PHASE TRANSITIONS IN LATTICE GAUGE THEORIES WITH SU(N) GAUGE GROUPS IN THE $N \to \infty$ LIMIT*

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Arguments supporting the existence and clarifying the physical interpretation of phase transitions in SU($N \to \infty$) lattice gauge theories are reviewed.

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1. Introduction

Some ten years ago most particle physicists had little interest in phase transitions. Today a conference on theoretical particle physics without somebody mentioning phase transitions is hardly possible. One reason is that phase transitions block the road to a very nice proof of confinement in quantum chromodynamics.

K. Wilson in 1974 pointed out [1] that, when a gauge theory is reformulated on a lattice, the strong coupling expansion is rather easy to obtain and that in the strong coupling approximation gauge theories (here and in the following we consider SU(N) and U(N) theories without fermions) confine. This result is not directly relevant to continuum theories, because continuum theories are obtained in the weak coupling limit of lattice gauge theories. If, however, the strong coupling theory could be analytically continued into the weak coupling region, one could use renormalization group arguments to prove confinement in the continuum theory.

A phase transition corresponds to a singularity in the dependence of the lattice energy E on the coupling constant g and, therefore, if present, makes the analytic continuation from strong coupling ($g \to \infty$) to weak coupling ($g \to 0$) impossible.

Phase transitions in particle theory are not quite the same thing as phase transitions in thermodynamics. In both cases they may be related to singularities of the function $E(\beta)$. In thermodynamics, however, β is proportional to the inverse temperature $1/T$, while in particle physics it is proportional to g^{-2}. In order to illustrate this difference

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consider the simple example shown in Fig. 1. Fig. 1a presents the ground state of N non--interacting fermions in a one dimensional potential field. All the levels below the Fermi level μ are occupied and all the levels above are empty. This ground state corresponds in mechanics to the minimum of energy and in thermodynamics to the minimum of temperature. The effect of increasing the temperature is shown in Fig. 1b. The Fermi surface at μ becomes blurred. Holes below μ and particles above appear. Fig 1c shows the corre-

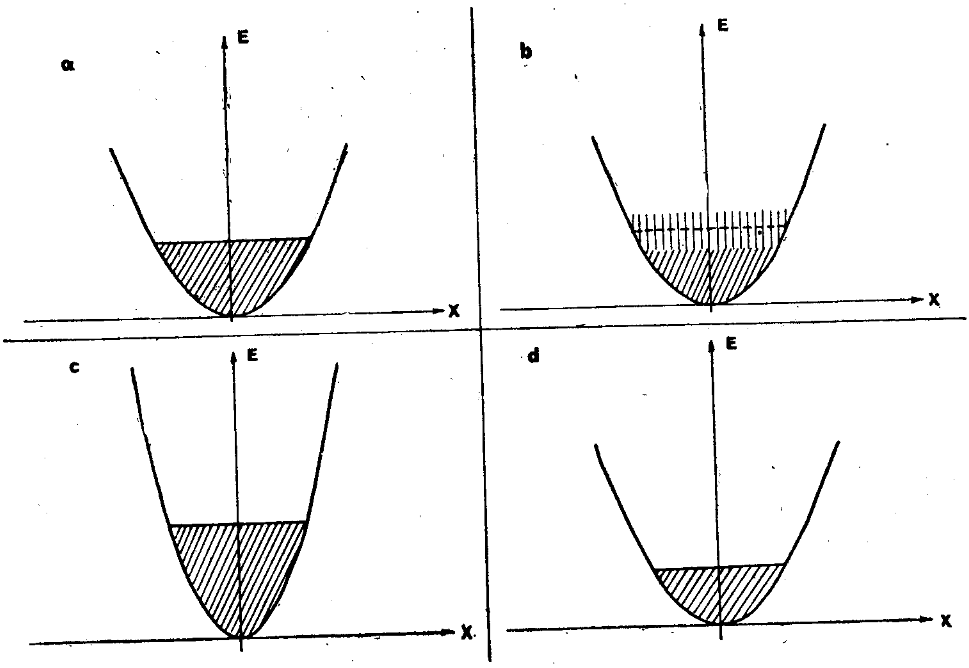

Fig. 1. Fermions in a one dimensional potential (see text); a) ground state, b) state at a higher temperature, c) state at a lower coupling constant, d) the same as (c) with the unit on the vertical axis rescaled by $1/\beta$

