RELATIVISTIC HYDRODYNAMICS IN A MEAN-FIELD THEORY OF NUCLEAR MATTER*

BY W. E. KLEPPINGER

Institute of Theoretical Physics, Department of Physics, Stanford University**

(*Received February 9, 1982*)

The relativistic wave equations describing the propagation of isentropic sound waves for an isotropic, ideal fluid moving with an arbitrary fluid velocity are derived. The Green's functions are given in closed analytic form, and for supersonic flow we find that there is a Mach cone. In the Walecka model of infinite nuclear matter, these sound waves are the elementary excitations of the system and the speed of sound can be calculated explicitly. Finally, we present a possible application to heavy-ion fission induced by an exactly central collision based upon a Cherenkov radiation mechanism.

PACS numbers: 21.65.+f, 03.40.Kf

We will present here the results one obtains for the elementary excitations (sound waves) of infinite nuclear matter, as formulated in the Walecka model [1–5], in the approximation that infinite nuclear matter can be treated as a classical, relativistic fluid. The Walecka model has three distinct advantages: the theory is explicitly Lorentz covariant (and thus includes the correct relativistic propagation of the particles); it incorporates the meson degrees of freedom ab initio; and, when extended to finite nuclei [6–9], successfully predicts all of nuclear structure except for the compressibility of nuclear matter. The relevance of the present approach is that one then has a consistent calculational framework – a relativistic quantum field theory (with all the advantages discussed above) underlies the relativistic classical approximation. Thus, the theory has the causal propagation of any disturbance already built in. Such an approach, in our opinion, is superior to treating nuclear matter in a hydrodynamic approximation with nonrelativistic interactions.

After a brief review of the Walecka model, we will therefore discuss the relativistic propagation of isentropic sound waves in a moving, ideal fluid. We will show that the

* Work supported in part by NSF Grant # NSF PHY 79-18046.

** Address: Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, CA 94305, USA.

exact Green's function for the propagation of sound waves has a simple analytic form, and that for supersonic flow the disturbance is limited to the interior of the Mach cone as expected from causality. To illustrate the application of the ideas presented here to heavy-ion physics, we will examine a simple model for induced fission reactions, proposed originally by Glassgold et al. [10], based on a Cherenkov radiation mechanism in the context of a relativistically consistent theory.

The Lagrangian density of the model field theory is given by[1]

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - M) + \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2) - \tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} + \tfrac{1}{2}m_v^2 V_\mu V^\mu - g_v\bar{\psi}\gamma_\mu\psi V^\mu + g_s\bar{\psi}\psi\phi, \tag{1}$$

where M, m_s, and m_v are the nucleon, scalar, and vector meson masses, and $G_{\nu\mu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is the vector meson field tensor. As discussed by Walecka, in the high density limit one can make the mean-field approximation and replace the meson fields with their classical expectation values [1]. With these approximations, the Euler-Lagrange equations obtained from the Lagrangian of equation (1) can be solved exactly. For a uniform system, with the Fermi sea filled up to wave number k_F, one obtains the following results for the baryon density ϱ, energy density ε, and pressure P [1]

$$\varrho = \gamma \int_0^{k_F} \frac{d^3k}{(2\pi)^3} \tag{2a}$$

$$\varepsilon = \tfrac{1}{2}\frac{C_v^2}{M^2}\varrho^2 + \tfrac{1}{2}\frac{M^2}{C_s^2}(M-M^*)^2 + \gamma\int_0^{k_F}\frac{d^3k}{(2\pi)^3}(k^2+M^{*2})^{1/2} \tag{2b}$$

$$P = \tfrac{1}{2}\frac{C_v^2}{M^2}\varrho^2 - \tfrac{1}{2}\frac{M^2}{m_s^2}(M-M^*)^2 + \tfrac{1}{3}\gamma\int_0^{k_F}\frac{d^3k}{(2\pi)^3}k^2(k^2+M^{*2})^{-1/2} \tag{2c}$$

M^* is the effective mass given self-consistently by

$$M^* = M - \gamma\frac{C_s^2}{M^2}\int_0^{k_F}\frac{d^3k}{(2\pi)^3}M^*(k^2+M^{*2})^{-1/2}, \tag{2d}$$

and γ is the spin-isospin degeneracy ($\gamma = 4$ for nuclear matter). The dimensionless coupling constants $C_i^2 = (g_i^2/m_i^2)M^2$ (i = s, v) are fit to the density and binding energy of bulk nuclear matter and have the values $C_v^2 = 195.7$ and $C_s^2 = 266.9$. With the coupling constants now determined, the model then predicts the energy density and pressure at all other densities.

