THE ELEMENTARY CHARGE AND THE CHARGE OF THE LONGITUDINAL ELECTROMAGNETIC FIELD ARE COMMENSURATE

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(*Received May 4, 1982*)

It is argued that the electron charge e and the charge e' of the longitudinal electromagnetic field are commensurate. The argument is based on the same elementary ideas which lead to the Dirac relation between the charge e and the magnetic charge g.

PACS numbers: 03.50.De, 03.50.Kk

1. *Electrodynamics of a gradient current*

Consider the Maxwell equations

$$\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu} = 0,$$

$$\partial^\mu F_{\mu\nu} = 4\pi j_\nu.$$

These equations allow to determine the field $F_{\mu\nu}$ if the current j_ν is known i.e. if it is given as a four-vector function of space-time points. This is the case of an external current and it is obviously an approximation. A more general scheme is to give the current as a function of certain dynamical variables whose equations of motion are chosen so that the whole system is dynamically closed. Classical electrodynamics of point particles is an example of more general scheme; a bad example because the complete system of equations is self-contradictory.

Now, the dynamical variables which the current depends on and their equations of motion cannot be arbitrarily given, they have to be consistent with the charge conservation law

$$\partial^\mu j_\mu = 0.$$

A similar situation appears in the general theory of relativity, where conservation laws of energy and momentum follow from the Einstein equations. It is also known from the

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general theory of relativity, that there are dynamical systems so simple that conservation laws of energy and momentum determine their motion completely. Let us put forward the following question: is it possible to have a system, whose motion is determined completely by the charge conservation law alone? Is it possible to have a pure charge not attached to a nonelectromagnetic piece of matter?

An affirmative answer to this question is provided by the following consideration. Assume that the electric current is a gradient of a scalar field F:

$$4\pi j_\nu = -\partial_\nu F.$$

It follows then from the charge conservation law that

$$\Box F = 0,$$

which is the equation of motion for F.

We come thus to the conclusion that the Maxwell electrodynamics of a gradient current is a closed dynamical system. The conclusion is further supported by the following circumstances:

(a) there exists a positive definite, conserved energy momentum tensor;
(b) there exists an action principle.

(a) The positive definite, conserved energy-momentum tensor was derived in [1]; it seems that Strazhev [2] was the first to discover this remarkable tensor. It has the form

$$T_\lambda{}^\mu = \frac{1}{4\pi}\left(\tfrac{1}{4} F^{\alpha\beta}F_{\alpha\beta}\delta^\mu_\lambda - F_{\lambda\nu}F^{\mu\nu} + \tfrac{1}{2} F^2\delta^\mu_\lambda - FF_\lambda{}^\mu\right).$$

(b) The action has the form

$$S = -\frac{1}{16\pi}\int d^4x\left(F_{\mu\nu}F^{\mu\nu} + \frac{2}{\varepsilon}F^2\right)$$

where ε is a dimensionless constant,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad F = \varepsilon\partial^\mu A_\mu$$

and A_μ is the dynamical coordinate to be varied. For $\varepsilon = 1$ S is the action first considered by Dirac, Fock and Podolsky [3]; Heisenberg and Pauli [4] considered arbitrary ε with intention of taking the limit $\varepsilon \to 0$.

The classical authors considered the additional term in the action as basically undesirable modification introduced ad hoc to make the theory canonically quantizable. We, on the other hand, interprete the modified action as the total action for the closed dynamical system consisting of the electromagnetic field $F_{\mu\nu}$ and its source $j_\nu = -(1/4\pi)\partial_\nu F$. This interpretation, which resembles that of Zwanziger [5], makes intelligible shrinking of the gauge group i.e. the very thing objectionable to most authors; there is less freedom in choosing the potential A_μ because the same dynamical coordinates are used to describe the electromagnetic field and its source.

2. *A comment on the Dirac relation between electric and magnetic charges*

The Dirac relation between electric and magnetic charges is commonly believed to be the only known principle of charge quantization [6]. There are numerous derivations of this relation, with varying degrees of sophistication, but all are ultimately based on two elementary ideas. The first idea is that angular momentum parallel to a distinguished axis should be a multiple of 1/2 (Dirac) or 1 (Schwinger). The second idea is that if there is to be a discontinuity of phase of a wave function, it should be a multiple of 2π.

