A LOCAL REPRESENTATION FOR FERMIONS ON A LATTICE

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Generalization of Jordan-Wigner transformation to three space dimensions is proposed. According to the new prescription the system of free fermions is mapped into a set of locally interacting spins with constraints. The whole procedure is performed in the Hamiltonian formulation of the theory.

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1. Introduction

Spectrum of hadrons, as predicted by QCD, is a long awaited, crucial test of the theory. Only recently some quantitative results on this subject became available [6, 7]. The main difficulty in obtaining these numbers is very time consuming procedure for performing Monte Carlo calculations with fermions [11–15]. This work is a small step towards developing a new technique of dealing with fermions on a lattice in three dimensions. The final goal is to map system of fermions $\{\phi\}$ into set of Ising-like variables $\{\gamma\}$. Having achieved that, one could do Monte Carlo with fermions by performing all statistical sums simultaneously in gluonic $\{U\}$ and new fermionic $\{\gamma\}$ variables. In this way the troublesome calculation of the fermionic determinant, for each configuration of U's, could be avoided.

In this paper we construct such a mapping, in three space dimensions, but in the Hamiltonian approach. To make above mentioned program realistic this work must be adapted to the Euclidean formulation of the theory.

In one space dimension, transformation between fermionic and spin operators is well known [3–5] as the Jordan-Wigner transformation [1]. It is a key of the algebraic solution of the Ising model in two $(1+1)$ dimensions. However in more than one dimension Jordan-Wigner prescription leads to non-local interactions. This is a consequence of lack of the natural numbering scheme of lattice sites in higher dimensions. Here we will present the transformation which maintains locality of interactions in arbitrary dimen-

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sions. This approach is a natural generalization of the method used by Nambu [2] in his solution of the Ising 2 model.

To begin with, let us solve the one dimensional problem using Nambu trick. Consider the Hamiltonian [9]

$$H = i\sum_n \phi^\dagger(n)\phi(n+1) - \phi^\dagger(n+1)\phi(n), \tag{1}$$

where the variables $\phi(n)$ fulfill standard algebra

$$\{\phi^\dagger(n), \phi(m)\} = \delta_{nm}, \tag{2}$$

and other anticommutators vanish. After changing variables to

$$X(n) = \phi^\dagger(n) + \phi(n), \tag{3}$$

$$Y(n) = i(\phi^\dagger(n) - \phi(n)),$$

the Hamiltonian takes the form

$$H = \tfrac{1}{2}\sum_n S^X(n) + S^Y(n), \tag{4}$$

where we have introduced link operators of two types

$$S^X(n) = iX(n)X(n+1) \quad \text{and} \quad S^Y(n) = iY(n)Y(n+1). \tag{5}$$

It follows from (2) and (3) that operators S have the algebra ($Z = X, Y$):

$$\begin{aligned} &[S^X(n), S^y(m)] = 0, \\ &[S^Z(n), S^z(m)] = 0 \qquad m \neq n-1,\ n+1, \\ &\{S^Z(n), S^z(m)\} = 0 \qquad m = n-1,\ n+1. \end{aligned} \tag{6}$$

Now Nambu transformation to spin operators (Pauli matrices) can be summarized in the following prescription: express operators (5) by bilinears of σ matrices in such a way that algebra (6) is preserved. One possible choice is

$$S^X(n) = \sigma^1(n)\sigma^2(n+1), \quad S^Y(n) = \sigma^2(n)\sigma^1(n+1), \tag{7}$$

where 1 and 2 are Pauli indices and n labels lattice sites. In this way we have transformed Fermi into spin operators without using Jordan-Wigner transformation explicitly.

Let us conclude this section with two remarks. Analytical solution of the model is easier when Hamiltonian is expressed in terms of Fermi operators. Actually, while solving Ising 2, one proceeds from (7) to (1). For computer calculations, however, form (7) is more convenient since elementary objects there commute at large distances. We may expect then, that their Euclidean representation will be given by numbers and not Grassmann variables.