In order to make the approximation that nuclear matter can be treated as a uniform, isotropic fluid, it is necessary that the baryon density be sufficiently large in order that

---

[1] We will use the notations and conventions of Ref. [11].

the energy density, pressure, and fluid velocity $\boldsymbol{v}$ are well-defined macroscopic quantities. Since the high density limit is the region where the mean-field approximation to the Lagrangian of equation (1) is valid, a hydrodynamic treatment of nuclear matter is consistent with the Walecka model. A suitable average of the microscopic stress-energy tensor must then be taken to obtain the macroscopic stress-energy tensor for an ideal, isotropic fluid [12]

$$T_{\mu\nu} = -Pg_{\mu\nu}+(\varepsilon+P)u_\mu u_\nu, \tag{3}$$

where $g_{\mu\nu}$ is the metric tensor and u_μ is the four-velocity of the fluid. For a moving fluid, the quantities ε, P, and ϱ are defined in the rest frame of the fluid, i.e., they are Lorentz scalar functions. Lorentz covariance then implies that the baryon current is given by

$$B_\mu = \varrho u_\mu. \tag{4}$$

The conservations laws

$$\partial^\mu T_{\mu\nu} = 0 \tag{5a}$$

$$\partial^\mu B_\mu = 0 \tag{5b}$$

then lead to the relativistic generalizations of Euler's equation. (See for example, Landau and Lifshitz [12] or McKee and Colgate [13].)

In addition, we must satisfy the laws of thermodynamics. At zero temperature, the thermodynamic potential Ω is given by

$$\Omega = -PV = E-\mu B, \tag{6}$$

where μ is the chemical potential. Converting equation (6) into densities, one obtains the relation

$$\mu = (\varepsilon+P)/\varrho. \tag{7}$$

Evaluating this expression in the mean-field approximation yields the chemical potential for nuclear matter in the Walecka model [14]

$$\mu = g_v V_0 + E_F, \tag{8}$$

with $E_F = (M^{*2}+k_F^2)^{1/2}$ the Fermi energy.

To complete the characterization of the dynamics, we will use the equation of state $P = P(\varepsilon)$ calculated in the mean-field approximation of the Walecka model [1]. In analogy with nonrelativistic fluid mechanics, we define the speed of first (isentropic) sound s in nuclear matter to be

$$s^2 = \left.\frac{dP}{d\varepsilon}\right|_{\mathrm{ad}} \tag{9}$$

in the rest frame of the fluid. Since at zero temperature, only zero sound is allowed to propogate in a Fermi liquid, we must assume that some mechanism exists which allows first sound to propagate in nuclear matter [15]. For example, the system could have a finite temperature or a finite viscosity. Using the standard model, we calculate for the speed of sound

$$s^2 = \frac{\varrho}{\mu}\left\{\frac{g_v^2}{m_v^2} + \frac{k_F^2}{3E_F\varrho} - \frac{g_s^2}{m_s^2}(M^*/E_F)^2\left[1+\frac{g_s^2}{m_s^2}\gamma\int_0^{k_F}\frac{d^3k}{(2\pi)^3}k^2(k^2+M^{*2})^{-3/2}\right]^{-1}\right\}. \tag{10}$$

This agrees with a similar calculation due to Matsui [16]. At nuclear matter densities, the predicted speed of sound is $s = 0.256$. The fact that the predicted speed of sound is approximately one-fourth the speed of light implies that relativistic effects will be important and thus justifies using relativistics hydrodynamics. In the high density limit $\varrho \to \infty$, $s^2 \to 1$ from below so that the speed of sound predicted by the Walecka model is always less than the speed of light.

Since the speed of sound is related to the compressibility K ($K^{-1} = 9\mu s^2$), we can obtain an estimate for the speed of sound using the experimentally measured excitation energy of the giant monopole resonance in ^{208}Pb, assuming the measured strength exhausts the sum rule. Taking into account finite size effects, the experimental estimate for the compressibility is $K^{-1} = 210$ MeV [17]. This gives for the speed of sound $s_{exp} = 0.159$. The discrepancy between the theoretical prediction and the experimental estimate implies that the nuclear equation of state calculated in the Walecka model is too stiff.