sponding change as interpreted in particle physics. A change in β means rescaling the potential. Consequently, the Fermi level moves, but the Fermi surface remains sharp. Another version of Fig. 1c is shown in Fig. 1d. Here the unit on the vertical axis is rescaled by $1/\beta$. Thus the potential curve looks the same as in Fig. 1a and 1b. As easily checked, when β decreases, the Fermi level μ is lower than in Fig. 1a. As shown by this example, some properties of thermodynamical phase transitions are different from the corresponding properties of the phase transitions discussed in particle physics. Nevertheless the analogy is close and all the terminology, as well as most physical intuitions, can be taken over.

From this point of view it may be surprising that Gross and Witten [2] studying two--dimensional gauge theories with SU(N) gauge groups on a lattice found in the limit $N \to \infty$ a third order phase transition. Before explaining why third order phase transitions are unusual let us comment on the $N \to \infty$ limit.

For quantum chromodynamics the relevant lattice gauge theory is of course that in four dimensions (x, y, z, t) and with the gauge group SU(3). This is a very difficult theory, however. For the continuum theory it has been suggested [3] that an instructive model is one with SU(N) gauge group in the limit $N \to \infty$, $g \to 0$, g^2N fixed. The simplifications are significant. In particular [3] only planar graphs occur in the weak coupling expansion. Unfortunately, even this simplified model is so complicated that only in two dimensions the solution has been found. Gross and Witten studied the two dimensional lattice version, which also can be solved analytically.

Our intuitive notion of a phase transition usually corresponds to a first order phase transition. When water boils, the volume of a unit of mass changes by three orders of magnitude. It is obvious that the phase has changed.

Second order transitions are more subtle, but still rather common. Consider for example a non-interacting, monoatomic Bose gas. Let us start at a temperature above the Einstein condensation point. When temperature decreases, the momentum distribution of the particles gradually changes — particles become slower on the average. At the condensation point the picture changes. The momentum distribution for non-zero momenta becomes frozen. Only its normalization decreases. The surplus particles condense i.e. drop to the lowest energy ($p = 0$) single particle state. Thus here nothing spectacular happens at the transition point. It is the evolution of the system below the transition point, which is different from the evolution above.

A third order transition differs from the second order transition in that the change in the evolution law at the transition point is infinitesimal. It is only at some distance from the transition point that the difference in evolution becomes significant. In thermodynamics, a third order transition remains a theoretical possibility without experimental realizations. Therefore, it is interesting that such transitions may be realized in particle physics.

In the following section, we present a simple model, which shows that the third order phase transition found in the SU($N \to \infty$) lattice gauge theories can be understood in terms of the discussion given above.

2. *Simple model*

The model presented here [4] is a simple extension of a model solved by Brézin, Itzykson, Parisi and Zuber [5] (further quoted BIPZ). It may be also interpreted as a generalization of the one-plaquette lattice gauge theory model considered by Wadia [6]. Wadia's plaquette had two spatial dimensions and time was a continuous parameter. Also in this lattice gauge theory model the third order phase transition in the limit $N \to \infty$ occurs. For N finite, there is no phase transition, however [6].