Secondly, Hamiltonian (8) is different than one for real fermions [8, 10]. Nevertheless (8) presents all essential difficulties to the standard Jordan-Wigner approach. Hence,

for the sake of simplicity, we will explain our construction in details on the toy model (8) in two space dimensions. Generalization to three dimensions is straightforward and will be given in Section 3.

2. *Jordan-Wigner transformation in two dimensions*

Consider the periodic lattice, of size L, in two dimensions. Hamiltonian (1) takes the form

$$H = i\sum_{\vec{n},\hat{e}} [\phi^\dagger(\vec{n})\phi(\vec{n}+\hat{e}) - \text{h.c.}], \tag{8}$$

where $\vec{n} = (n_x, n_y)$, $\hat{e} = \hat{e}_x, \hat{e}_y$. After the algebra similar to (1)–(4) [4] one obtains

$$H = \tfrac{1}{2}\sum_{\vec{n},\hat{e}} S^X(\vec{n}, \hat{e}) + S^Y(\vec{n}, \hat{e}). \tag{9}$$

We have now two links of each type ($Z = X, Y$):

$$S^z(\vec{n}, \hat{e}) = iZ(\vec{n})Z(\vec{n}+\hat{e}), \quad \hat{e} = \hat{e}_x, \hat{e}_y. \tag{10}$$

The algebra of link operators is a natural generalization of (6)

$$[S^X(\vec{n}, \hat{e}), \quad S^Y(\vec{m}, \hat{k})] = 0$$

$$[S^Z(\vec{n}, \hat{e}), \quad S^Z(\vec{m}, \hat{k})] = 0 \quad \text{if corresponding links do not overlap and}$$

$$\{S^Z(\vec{n}, \hat{e}), \quad S^Z(\vec{m}, \hat{k})\} = 0 \quad \text{if corresponding links overlap at one end.} \tag{11}$$

The crucial point of Nambu transformation, from (5) to (7), was changing *non-local* anticommutation of $Z(n)$ to the *local* anticommutation of $\sigma^1(n)$ and $\sigma^2(n)$. To achieve this we needed two anticommuting sigmas at each lattice site, because only two nearest neighbours link operators anticommute.

Generalization to arbitrary dimensions is now simple. We need bigger matrices. In M dimensions there are $2M$ links overlapping at one end. Hence, in order to satisfy (11), one must replace σ's by matrices among which there is a set of 2M anticommuting ones. This can be done with generalized Euclidean Dirac matrices $\Gamma^{(M)}$[1].

In two dimensions Euclidean Dirac matrices Γ^i, $i = 1, 2, 3, 4$, form a required set. Formula (7) is replaced by (Fig. 1)

$$\begin{aligned} S^X(\vec{n}, \hat{e}_x) &= \Gamma^1(\vec{n})\Gamma^3(\vec{n}+\hat{e}_x), \\ S^X(\vec{n}, \hat{e}_y) &= \Gamma^2(\vec{n})\Gamma^4(\vec{n}+\hat{e}_y), \\ S^Y(\vec{n}, \hat{e}_x) &= \tilde{\Gamma}^1(\vec{n})\tilde{\Gamma}^3(\vec{n}+\hat{e}_x), \\ S^Y(\vec{n}, \hat{e}_y) &= \tilde{\Gamma}^2(\vec{n})\tilde{\Gamma}^4(\vec{n}+\hat{e}_y), \end{aligned} \tag{12}$$

[1] Nambu [2] also discusses a possible (though different) transformation in two space dimensions. Nevertheless he abandoned the idea since it led to four-fermion interactions, in case of Ising 3 model.

where $\tilde{\Gamma}^k = i\prod_{\substack{j=1\\ j\neq k}}^{4} \Gamma^j$. Using anticommutation rules $\{\Gamma^i, \Gamma^k\} = 2\delta^{ik}$, one readily verifies that (12) fulfill the algebra (11).