We wish to study the behavior of nuclear matter for small oscillations about equilibrium. To this end, we will expand the dynamical variables about their equilibrium values – specifically, we take

$$\begin{aligned} \varrho &= \varrho_0+\delta\varrho, \\ \varepsilon &= \varepsilon_0+\delta\varepsilon, \\ P &= P_0+\delta p, \\ \boldsymbol{v} &= \boldsymbol{u}+\delta\boldsymbol{v}, \end{aligned} \tag{11}$$

and we will work to first order in the small quantities $\delta\varrho$, $\delta\varepsilon$, δp, and $\delta\boldsymbol{v}$. We will also separate $\delta\boldsymbol{v}$ into longitudinal and transverse parts

$$\delta\boldsymbol{v} = -\nabla\phi+\boldsymbol{\sigma}, \tag{12}$$

where the vector $\boldsymbol{\sigma}$ is divergenceless. The resulting linearized equations of motion are then

$$\delta\varepsilon = \mu_0\delta\varrho, \tag{13a}$$

$$\delta p = s^2\delta\varepsilon = s^2\mu_0\delta\varrho, \tag{13b}$$

$$\left[\Box + \frac{1}{s^2}\frac{(1-s^2)}{(1-u^2)}\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)^2\right]\delta\varrho = 0, \tag{13c}$$

$$\nabla^2\phi = \left[(1-s^2u^2)\frac{\partial}{\partial t} + (1-s^2)\boldsymbol{u}\cdot\nabla\right]\frac{\delta\varrho}{\varrho_C}, \tag{13d}$$

$$\frac{1}{1-u^2}\left[\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right](\nabla\times\boldsymbol{\sigma}) = s^2\boldsymbol{u}\times\nabla\frac{\partial}{\partial t}\frac{\delta\varrho}{\varrho_0}, \tag{13e}$$

where μ_0 is the equilibrium value of the chemical potential.

From the structure of the equations of motion (13a) through (13e), we observe that the baryon density fluctuation $\delta\varrho$ completely characterizes the excitation or sound wave. The energy density and pressure fluctuations are related to the density fluctuation $\delta\varrho$ by thermodynamic relations while $\delta\varrho$ acts as the source for the velocity field equations. Thus, we will concentrate mainly on equation (13c), which we will refer to as the wave equation for brevity. We do note, however, that the equation of motion for the transverse velocity field – equation (13e) – implies that, in general, the fluid motion will be rotational. In the nonrelativistic limit $u^2 \ll 1$ and $s^2 \ll 1$, the wave equation takes the form

$$\left[\nabla^2 - \frac{1}{s^2}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)^2\right]\delta\varrho = 0, \tag{14}$$

which is identical to what one obtains by making a Galilean transformation on the wave equation in the rest frame [18]. In the high density limit, the Walecka model predicts $s^2 \to 1$ [1], and the wave equation becomes

$$\Box\,\delta\varrho = 0 \tag{15}$$

as expected.

The Green's function for the wave equation (13c) is defined by

$$\left[\Box + \frac{1}{s^2}\frac{(1-s^2)}{(1-u^2)}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)^2\right]G(x-x') = \delta^4(x-x'), \tag{16}$$

where we have used translational invariance to write $G(x, x') = G(x-x')$. Using Fourier transforms and properties of Bessel functions [19], the Green's function can be obtained in closed analytic form in cylindrical coordinates where $\boldsymbol{u}$ defines the z-axis. It is simplest to give the analytic form for $G(\omega, \boldsymbol{r})$, where

$$G(\omega, \boldsymbol{r}) \equiv \int_{-\infty}^{\infty} dt e^{i\omega t} G(x). \tag{17}$$

