} J^{db} - I^a I^d = |\det \hat{u}_0|^2 \delta^{ad}. \tag{69}$$

Another set of relations is obtained from the fact that if $L^{\nu}{}_{\mu}$ is a Lorentz transformation, then $(L^{\mathrm{T}})^{\nu}{}_{\mu}$ is a Lorentz transformation too. The difference between L and L^{T} is equivalent to the interchange of $\vec{S}$ and $\vec{I}$, and to the replacement of J^{ab} by J^{ba}. Thus, we obtain

$$\tfrac{1}{4} - \vec{I}^2 = |\det \hat{u}_0|^2, \tag{70}$$

$$\tfrac{1}{4} S^b - J^{ab} I^a = 0, \tag{71}$$

$$4J^{db} J^{dc} - S^b S^c = |\det \hat{u}_0|^2 \delta^{bc}. \tag{72}$$

In the degenerate case these relations can be easily verified by a direct calculation.

Equations (67), (70) are equivalent to (49). Equations (68), (69), are the nine constraint equations. Equations (71), (72) are equivalent to (68), (69) because (68), (69) together form the sufficient condition for $L^{\nu}{}_{\mu}$ to be the Lorentz transformation. The previously found relations (50), (50′) also follow from (67)–(69).

The next problem to be investigated is the question whether the classical equations of motion (36), (38), (40) respect the constraints, i.e., whether the above relations are conserved in time if $\vec{I}$, $\vec{S}$, $[J^{ab}]$ evolve in time according to the equations of motion.

It seems that the most illuminating way to find the answer to this question is to observe that the three classical equations of motion (36), (38), (40) can be derived from a single equation for $\hat{u}_0(t)$. Then, the solutions to (36), (38), (40) can be regarded as the expectation values (44)–(46) in the state $u_0(t)$ for all t — this would guarantee that together they form the matrix (64) for all t, i.e., that the constraints are conserved.

Such an equation for $\hat{u}_0$ can be derived in the following manner. From the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi = H_2 \Psi$$

we obtain for the wave packet (24) that

$$\Psi = u(\vec{x}, t)\varphi(\vec{x}-\vec{x}(t)) \equiv \tilde{\varphi}(\vec{x}, t)u_0(t),$$

$$i\hbar \frac{\partial \tilde{\varphi}}{\partial t} u_0 + i\hbar \tilde{\varphi} \frac{\partial u_0}{\partial t} = H_2 \tilde{\varphi} u_0. \tag{73}$$

Here we regard u_0 as the 2-spinor, not as a 2×2 matrix.

Now we assume that

$$u_0^\dagger \frac{\partial}{\partial t} u_0 = 0. \tag{74}$$

In fact, this assumption does not restrict the generality of our arguments, because if a certain $\bar{u}_0$ does not obey (74), then

$$u_0(t) = \exp\left[-\int_0^t \left(\bar{u}_0^\dagger \frac{\partial}{\partial t'} \bar{u}_0 \right) dt' \right] \bar{u}_0(t)$$

obeys (74). Because $\bar{u}_0^\dagger \bar{u}_0 = 1$, the above exponential is a time-dependent phase factor (the integrand is imaginary). Therefore, (74) is merely a restriction on the overall phase of u_0, which has no effect on $\vec{I}$, $\vec{S}$, $[J^{ab}]$, as is clear from (31), (37), (39).

From (73), (74) we obtain that

$$i\hbar \frac{\partial \varphi}{\partial t} = (u^\dagger H_2 u)\varphi - \frac{g}{c} A_k^a \dot{x}^k I^a \varphi, \tag{75}$$

and therefore

$$i\hbar \frac{\partial}{\partial t} u_0 = \bar{H}_2 u_0 - (u_0^\dagger \bar{H}_2 u_0) u_0, \tag{76}$$

where

$$\bar{H}_2 = \int d^3x \varphi^* \left(1 - \frac{ig}{\hbar c} \hat{A}^i \Delta x^i\right) H_2 \left(1 + \frac{ig}{\hbar c} \hat{A}^k \Delta x^k\right) \varphi - \frac{g}{c} \hat{A}^i \dot{x}^i + \frac{g}{c} A^{ai} I^a \dot{x}^i \tag{77}$$

is the "effective" Hamiltonian for spin and color degrees of freedom. It is easy to check that $\bar{H}_2$ has the form $\Bigl($we neglect terms of order $\hbar^2$ because we neglect the spreading out of the wave packet which gives the contribution of order $\hbar$ to $\frac{\partial \varphi}{\partial t}\Bigr)$:

$$\bar{H}_2 = \tfrac{1}{2} m \dot{\vec{x}}^2 - \frac{g}{c} \hat{A}^i \dot{x}^i + g\hat{A}_0 + \frac{g\hbar}{2mc} \varepsilon^{iks} \hat{S}^s \hat{F}_{ik} - \frac{g\hbar}{2mc^2} \varepsilon^{iks} \dot{x}^k \hat{S}^s \hat{E} + \frac{g}{c} A_i^a I^a \dot{x}^i. \tag{78}$$

Of course, $\bar{H}_2$ is a hermitian matrix.