Thus, we have obtained a system of quantum spins of the more complicated structure. The Hilbert space is the tensor product of the four dimensional spaces for each spin. $\Gamma^i(\vec{n})$, $i = 1 \ldots 4$, act only in the space of spin sitting at lattice site $\vec{n}$.

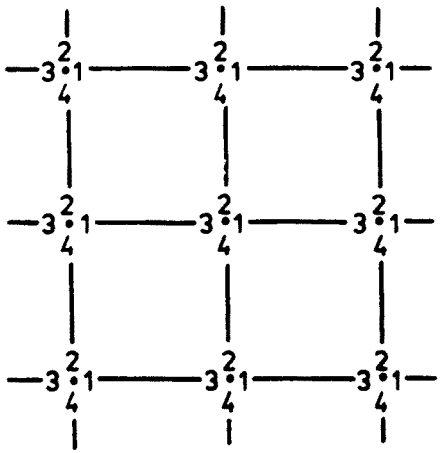

Fig. 1. Arrangement of Dirac matrices for the two dimensional lattice. Numbers are Dirac indices of Γ matrices attached to given lattice sites

From (12) and (9) one gets the Hamiltonian

$$H = \tfrac{1}{2}\sum_{\vec{n}} \Gamma^1(\vec{n})\Gamma^3(\vec{n}+\hat{e}_x)+\Gamma^2(\vec{n})\Gamma^4(\vec{n}+\hat{e}_y)+(\Gamma \to \tilde{\Gamma}). \tag{13}$$

However the set of spins described by (13) is bigger than the original fermionic system. The situation is similar to the multiplication of modes in Susskind approach [8]. Nevertheless it is not clear if it is really the same phenomenon. At least our solution of the problem is different.

To extract fermionic modes from (13) one has to project out redundant degrees of freedom. I would like to thank professor C. Itzykson for showing me how to identify necessary projectors. In the representation (10) link operators fulfill the following relations

$$P(C) \equiv \prod_{\substack{\text{closed}\\ \text{path } C}} S^Z(\vec{n}, \hat{e}) = 1, \tag{14}$$

since $Z(\vec{n})^2 = 1$. Eqs. (14) are not satisfied in the representation (12). However $P(C)^2 = 1$ also in our representation, hence the proper projection operators are

$$\sum(C) = \tfrac{1}{2}(1+P(C)). \tag{15}$$

It turns out, that only operators corresponding to elementary plaquettes are independent. More complicated plaquette operators can be built from elementary ones just by multiplication. Ignoring boundary conditions, for a moment, we see that $\mathcal{N}(\mathcal{N} = L^2)$ constraints $P(C_{el}) = 1$ reduce our system to the required size.

With periodic boundary conditions (p.b.c.) taken into account we have only $\mathcal{N}-1$ independent elementary plaquettes because there is an identity

$$\prod_{\text{all } C_{el}} P(C_{el}) = 1. \tag{16}$$

However there are two other independent constraints

$$L_x \equiv \prod_{n_x=1}^{L} S^X(\vec{n}, \hat{e}_x) = 1 \quad \text{and} \quad L_y \equiv \prod_{n_y=1}^{L} S^X(\vec{n}, \hat{e}_y) = 1. \tag{17}$$

Hence our final system has $2^{\mathcal{N}-1}$ degrees of freedom[2]. The physical meaning of the additional constraint will be clear if we consider the operator of the total number of fermions. In terms of standard variables

$$N = \sum_{\vec{n}} \phi^\dagger(\vec{n})\phi(\vec{n}) = \sum_{\vec{n}} \tfrac{1}{2}(1+S^{XY}(\vec{n})), \tag{18}$$

where $S^{XY}(\vec{n}) = iX_{\vec{n}}Y_{\vec{n}}$. Corresponding expression, in our representation, is given by Eq. (24). However, using (17) one obtains

$$\prod_{\vec{n}}^{\mathcal{N}} \Gamma^5(\vec{n}) = \prod_{n_x=1}^{L} L_y(n_x) \prod_{n_y=1}^{L} L_x(n_y) = 1. \tag{19}$$

It is clear, from (19) and (24) that p.b.c. force us to consider only even (odd) number of fermions for even (odd) lattice size L. Sector of Hilbert space with odd (even) number of fermions can be described independently if we required antiperiodicity at one (generally at odd) number of links [2, 3].