There are two different regimes of motion – the subsonic case $u < s$ and the supersonic case $u > s$. In the subsonic case, the disturbance will eventually propagate to all points in space and the explicit form of the Green's function is

$$G(\omega, \boldsymbol{r}) = \frac{b}{(b^2-u^2)^{1/2}}\frac{1}{4\pi R}\exp\left\{-\frac{i\omega uz}{b^2-u^2} + i\omega R\left(1+\frac{1}{b^2-u^2}\right)^{1/2}\right\}, \tag{18}$$

where

$$\frac{1}{b^2} \equiv \frac{1}{s^2}\frac{(1-s^2)}{(1-u^2)}, \tag{19a}$$

and

$$R^2 = x^2+y^2+\frac{b^2}{b^2-u^2}z^2. \tag{19b}$$

In the supersonic case, the disturbance is limited to the interior of the forward Mach cone with opening angle θ_m

$$\sin\theta_m = \frac{s}{u}\left[\frac{1-u^2}{1-s^2}\right]^{1/2}. \tag{20}$$

Outside the forward Mach cone the Green's function is zero, and inside it is given by

$$G(\omega, \boldsymbol{r}) = \frac{b}{(u^2-b^2)^{1/2}}\frac{1}{2\pi|R|}\exp\left(\frac{i\omega uz}{u^2-b^2}\right)\cos\left\{\omega|R|\left(\frac{1}{u^2-b^2}-1\right)^{1/2}\right\}. \tag{21}$$

In the nonrelativistic limit, the above Green's functions agree with the result given by Fetter and Walecka [18]. For both the subsonic and supersonic cases, the surfaces of con-

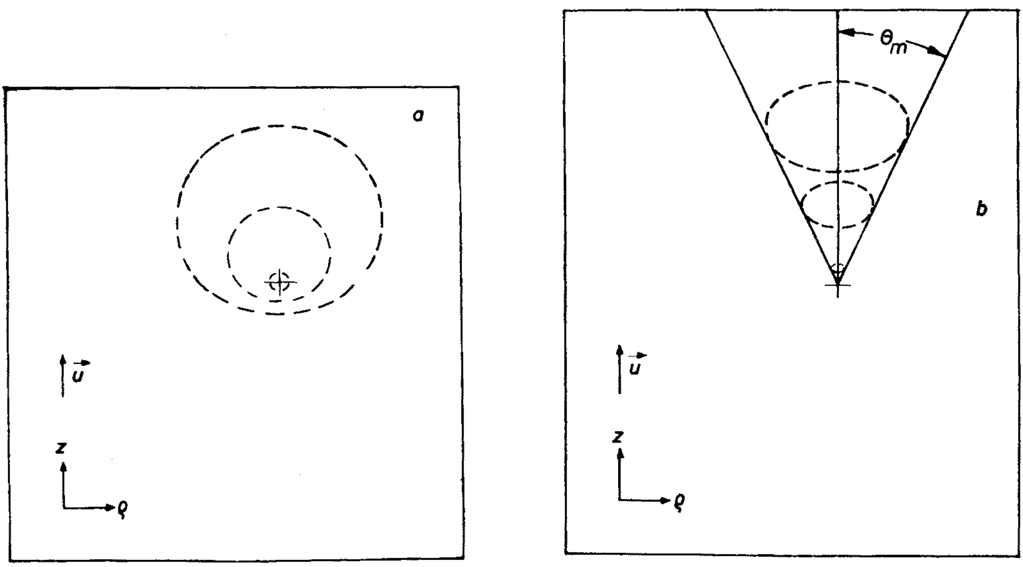

Fig. 1. Surface of constant phase of the Green's function for three successive times. (a) Subsonic case (b) Supersonic case with the Mach cone and opening angle θ_m indicated with solid lines

stant amplitude are ellipsoidal and figures 1a and 1b show how the Green's function evolves in time for two representative cases.

As an application of the above ideas, we will calculate the density fluctuation induced by a source moving with velocity $\boldsymbol{u}$ in the rest frame of the fluid, i.e. Cherenkov radiation

[20]. The relevant wave equation is

$$\left[\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right]\delta\varrho = 4\pi\varrho_0 g\delta^3(\boldsymbol{x}-\boldsymbol{u}t), \tag{22}$$

which is just the wave equation (13c) in the rest frame of the fluid with an additional source term. The constant g is a model-dependent coupling constant that we will not evaluate. As for the Green's functions, the solution can be obtained in closed analytic