Consider a Schrödinger equation with the Hamiltonian

$$H = -\nabla^2 + W(M). \tag{2.1}$$

The only difference between this Hamiltonian and Hamiltonians found in elementary textbooks on quantum mechanics is that M is not a simple variable, but an $N \times N$ Hermitian matrix. Of course the wave functions $\psi(M)$ also depend on matrices. The Laplace operator

$$\nabla^2 = \sum_i \frac{\partial^2}{\partial M_{ii}^2} + \frac{1}{2} \sum_{i<j} \left(\frac{\partial^2}{\partial (\mathrm{Re}\, M_{ij})^2} + \frac{\partial^2}{\partial (\mathrm{Im}\, M_{ij})^2} \right). \tag{2.2}$$

The potential W is supposed to satisfy the relation

$$W(M) = \beta \,\mathrm{Tr}\, V(M) = \beta \sum_{i=1}^{N} V(\lambda_i), \tag{2.3}$$

where β is a constant, and λ_i are eigenvalues of matrix M. M being Hermitian has of course N real eigenvalues. For instance BIPZ have

$$\beta V(M) = \tfrac{1}{2} M^2 + \frac{g}{N} M^4. \tag{2.4}$$

Since $W(M)$ is invariant with respect to unitary transformations of matrix M, we may choose the representation, where M is diagonal and relation (2.3) follows. The class of functions $V(M)$ satisfying (2.3) is, however, much broader than the class of analytic functions. For instance, $V(M)$ could have different power series expansions in different regions of the $\{\lambda_1, \ldots, \lambda_N\}$ space. We shall need this generality to illustrate some interesting points in physics. Note that the Hamiltonian (2.1) is invariant under all U(N) transformations and not only under SU(N) transformations. Thus our model is closer to a U(N) gauge theory than to an SU(N) gauge theory. This is harmless, because it is generally believed that for $N \to \infty$ the differences between SU(N) and U(N) gauge theories become irrelevant.

Matrix M is a simple (zero dimensional) model for intermediate boson fields. The trace, which is an SU(N) singlet, corresponds to the photon and the N^2-1 other independent parameters can be related to the coloured gluons.

The solution of the Schrödinger equation with Hamiltonian (2.1) can be obtained [5] using the standard variational method. For the ground state

$$E = \mathrm{Min} \frac{\int d^{N^2} M [(\nabla \psi)^2 + W(M) \psi^2]}{\int d^{N^2} M \psi^2}. \tag{2.5}$$

The operators in the square bracket are invariant under unitary transformations. Therefore it is legitimate to assume that $\psi(M)$ is a symmetric function of the eigenvalues $\lambda_1, \ldots, \lambda_N$. Integrating over all the "angular variables" one finds

$$E = \mathrm{Min} \frac{\displaystyle\int d^{N^2} \lambda \prod_{i<j} (\lambda_i - \lambda_j)^2 \sum_k \left[\left(\frac{\partial \psi}{\partial \lambda_k} \right)^2 + \beta V(\lambda_k) \psi^2 \right]}{\displaystyle\int d^N \lambda \prod_{i<j} (\lambda_i - \lambda_k)^2 \psi^2}. \tag{2.6}$$

Here BIPZ suggest the substitution

$$\varphi(\lambda_1, \ldots, \lambda_N) = \prod_{i<j} (\lambda_i - \lambda_j) \psi(\lambda_1, \ldots, \lambda_N) \tag{2.7}$$

and obtain a variational principle corresponding to N non-interacting fermions (φ is totally antisymmetric)

$$E = \text{Min} \frac{\int d^N\lambda \sum_k \left[\left(\frac{\partial\varphi}{\partial\lambda_k}\right)^3 + \beta V(\lambda_k)\varphi^2\right]}{\int d^N\lambda\varphi^2} . \tag{2.8}$$

It is remarkable that the problem of N^2 bosons (M) with a gauge group, or of N bosons $\psi(\lambda_1, ..., \lambda_N)$, has been transformed here into a problem of N fermions $\varphi(\lambda_1, ..., \lambda_N)$. This can be avoided. E.g. Jevicki and collaborators (references may be traced from Ref. [7]) use a collective field formalism and remain with a bosonic problem. In the bosonic interpretation (e.g. putting moduli of $\lambda_i - \lambda_j$ in (2.7)) the Pauli principle for fermions is replaced by highly singular potentials giving short range repulsion between bosons. A discussion of this problem may be found in Ref. [8]. We keep the version with fermions, because in the present model it is the simplest.

