RELATIVISTIC SOLUTION FOR ONE SPIN-1/2 AND ONE SPIN-0 PARTICLE BOUND BY COULOMB POTENTIAL: PART TWO

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The fine-structure formula is derived for Coulombic bound states of one spin-1/2 and one spin-0 particle with arbitrary masses.

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Some time ago a relativistic two-body wave equation for one spin-1/2 and one spin-0 particle was derived on the potential level [1]. The original motivation for this work was to get a tool for describing the bound states of a supersymmetric pair of preons. The obtained equation, however, can be applied as well to other spin-1/2+spin-0 systems like e.g. $e^-\alpha$, $\mu^-\alpha$, $e^-\pi^+$, $\mu^-\pi^+$, π^-p, K^-p, e^-K^+, etc. In the present note we would like to establish the Coulombic fine-structure formula following from this equation.

Denoting by 1 and 2 the spin-1/2 and spin-0 particle, respectively, and using the centre-of-mass frame, where $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$ and $\vec{r}_1 - \vec{r}_2 \equiv \vec{r}$, we can write the considered equation in the form [1]

$$\left[\tfrac{1}{2}\left(E-V+\frac{m_1^2-m_2^2}{E-V}\right)-(\vec{\alpha}_1\cdot\vec{p}+\beta_1 m_1)\right]\sqrt{E-V}\,\psi(\vec{r}) = 0. \tag{1}$$

In the one-body limit, where $m_1/m_2 \to 0$ and $V/m_2 \to 0$ or $m_2/m_1 \to 0$ and $V/m_1 \to 0$, Eq. (1) transits into the usual Dirac equation

$$[\varepsilon_1 - V - (\vec{\alpha}_1 \cdot \vec{p} + \beta_1 m_1)]\psi(\vec{r}) = 0 \quad (\varepsilon_1 \equiv E - m_2) \tag{2}$$

or the usual Klein-Gordon equation

$$[(\varepsilon_2 - V)^2 - (\vec{p}^2 + m_2^2)]\psi(\vec{r}) = 0 \quad (\varepsilon_2 \equiv E - m_1), \tag{3}$$

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respectively. In the latter transition the small components of $\psi(\vec{r})$ are eliminated. In the case of equal masses $m_1 = m_2 \equiv m$ Eq. (1) takes the Dirac-like form

$$[\tfrac{1}{2}(E-V)-(\vec{\alpha}_1 \cdot \vec{p}+\beta_1 m)]\sqrt{E-V}\psi(\vec{r}) = 0 \tag{4}$$

(but describes the internal motion).

In the case of Coulombic bound states with $V = -\alpha/r$ we have in Eq. (1).

$$\tfrac{1}{2}\left(E-V+\frac{m_1^2-m_2^2}{E-V}\right) = \tfrac{1}{2}\left(E_0+\frac{m_1^2-m_2^2}{E_0}\right)+\frac{m_2}{M}\Delta E-V_{\text{eff}}-\Delta V_{\text{eff}}+O(\alpha^6), \tag{5}$$

where $E = E_0+\Delta E+O(\alpha^6)$ with $E_0 = M-\alpha^2\mu/2n^2+O(\alpha^4)$ and $\Delta E = O(\alpha^4)$, determined by

$$V_{\text{eff}} = \frac{m_2}{M}V \tag{6}$$

and

$$\Delta V_{\text{eff}} = -\frac{m_1-m_2}{2M^2}\left(V^2-\frac{\alpha^2\mu}{n^2}V\right), \tag{7}$$

respectively. Here, $M = m_1+m_2$ and $\mu = m_1 m_2/M$.

Neglecting in Eq. (5) the correction (7) to the effective Coulombic potential (6) (and hence also ΔE) we solve Eq. (1) exactly [2]. Expanding the resulting E_0 into powers of α^2 we obtain

$$E_0 = M-\frac{\alpha^2\mu}{2n^2}-\frac{\alpha^4\mu}{2n^4}\frac{m_2^2}{M^2}\left[\frac{n}{j+\frac{1}{2}}-\frac{3}{4}+\frac{1}{4}\frac{m_1(m_1-m_2)}{M^2}\right]+O(\alpha^6), \tag{8}$$

where $n = n_r+j+1/2 = 1, 2, 3, \ldots$ and $j = 1/2, 3/2, 5/2, \ldots$

Now, we treat this solution as a ladder approximation. Then perturbing it by ΔV_{eff} we get (see Eq. (5))[1]:

$$\Delta E = \frac{M}{m_2}\langle\Delta V_{\text{eff}}\rangle = -\frac{\alpha^4\mu}{2n^4}\frac{m_1(m_1-m_2)}{M^2}\left(\frac{n}{l+\frac{1}{2}}-1\right)+O(\alpha^6)(m_1-m_2). \tag{9}$$

where $l = 0, 1, 2, \ldots$. We can see that $\Delta E = 0$ for $m_1/m_2 \to 0$ and for $m_1 = m_2$. Combining Eqs. (8) and (9) we can write the following fine-structure formula for Coulombic bound states of one spin-1/2 and one spin-0 particle:

$$E = M-\frac{\alpha^2\mu}{2n^2}-\frac{\alpha^4\mu}{2n^4}\left[\frac{m_2^2}{M^2}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)+\frac{m_1(m_1-m_2)}{M^2}\left(\frac{n}{l+\frac{1}{2}}-\frac{3}{4}\right)\right]+O(\alpha^6). \tag{10}$$

The masses m_1 and m_2 are here arbitrary.

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[1] In the footnote 1 in Ref. [2] the factor appearing in Eq. (9), M/m_2, is omitted, so the resulting equation is valid only in the limit of $m_1/m_2 \to 0$. I am indebted to Sławomir Wycech for his calling my attention to this point.

In the one-body limit of $m_1/m_2 \to 0$ or $m_2/m_1 \to 0$ Eq. (10) reduces to the usual fine-structure formula

$$\varepsilon_1 = m_1 - \frac{\alpha^2 m_1}{2n^2} - \frac{\alpha^4 m_1}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) + O(\alpha^6) \tag{11}$$

or

$$\varepsilon_2 = m_2 - \frac{\alpha^2 m_2}{2n^2} - \frac{\alpha^4 m_2}{2n^4}\left(\frac{n}{l+\frac{1}{2}} - \frac{3}{4}\right) + O(\alpha^6) \tag{12}$$

for the Dirac or Klein-Gordon equation, respectively. In the case of equal masses $m_1 = m_2 \equiv m$ Eq. (10) gives

$$E = 2m - \frac{\alpha^2 m}{4n^2} - \frac{\alpha^4 m}{16n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) + O(\alpha^6). \tag{13}$$

In this case Eq. (1) taking the form (4) can be solved exactly giving the Sommerfeld-like formula [1]

$$E = 2m\left[1+\left(\frac{\alpha/2}{n_r+\gamma}\right)^2\right]^{-1/2}, \quad \gamma = [(j+\tfrac{1}{2})^2-(\alpha/2)^2]^{1/2}, \tag{14}$$

where $n_r = 0, 1, 2, \ldots$.

REFERENCES

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