At present we are lacking the proof that our description of fermions is equivalent to the standard one. To check this hypothesis, however, we have calculated two simple observables obtaining exactly the same answers in both representations. Let us define the "moments" of the Hamiltonian

$$M_k = \operatorname{Tr} H^k, \quad K = 1, 2, \ldots . \tag{20}$$

From exact solution of system (8), with p.b.c., one obtains

$$M_2 = 2^{\mathcal{N}-1}\mathcal{N} \tag{21a}$$

$$M_4 = 2^{\mathcal{N}-2}\mathcal{N}(6\mathcal{N}-10) \quad \text{for } L = 4, \text{ and} \tag{21b}$$

$$M_4 = 2^{\mathcal{N}-2}\mathcal{N}(6\mathcal{N}-9) \quad \text{for } L > 4. \tag{21c}$$

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[2] To simplify writing we will (incorrectly) use a term number of degrees of freedom for dimensionality of the corresponding Hilbert space.

The same moments were calculated using Hamiltonian (13) with constraints (15) and (17), e.g. we used the formula

$$M_K = \mathrm{Tr}\, \Xi H^K. \tag{22}$$

Ξ is the projection operator

$$\Xi = \Lambda_x \Lambda_y \prod_{\vec{n}}^{\mathcal{N}-1} \sum (C_{\vec{n}}), \tag{23}$$

where $\Lambda_{x,y} = \frac{1}{2}(1+L_{x,y})$ and $C_{\vec{n}}$ is the elementary plaquette with lower-left corner sitting at site $\vec{n}$.

Results obtained from (22) were again those given by (21). The difference between (21b) and (21c) is due to the p.b.c. Namely, for small ($L = 4$) lattice, one can build from H^4 loops which close through the boundary. The same loops can be found in the expansion of Ξ. Corresponding operators match together giving non-zero contribution to the trace. For higher moments, we would have different results for $L = 2, \ldots, K$ and for $L > K$. The fact that representation (13) has these regularities is very appealing.

The operator of total number of particles can also be easily constructed. Using (18) and substitution rules (12) one gets (after little guessing)

$$N = \sum_{\vec{n}} \tfrac{1}{2}[1+\Gamma^5(\vec{n})]. \tag{24}$$

One can assure himself that (24) is a good choice by calculating $[H, N]$. After some algebra one obtains

$$[H, N] = 0. \tag{25}$$

Results (25) and (24) are very interesting since they might be a clue to the complete diagonalization of the Hamiltonian (13). Because of (24) we can simply choose a basis in which N is diagonal. In that basis the Hamiltonian is quasi-diagonal. Hence instead of the problem of diagonalizing $2^{\mathcal{N}} \times 2^{\mathcal{N}}$ [3] matrix we are left with task of finding eigenvalues for smaller $\binom{\mathcal{N}}{N} \times \binom{\mathcal{N}}{N}$ matrices, $N = 0, \ldots, \mathcal{N}$. In particular, the one-particle spectrum is given by eigenvalues of $\mathcal{N} \times \mathcal{N}$ matrix.

3. Generalization to three dimensions

In our space there are six links which meet at one site of a cubic lattice. Hence we need, 8×8, $\Gamma^{(3)}$ Dirac matrices in order to satisfy algebra (11). Extending accordingly substitution rules (12) one obtains the following Hamiltonian (c.f. Fig. 2).

$$H = \tfrac{1}{2} \sum_{\vec{n}} \Gamma^1(\vec{n})\Gamma^3(\vec{n}+\hat{e}_x) + \Gamma^2(\vec{n})\Gamma^4(\vec{n}+\hat{e}_y) + \Gamma^5(\vec{n})\Gamma^6(\vec{n}+\hat{e}_z) + (\Gamma \to \tilde{\Gamma}), \tag{26}$$

where $\tilde{\Gamma}^k = \prod_{\substack{i=1 \\ i \neq k}}^{6} \Gamma^i$.