The variational principle (2.8) corresponds to the single particle Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + \beta V(x)\right]\varphi(x) = e\varphi(x). \tag{2.9}$$

The ground state energy for Hamiltonian (2.1) is the sum of the N lowest eigenvalues of equation (2.9), where a k times degenerated eigenvalue is counted as k equal eigenvalues.

Wadia's problem [6] is obtained putting

$$\beta V(x) = \frac{4}{g^2} \sin^2 x. \tag{2.10}$$

Since we are interested in the solution for N large, one may use the WKB approximation, or (equivalently) the Thomas-Fermi approximation. E.g. the Thomas Fermi approximation yields

$$N = \int_{a_-}^{a_+} \sqrt{\mu - \beta V(x)}\, \frac{dx}{\pi} . \tag{2.11}$$

Here a_+ and a_- are the right hand side and the left hand side classical turning points. The total energy is

$$E = \mu N - \tfrac{2}{3} \int_{a_-}^{a_+} \frac{dx}{\pi} \sqrt{\mu - \beta V(x)}^3\, dx. \tag{2.12}$$

From formulae (2.11) and (2.12) it is possible to calculate the derivatives of the energy E with respect to β at fixed particle number N. By definition, a k-th order phase transition occurs at $\beta = \beta_c$, when the k-th derivative is discontinuous there, while E and the lower order derivatives are all continuous.

3. Discussion of the model

Differentiating E with respect to β one finds

$$\frac{\partial E}{\partial \beta} = \frac{3E-\mu N}{\beta}\,. \tag{3.1}$$

For $\beta \neq 0$ this is clearly continuous. The second derivative

$$\frac{\partial^2 E}{\partial \beta^2} = \frac{3(E-\mu N)}{4\beta^2} + \frac{\pi N^2}{2\beta^2}\frac{1}{I(\beta)}\,, \tag{3.2}$$

where

$$I(\beta) = \int_{a_-}^{a_+} \frac{dx}{\sqrt{\mu-\beta V(x)}} \tag{3.3}$$

is the classical time necessary for particle with energy μ to go from a_- to a_+. For continuous potentials $V(x)$, $I(\beta)$ may be infinite, but its inverse occurring in formula (3.2) is continuous. This explains why the phase transition, if any, must be of at least third order.

The third derivative is discontinuous if and only if

$$\frac{dI^{-1}}{d\beta} = I^{-2}\frac{dI}{d\beta} \tag{3.4}$$

is discontinuous. Then a third order transition occurs.

4. Example I

An example is shown in Fig. 2. The phase transition here will be caused by the discontinuity of dV/dx at $x = x_0$. Everywhere else the potential is assumed regular. Let us check first that, according to the qualitative argument given in the introduction, a third order phase transition is expected for this potential.

When the Fermi level rising arrives at μ_c (see figure, one may use the interpretation from Fig. 1d) there is no jump in energy, i.e. no first order transition, but the evolution law changes. The change, related to the deviation of the continuous line from the dotted one in the figure, is first very small and increases linearly with $\mu-\mu_c$. According to our discussion this causes a third order phase transition.

The same conclusion may be reached more rigorously by an analytic argument. Suppose that the break occurs at $x = x_0$. Then

$$V(x) = \frac{\mu_c}{\beta} + V_1'|x-x_0| + V_1(x-x_0) + O(|x-x_0|^2) \tag{4.1}$$

and

$$I(\beta) = C(\beta) - \frac{8V_1'}{V_1^2+(V_1')^2}\sqrt{\mu-\mu_c}\;\theta(\mu-\mu_c), \tag{4.2}$$

where $C(\beta)$ is regular at $\beta = \beta_c$. Substituting

$$\mu_c - \mu \approx \left(\frac{\partial\mu}{\partial\beta}\right)_{\beta=\beta_c}(\beta-\beta_c), \tag{4.3}$$

which is valid for μ close to μ_c, one clearly sees the discontinuity in expression (3.4). For $V_1' = 0$ there is no break and the discontinuity disappears.