Thus, the time evolution of u_0 is governed by the nonlinear equation (76). In fact, the nonlinear term in (76) is superficial. The simple change of phase of u_0, performed by passing to

$$w(t) = \exp\left[\frac{i}{\hbar}\int_0^t (u_0^\dagger \bar{H}_2 u_0)dt'\right] u_0(t) \tag{79}$$

removes this term. Namely, $w(t)$ obeys the linear equation

$$i\hbar \frac{\partial w}{\partial t} = \bar{H}_2 w. \tag{80}$$

The nonlinear term in (76) is necessary in order to ensure (74). Of course, $w(t)$ does not obey (74) in general.

From (79), (80) we see that

$$\frac{d}{dt}(u_0^\dagger \hat{P} u_0) = \frac{i}{\hbar} u_0^\dagger[\bar{H}_2, \hat{P}]u_0, \tag{81}$$

where $\hat{P}$ denotes $\hat{T}^a$, $\hat{S}^b$ or $\hat{J}^{ab}$. It is easy to check that this equation leads to equations (36), (38), (40) for I^a, S^b, J^{ab}.

Now we can prove that the constraints are conserved in time. The proof is based on the plausible assumption that equations (36), (38), (40) for the fixed trajectory $\vec{x}(t)$ have a unique solution determined by the initial data $\vec{I}(t_0)$, $\vec{S}(t_0)$, $[J^{ab}(t_0)]$. If the initial data are specified in such a way that the constraints are satisfied, then there exists $u_0(t_0)$ such that (44–46) are true for $t = t_0$. Next, we solve (76) for u_0 (t) with the $u_0(t_0)$ as the initial data. Applying (44–46) again with the calculated $u_0(t)$ we obtain the solution $\vec{I}(t)$ $\vec{S}(t)$, $[J^{ab}(t)]$ of equations (36), (38), (40) with the chosen initial values. As for this solution (44–46) are true for all t, the constraints are conserved in time.

Finally, let us state once more the most interesting result of this Section: the internal degrees of freedom for the classical particle with spin and SU(2) color spin are described by a 4×4 matrix $M^{\nu}{}_{\mu}$, which is closely related to an element of the SO(3, 1) group, due to the constraint equations.

5. *Remarks*

Since a summary of this paper has been presented in the Introduction, let us here present only certain general remarks.

First, we note that in the case of the abelian gauge group it is always possible to adjust the gauge in such a way that, in addition to $\hat{A}^{\perp}$, also $\hat{A}^{\parallel}$ vanishes along the trajectory. The difference between the nonabelian and abelian cases is essentially due to the fact that the nonabelian, constant gauge potential can yield $\hat{F}_{\mu\nu} = \neq 0$, so it cannot be gauged away to zero, while the abelian constant gauge potential gives $F_{\mu\nu} = 0$, and therefore it can be gauged away to zero.

Therefore, in the case of the abelian gauge group equation (35) for the trajectory $\vec{x}(t)$ is legitimate for any velocity, as it should be because in this case we have to obtain the well-established Newton-Lorentz equation.

In the nonabelian case we have found that the classical dynamical variables obey gauge invariant equations of motion only asymptotically, when condition (29) is satisfied. We expect that in general it is impossible to extract from the quantum theory a satisfactory notion of the classical trajectory of colored, point-like, particle. This would suggest that the concept of a classical colored particle is of limited relevancy for the description of color interactions. Nevertheless, the classical mechanics of colored particles remains interesting on more autonomous grounds, as a very interesting extension of ordinary classical mechanics.

Let us mention here that we have also investigated the time evolution of a localized wave packet in the case when condition (29) is not satisfied. It turns out that initial wave packet placed in the external nonabelian gauge field behaves like a superposition of two wave packets, each of them moving with different group velocity. These investigations will be published in the form of a separate paper.

Our results were obtained with the approximate Hamiltonian H_2 in the Foldy-Wouthuysen representation. The obtained classical theory is a nonrelativistic one. Moreover equations (36), (38), (40) for the internal degrees of freedom cannot be rewritten in the relativistic form by merely introducing the proper time by $\gamma d/dt = d/d\tau$. The equations have truly nonrelativistic form.

Finally, let us stress that in our investigation we have made extensive use of the fact that we consider just SU (2) gauge fields. However, we expect that the difficulty with the concept of the classical colored particle will persist for other nonabelian gauge groups.

REFERENCES

[1] S. K. Wong, *Nuovo Cimento* LXVA, 689 (1970).
[2] W. Czyż, P. K. Kabir, *Phys. Rev.* **D11**, 2219 (1975); J. Schechter, *Phys. Rev.* **D14**, 524 (1976); A. Stern, *Phys. Rev.* **D15**, 3672 (1977); H. Arodź, A. Górski, *Phys. Rev.* **D20**, 1022 (1979).
[3] A. P. Balachandran, P. Salomonson, B. S. Skagerstam, J. O. Winnberg, *Phys. Rev.* **D15**, 2308 (1977); A. Barducci, R. Casalbuoni, L. Lusanna, *Nucl. Phys.* **B124**, 521 (1977).
[4] L. L. Foldy, S. A. Wouthuysen, *Phys. Rev.* **78**, 29 (1950).
[5] R. Jackiw, *Rev. Mod. Phys.* **49**, 681 (1977) (and references therein).
[6] P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford at the Clarendon Press, 1958.
[7] P. Ehrenfest, *Z. Phys.* **45**, 455 (1927).
[8] H. Arodź, *Phys. Lett.* **77B**, 69 (1978).
[9] N. N. Bogoljubov, A. A. Logunov, I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, Ch. 5. W. A. Benjamin Inc., 1975.