[3] After projections.

Let us count number of idependent constraints forgetting boundary conditions for a moment. There are $3\mathcal{N}$ elementary plaquettes in the cubic lattice of size $L(\mathcal{N} = L^3)$. But for each elementary cube K (Fig. 3a) we have an identity

$$\prod_{\partial K} P(C) = 1. \tag{27}$$

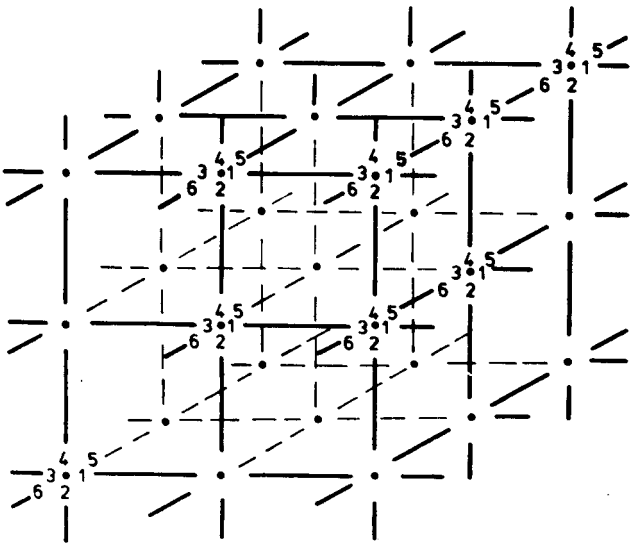

Fig. 2. Same as on Fig. 1 but for three dimensional lattice. Assignment of $\Gamma^{(3)}$ matrices is shown on a few examples

Hence we will make the convention that the "ceiling" plaquette is determined (via (27)) by the floor and four walls. Therefore, in a whole lattice, there are $2\mathcal{N}$ independent wall plaquettes and L^2 ones from the first floor. For large L one is left with $2\mathcal{N}$ constratnts which reduce number of degrees of freedom to the required $2^{\mathcal{N}}$.

However, even with p.b.c. included, our system has exactly the same number of degrees of freedom, as the original set of fermions. This we will prove now. Due to the p.b.c., the identity (16) must be satisfied for each plane in the three dimensional lattice. Applying it to the first floor we get $2\mathcal{N}+L^2-1$ candidates for independent plaquettes (Fig. 3c). Planar conditions (16) for the walls will be considered later in order to avoid double counting. In three dimensions there exists yet another set of relations between plaquettes. Consider a subset of elementary cubes arranged in a tower (Fig. 3b). All, higher than first, floors are removed, according to (27). Upper vertical links are connected to the lowest sites, via p.b.c. giving to the tower topology of the torus. For such an object the similar identity holds

$$\prod_{\partial T} P(C) = 1. \tag{28}$$

Hence we will regard uppermost backward plaquette as dependent on the rest (Fig. 3b). Conditions (28) can now be used to remove $(L-1)\times(L-1)$ uppermost backward facing

plaquettes (A) from the lattice (Fig. 3c). Then we apply planar identity (16) to each vertical plane of walls to remove plaquettes B and C. It is clear now that using (28) for all L^2 towers would lead to double counting. Finally we are left with

$$2L^3+(L^2-1)-(L-1)(L-1)-2L = 2L^3-2 \tag{29}$$

independent plaquettes. In addition to (29) there are also three independent "linear" loops L_x, L_y and L_z. Hence we have to impose $2\mathcal{N}+1$ constraints on the system (26) what reduces original $8^{\mathcal{N}}$ to the required $2^{\mathcal{N}-1}$ degrees of freedom.