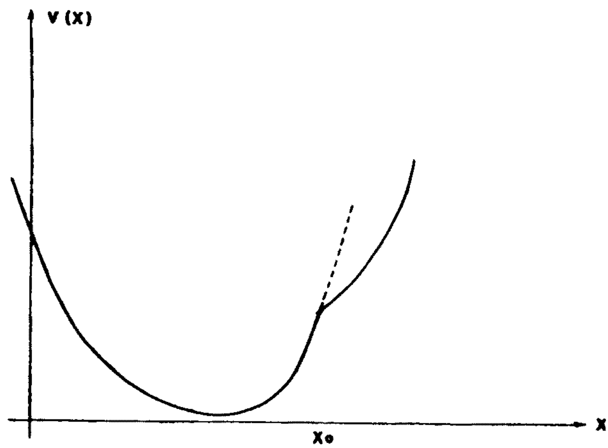

Fig. 2. Potential with a break at $x = x_0$

This example suggests two remarks. At the transition point the weak coupling expansion (in powers of β^{-1}) and the strong coupling expansion (in powers of β) diverge by definition. There are, however, many methods of improving convergence. The methods of Padé and Borel have been particularly popular among physicists. There are many others. The question arises: could one find a method, which would make the continuation across the transition point $\beta = \beta_c$ possible. As seen from our example, in the general case the answer is negative. Full knowledge about the potential for $x \leqslant x_0$ is not enough to predict its behaviour for $x > x_0$. Consequently also the dependence $E(\beta)$ for $\beta > \beta_c$ is unpredictable.

The second remark is related to the fact that, since in the weak coupling limit μ in Fig. 1d is very close to zero, only the bottom of the potential is relevant for the transition to the continuum limit. Therefore, replacing for $x > x_0$ in the potential shown in Fig. 2 the continuous line by the dotted one, gives a different lattice theory with the same continuum limit. The new potential has no break and consequently, there is no phase transition. This idea works. Manton [9] replaced Wilson's formula for the action (the action roughly corresponds to the potential in our example) by another one giving the same continuum limit. Calculations indicate that using Manton's action instead of Wilson's one can indeed kill the phase transition in some models [10], [11].

5. *Example II*

Another potential giving a third order phase transition is shown in Fig. 3. The physical region is limited to the range $|x| \leqslant x_0$ and for simplicity $V(x_0) = 1$. The boundary conditions for the wave functions do not affect qualitatively our conclusions. Here the singu-

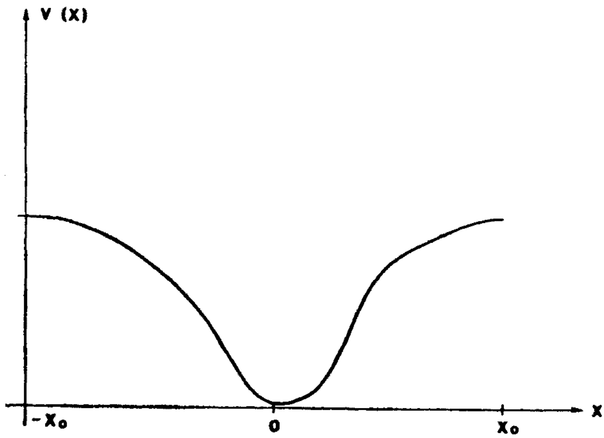

Fig. 3. Potential with maxima at $x = \pm x_0$

larity of expression (3.4) results not from breaks, but from maxima at $|x| = x_0$. The analytic formula is

$$V(x) = 1 - V_2(x-x_0)^2 + O(|x-x_0|^3) \tag{5.1}$$

for $|x| \leqslant x_0$. For $|x| > x_0$, we may assume that the potential is infinite. For simplicity we also assume that the potential is symmetric $V(x) = V(-x)$. In this case

$$I(\beta) = C(\beta) + \frac{\ln|\mu-\mu_c|}{\sqrt{\beta V_2}}. \tag{5.2}$$

Thus there is a third order phase transition.