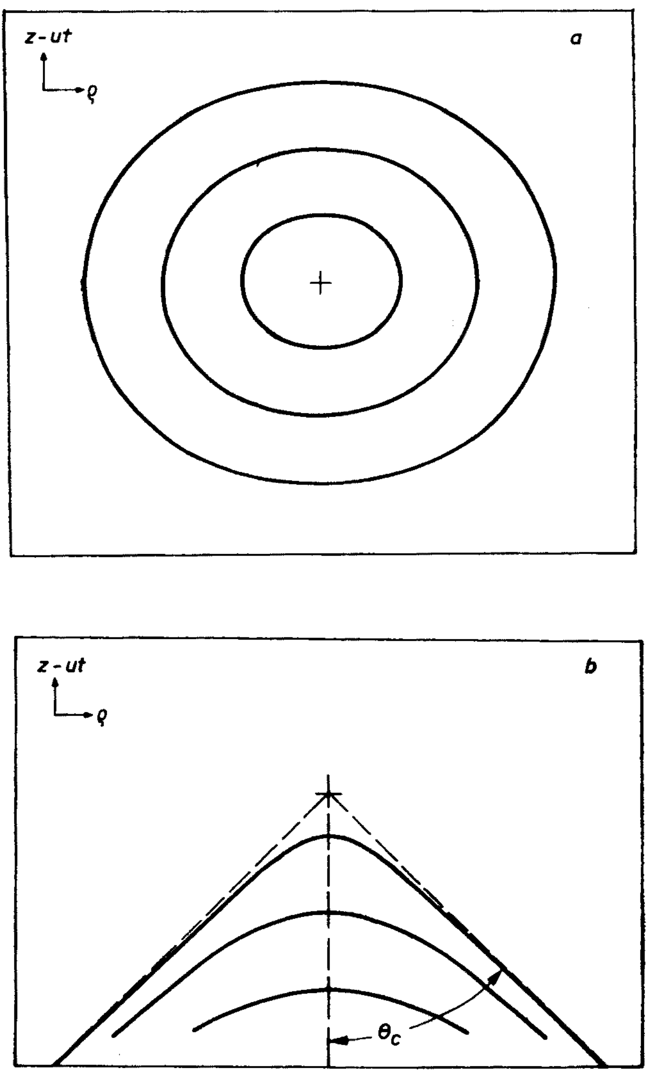

Fig. 2. Typical Cherenkov radiation patterns. Solid lines indicate surfaces of constant amplitude. The cross indicates the position of the source. (a) Subsonic case (b) Supersonic case with the Cherenkov cone and opening angle θ_C indicated with dashed lines

form with two distinct regimes of flow: subsonic and supersonic. As in the electromagnetic case, for supersonic flow the disturbance is restricted to the interior of the Cherenkov cone, with opening angle θ_C given by

$$\sin \theta_C = \frac{s}{u} . \tag{23}$$

The surface of the Cherenkov cone is a shock front for the disturbance. In figure 2, we show some representative radiation patterns.

As suggested originally by Glassgold, Heckrotte, and Watson, we can use the Cherenkov radiation process as a simple reaction mechanism for heavy-ion fission induced by bombardment with relativistic protons [10]. The difference between their work and ours is that we have a consistent relativistic microscopic theory. The model for the reaction

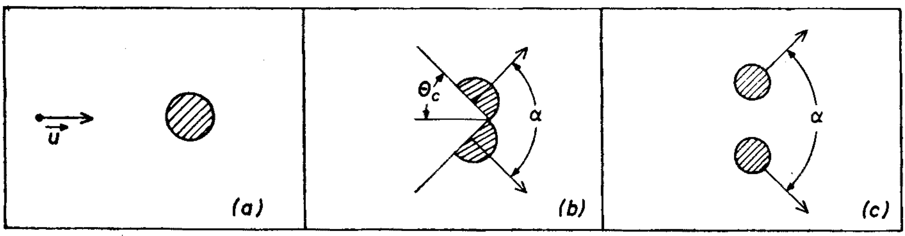

Fig. 3. Proposed reaction mechanism for heavy-ion fission induced by a relativistic projectile in an exactly central collision. (a) Before the collision (b) During the collision showing the Cherenkov shock front (c) After the collision showing the correlation angle α between the two fragments

mechanism is that, in an exactly central collision, the target nucleus fissions into two equal-mass fragments whose momenta are normal to the induced shock fronts (see figure 3). The incident proton is then the source for the Cherenkov radiation. The angle α between the momenta of the two fragments is then related to the Cherenkov angle

$$\cos \frac{\alpha}{2} = \sin \theta_C = \frac{s}{u} \tag{24}$$

where u is now the velocity of the incident proton. For 11.5 GeV protons and taking for the speed of sound the value predicted in the Walecka model, the predicted angle is $\alpha_{th} = 150.3°$. Wilkins et al. have measured α for 11.5 GeV protons on ^{238}U [21]. For equal mass fragments, the experimental distribution for α peaks at $\alpha_{exp} = 179°$. A smaller value for the speed of sound than the one predicted by the Walecka model (which is indicated from experiment) will increase the theoretical prediction for α. One would expect that the distribution for α will be smeared out due to the Fermi motion of the nucleons and from surface effects in finite nuclei [10]. A calculation of the effect of the Fermi motion is given by Glassgold et al. [10] while the effects of surface refraction are discussed in detail by Gleeson and Raha [22] and by Amsden et al. [23].