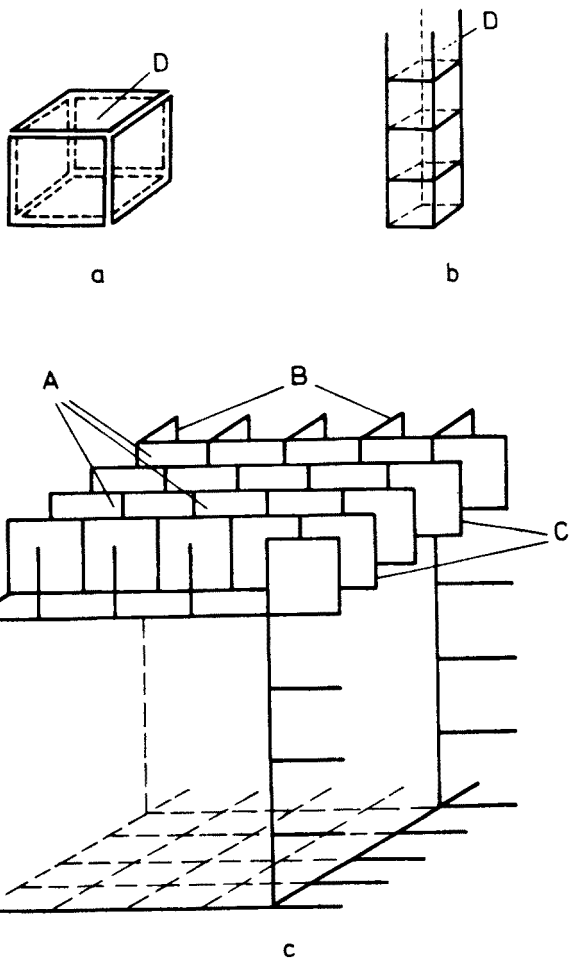

Fig. 3a, b. A representation of the identities (27) and (28) for the three dimensional lattice. By convention, plaquettes marked by D are considered to be dependent on the rest. c. Periodic lattice of size 5 in three dimensions. First floor contains already 24 independent plaquettes. Plaquettes A and B, C depend on the rest due to the conditions (28) and (16) respectively. Examples of links connecting last with the first site, in given direction, are also shown

Finally, generalization to Kogut-Susskind Hamiltonian [8, 10] should be straightforward because it is also built from bilinears in ϕ field[4]. One would have also to impose the constraints (27). Another interesting possibility is not to use constraints at all and reinterpret additional modes as in Ref. [8]. This approach requires more knowledge about the spectrum of (26). We defer detailed study of this question as well as of the transition to the Euclidean formulation, for the future.

4. *Conclusions*

A new representation for fermions on a lattice is proposed in framework of the Hamiltonian approach. Anticommuting Fermi operators are replaced by spin operators. The program was made possible thanks to generalization of Nambu trick to more than one dimensions. In this way standard difficulty of Jordan-Wigner transformation was avoided.

The price of the locality of our representation is that resulting system is an Ising-like one but with constraints. Nevertheless we think that they can be implemented locally on a system.

It was proved that the number of independent constraints is exactly the one which is needed for remaining system to describe the set of even or odd number of fermions.

Other consistency checks were also performed. They support the hypothesis that our representation is equivalent to the standard one.

Further studies of the new representation are important for at least two reasons.

1. Generalization to the Euclidean approach would allow to perform Monte Carlo calculations with fermions but *without* determinant.

2. It provides examples of systems of spins in higher dimensions which can be solved exactly via their fermionic representations.

Finally the structure of constraints is very suggestive by its similarity to the standard constructions in gauge theories.

I would like to thank Professor C. Itzykson for showing me how to construct relevant projection operators. I am also glad to thank the Theoretical Physics Group at University of Wuppertal, where this work was completed, for their hospitality.

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