Let us take for example

$$V(x) = \sin^2 x \quad \text{for } |x| < x_0. \tag{5.3}$$

The integral $I(\beta)$ may in this case be expressed by a well-known elliptic integral [12]

$$I(\beta) = \begin{cases} \dfrac{4}{\beta} K\left(\dfrac{\mu}{\beta}\right) & \text{for } \beta \geqslant \mu \\ \dfrac{4}{\mu} K\left(\dfrac{\beta}{\mu}\right) & \text{for } \beta \leqslant \mu \end{cases}, \tag{5.4}$$

where

$$K(m) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-m\sin^2 x}} \,. \tag{5.5}$$

The strong coupling expansion valid for $\beta < \mu$ is

$$I(\beta) = \frac{2\pi}{\sqrt{\mu}}\left[1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\left(\frac{\beta}{\mu}\right)^n\right]. \tag{5.6}$$

Strictly speaking, in order to obtain what is usually called the strong coupling expansion it would be necessary to substitute into expansion (5.6) an expansion of μ^{-1} in powers of β. Since expansion (5.6) converges whenever the standard strong coupling expansion converges, while it is simpler, it may be considered an improved strong coupling expansion. Similarly the weak coupling expansion valid for $\beta > \mu$ is

$$I(\beta) = \frac{2\pi}{\sqrt{\beta}}\left[1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\left(\frac{\mu}{\beta}\right)^n\right]. \tag{5.7}$$

Both expansions diverge for $\beta \to \mu$. For β close, but not equal to μ the approximation by truncated series may be somewhat improved by introducing corrections for tunneling [13].

We conclude this section with two remarks. In the present example there is no argument to forbid the continuation of the theory from the strong coupling to the weak coupling region. Indeed, in the following section we present a version of the WKB approximation, which successfully bridges the transition region.

The transition from Wilson's action to Manton's corresponds for the present problem to the replacement of $\sin^2 x$ by x^2. This kills the maxima, but breaks due to the boundedness of the physical region remain. Such breaks cause a phase transition as described in the preceding section. Thus in this case the phase transition is made milder, but not eliminated.

6. *WKB approximation*

Consider the Schrödinger equation

$$\left[-\frac{d^2}{dx^2}+\beta V(x)\right]\psi(x) = \beta\varepsilon\psi(x). \tag{6.1}$$

The potential $V(x)$ is assumed periodic with period π, twice differentiable and having exactly one maximum and one minimum per period. We look for solutions $\psi(x)$ periodic with period π. For $V(x) = \sin^2 x$, this is the problem considered by Wadia [6] and Neuberger [13].

Using the standard WKB formulae one can find the approximate eigenvalues in the range $\varepsilon < 1$ from the Bohr-Sommerfeld formula and for $\varepsilon > 1$ from the condition that the phase of $\psi(x)$ must increase by 2π over each period. Both these approximations, however, fail near $\varepsilon = 1$. In this respect they are similar to the weak coupling and strong coupling expansions.

An alternative WKB formula valid for $\varepsilon > 1$ and for $\varepsilon \approx 1$ is known, however [14]. It reads

$$\int_{a_-}^{a_+} \sqrt{\beta(\varepsilon - V(x))}\, dx = [2n - \varphi(a) \pm (\delta - \tfrac{1}{2})]\pi. \tag{6.2}$$

Here $a_\pm$ depend on ε. For $\varepsilon < 1$ they are the adjacent classical turning points and for $\varepsilon \geqslant 1$, a_- is arbitrary and $a_+ = a_- + \pi$. The function of ε

$$a = \sqrt{\frac{\beta}{2\left|\frac{\partial^2 V}{\partial x^2}\right|_m}}\,(1-\varepsilon), \tag{6.3}$$

where the subscript m means that the second derivative of the potential is evaluated at the maximum. The function

$$\varphi(a) = \frac{1}{\pi}\left[\arg \Gamma(\tfrac{1}{2} + ia) + a - a \ln |a|\right]. \tag{6.4}$$

The function of ε denoted δ is the smallest positive root of the equation

$$\sin \delta = \frac{1}{\sqrt{1 + e^{2\pi a}}}\,. \tag{6.5}$$

A detailed derivation of formula (6.2) may be found in Refs. [14] and [15].