In an attempt to study the excitations of nuclear matter as formulated in the Walecka model [1], we have investigated the relativistic propagation of sound waves in a uniform, isotropic fluid moving with an arbitrary velocity. We have calculated the Green's function in closed analytic form and shown that for supersonic flow, the disturbance is limited to interior of the Mach cone. Finally, as an application of these ideas to heavy ion reactions, we have considered a simple model for central collision induced fission reactions based on a Cherenkov radiation mechanism that agrees with experiment to about 20%.

The author wishes to thank J. D. Walecka, A. L. Fetter, and B. D. Serot for many valuable discussions.

REFERENCES

[1] J. D. Walecka, *Ann. Phys.* **83**, 491 (1974).

[2] S. A. Chin, J. D. Walecka, *Phys. Lett.* **52B**, 24 (1974); S. A. Chin, *Ann. Phys.* **108**, 301 (1977).

[3] J. D. Walecka, *Phys. Lett.* **59B**, 109 (1975); R. A. Freedman, *Phys. Lett.* **71B**, 369 (1977).

[4] M. Brittan, *Phys. Lett.* **79B**, 27 (1978).

[5] A. F. Bielajew, *Nucl. Phys.* **A367**, 358 (1981).

[6] B. D. Serot, *Phys. Lett.* **86B**, 146 (1979).

[7] F. E. Serr, J. D. Walecka, *Phys. Lett.* **79B**, 10 (1978); B. D. Serot, J. D. Walecka, *Phys. Lett.* **87B**, 172 (1979).

[8] J. D. Walecka, *Phys. Lett.* **94B**, 293 (1980); C. J. Horowitz, J. D. Walecka, *Nucl. Phys.* **A364**, 429 (1981).

[9] C. J. Horowitz, B. D. Serot, *Nucl. Phys.* **A368**, 503 (1981).

[10] A. E. Glassgold, W. Heckrotte, K. M. Watson, *Ann. Phys.* **6**, 1 (1959).

[11] J. D. Bjorken, S. D. Drell, *Relativistic Quantum Fields*, McGraw-Hill, New York 1965.

[12] L. D. Landau, E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, London 1959.

[13] C. R. McKee, S. A. Colgate, *Ap. J.* **181**, 903 (1973).

[14] J. D. Walecka, *Rice University Studies* **66**, No. 3, 217 (1980).

[15] I. Ia. Pomeranchuk, *Zh. Eksp. Teor. Fiz.* **20**, 919 (1950); L. D. Landau, *Sov. Phys. — JETP* **3**, 920 (1957); **5**, 101 (1957).

[16] T. Matsui, *Nucl. Phys.* **A370**, 365 (1981).

[17] D. H. Youngblood, *The Giant Monopole Resonance — An Experimental Review*, Proc. Giant Multipole Resonance Topical Conf., Oak Ridge, Oct. 1979; J. P. Blaizot, *The Breathing Mode and Nuclear Matter*, Proc. Giant Multipole Resonance Topical Conf., Oak Ridge, Oct. 1979.

[18] A. L. Fetter, J. D. Walecka, *Theoretical Mechanics of Particles and Continua*, McGraw-Hill, New York 1980.

[19] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 2nd ed., London 1966.

[20] J. D. Jackson, *Classical Electrodynamics*, John Wiley and Sons, 2nd ed., New York 1975.

[21] B. D. Wilkins, S. B. Kaufman, E. P. Steinberg, J. A. Urbon, D. J. Henderson, *Phys. Rev. Lett.* **43**, 1080 (1979).

[22] A. M. Gleason, S. Raha, *Phys. Rev.* **C21**, 1065 (1980).

[23] A. A. Amsden, F. H. Harlow, J. R. Nix, *Phys. Rev.* **C15**, 2059 (1977).