7. *Extension to $d > 2$ dimensions*

Two dimensional $SU(N \to \infty)$ lattice gauge theory is soluble and the phase transition is seen from analytic formulae [2]. Gross and Witten [2] conjectured, moreover, that a similar transition should occur in the more interesting four dimensional case.

This conjecture has been proved true by F. Green and S. Samuel [16], [17], who moreover found the transition points β_c for $d = 3$ and $d = 4$ dimensions. Here we present a short sketch of their argument.

The crucial remark is that, since everybody believes that in the $N \to \infty$ limit differences between $SU(N)$ and $U(N)$ gauge theories are negligible, it is legitimate to consider the series of $U(N)$ gauge theories instead of the series of $SU(N)$ gauge theories.

For a $U(N)$ lattice gauge theory the Wilson operator $\hat{W}$ corresponding to a path C is defined by (cf. Fig. 4)

$$\hat{W}[C] = \prod_{l \in C} U_l. \tag{7.1}$$

Here U_l is the unitary $N \times N$ matrix ascribed to link l on a lattice. The oriented contour C begins at some lattice node x, goes over a certain number of links (cf. Fig. 4) and ends in x. Wilson [1] used the trace of $\hat{W}$ to derive an interesting criterion for confinement. Green and Samuel propose to study the determinant.

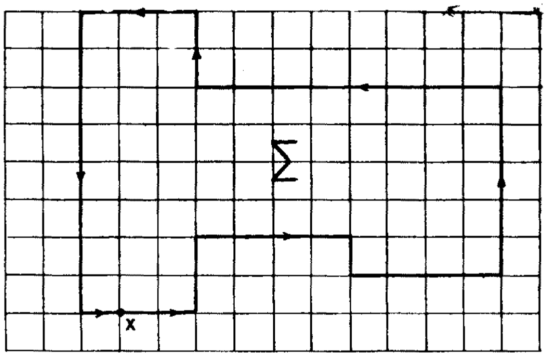

Fig. 4. Two-dimensional lattice with a Wilson loop C enclosing surface Σ

Define

$$P[C] = \langle \mathrm{Det}\, W \rangle^{\frac{1}{N}} \sim e^{-\alpha \Sigma}. \tag{7.2}$$

Here the averaging is over all possible matrices U_l. This can be given a rigorous meaning (cf. e.g. [1]). Σ is the surface (e.g. the number of plaquettes) enclosed by the contour C. For two dimensions this definition is unambiguous. For the other cases the minimal surface should be taken. The sign $\sim$ means that more slowly varying factors, e.g. exponentials of the length of C, are omitted. α is a constant coefficient.

Note that $P[C]$ for SU(N) theories would be identically equal one and certainly would not be suitable for distinguishing between phases. Green and Samuel propose that α is an order parameter.

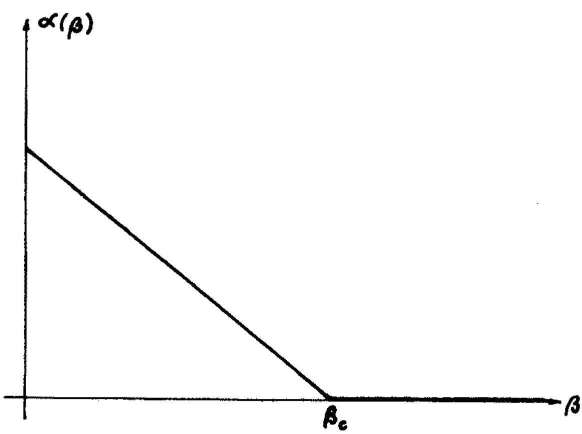

Fig. 5. Order parameter $\alpha(\beta)$. The scale on the vertical axis is arbitrary

An order parameter should change at a phase transition between zero and a non zero value. For instance, if a non magnetized piece of iron is put into a magnetic field and then the field is switched off, the final magnetization is an order parameter: it is zero above the Curie temperature and non zero below.

A qualitative graph representing the dependence of α on β, if α is an order parameter, is shown in Fig. 5. The scale on the vertical axis is arbitrary, so that no importance should be attached to the fact that $\alpha(\beta)$ for $\beta < \beta_c$ is drawn as a straight line. It is important, however, that $\alpha(\beta) = 0$ for $\beta \geqslant \beta_c$. Green and Samuel have checked on a number of exactly soluble models (i.e. for a number of lattices) that $\alpha(\beta)$ is indeed an order parameter for U(N) lattice gauge theories in the limit $N \to \infty$. They have moreover given qualitative arguments that this should be the case also for lattices, where the solution of the U($N \to \infty$) gauge theory is unknown. Further they found that in the strong coupling limit $\alpha \neq 0$, while in the weak coupling limit $\alpha = 0$.

The strong coupling limit involves no problems of principle. The weak coupling limit, however, deserves some discussion. In two dimensions the Coulomb potential is proportional to r and therefore confines. Accordingly, for finite N the authors find $\alpha \neq 0$. The N dependence, however is

$$\alpha = O\left(\frac{1}{N}\right) \quad \text{for} \quad \beta > \beta_c, \tag{7.3}$$

so that for $N \to \infty$, $\alpha = 0$ as it should. This result can be obtained perturbatively. For three dimensions the perturbative (weak coupling) expansion yields $\alpha = 0$. It is known, however that in this case non-perturbative effects (instantons) dominate. Including instantons one finds

$$\alpha = O(e^{-cN}) \quad \text{for} \quad \beta > \beta_c, \tag{7.4}$$

where c is a constant. Thus again $\alpha = 0$ in the high N limit, but the physics is very different. Finally in four dimensions

$$\alpha = 0 \tag{7.5}$$

for any N.

Once the interpretation of α as an order parameter is accepted and it is known that $\beta \to 0$ corresponds to the $\alpha \neq 0$ phase and $\beta \to \infty$ to the $\alpha = 0$ phase, it is possible to calculate the transition point β_c. It is enough to calculate α in the strong coupling approximation and to extrapolate the function $\alpha(\beta)$ to the point where it vanishes. Note that in this approach it is not necessary to deal with the weak coupling range, where perturbative and nonperturbative effects compete.

Using a strong coupling expansion up to β^6 Green and Samuel found for dimensions $d = 2, 3, 4$ the transition points $\beta_c = 0.500$, 0.435 and 0.396 respectively. The normalization here is $\beta = 1/g^2N$. They estimate by comparison with shorter expansions that at least the first two digits of each β_c are reliable. Of course the result for $d = 2$ is exact and had been known [2].

8. Summary and remarks

Let us summarize the results and report on some remarks from Ref. [17]. It has been made plausible (a physicist might perhaps say proved) that SU($N \to \infty$) gauge theories on two-, three-, and four-dimensional lattices have phase transitions and the transition points have been found.

The physical origin of the transition depends on the number of dimensions. The weak coupling phase may be described perturbatively for $d = 2$, is dominated by (non-perturbative) instantons for $d = 3$ and has still some other structure for $d = 4$. This makes extrapolations from lower dimensionality to $d = 4$ risky.

For $d = 4$ Green and Samuel argue that the $\alpha \neq 0$ phase is dominated by monopoles, which cause confinement, while in the $\alpha = 0$ phase the monopoles play no important role.

Going from $N = \infty$ to finite N, for $d = 4$, along the U(N) sequence, one always has the phase transition. For the SU(N) sequence the situation is less clear. It seems, however, [17] that even if there is no phase transition, singularities for complex β close to the real axis should occur. They limit the convergence radius of the strong coupling expansion just as efficiently as a phase transition would.

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