In Section 4 we present arguments to the effect that the total charge e' of a gradient current must be a multiple of the charge e of a point particle. We realize that our arguments are not convincing. For this reason we wish to stress in advance that our arguments, albeit imperfect, are based solely on the two elementary ideas mentioned above.

3. *Two lemmas on solutions of the wave equation*

Consider the integral

$$\int d^4x f(x) g(x),$$

where f and g are solutions of the wave equation:

$$\Box f = 0 = \Box g.$$

In general this integral does not exist; below we describe a situation in which this integral is perfectly well determined.

Suppose that Cauchy data for f are given on the hyperplane Σ_f while Cauchy data for g are given on the hyperplane Σ_g (Fig. 1); suppose further that the Cauchy data both

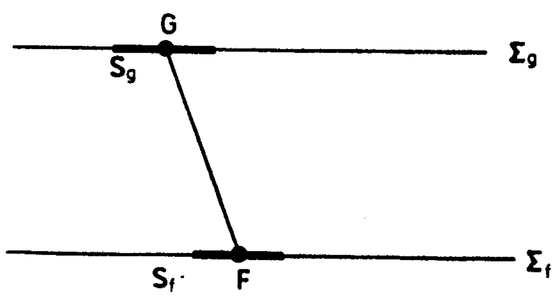

Fig. 1

for f and for g have compact supports S_f and S_g respectively. I say that solutions f and g are well separated, when all segments FG, where F is in S_f and G is in S_g, are timelike.

Lemma 1. If two solutions of the wave equation, f and g, are well separated, then

$$\int d^4x f(x) g(x) = -\frac{1}{8\pi} \int d^3x \frac{\partial f}{\partial x^0} \cdot \int d^3x \frac{\partial g}{\partial x^0}.$$

Proof. We put on the left hand side

$$f(x) = \int_{S_f} dS^\mu \left\{ D(x-y) \frac{\partial f(y)}{\partial y^\mu} - f(y) \frac{\partial D(x-y)}{\partial y^\mu} \right\},$$

$$g(x) = \int_{S_g} dS^\mu \left\{ D(x-z) \frac{\partial g(z)}{\partial z^\mu} - g(z) \frac{\partial D(x-z)}{\partial z^\mu} \right\},$$

where D is the Pauli-Jordan function. If f and g are well separated, all segments $y-z$ are timelike and we can integrate over x first. Using the formula

$$\int d^4x D(x-y) D(x-z) = -\frac{1}{8\pi},$$

valid for timelike $y-z$, one obtains the desired result.

Lemma 2. If a real solution $f(x)$ of the wave equation has Cauchy data of compact support, then its Fourier transform

$$f(k) \stackrel{\mathrm{df}}{=} \frac{i}{8\pi^2} \int_{S_f} dS^\mu \{ e^{ikx} \partial_\mu f(x) - f(x) \partial_\mu e^{ikx} \},$$

k being the null, future oriented wave vector, has the zero frequency limit

$$\lim_{k^0 \to 0} f(k) = \frac{i}{8\pi^2} \int d^3x \frac{\partial f}{\partial x^0} = \mathrm{const.}$$

Proof is given in the Appendix in Ref. [7].

4. *Commensurability of e and e'*

The positive definite, conserved energy momentum tensor is not symmetric, which means that the angular momentum and the centre of mass motion are not conserved:

$$\partial_\mu (x_\varrho T_\lambda{}^\mu - x_\lambda T_\varrho{}^\mu) = \frac{1}{2\pi} F F_{\varrho\lambda}.$$

The change of angular momentum and centre of mass motion between plus and minus infinity is

$$\Delta M_{\varrho\lambda} = \frac{1}{2\pi} \int d^4x F F_{\varrho\lambda}.$$

If the fields F and $F_{\varrho\lambda}$ are well separated, we have from Lemma 1

$$\Delta M_{\varrho\lambda} = -\frac{1}{16\pi^2} \int d^3x \frac{\partial F}{\partial x^0} \cdot \int d^3x \frac{\partial F_{\varrho\lambda}}{\partial x^0} = \frac{1}{4\pi} e' \int d^3x \frac{\partial F_{\varrho\lambda}}{\partial x^0},$$

where e' is the total charge.

Now, only two components of $F_{\varrho\lambda}$, say F_{12} and F_{03}, can be well separated with F. This can be seen as follows.

Assuming that the potential A_μ fulfils the wave equation (the Dirac-Fock-Podolsky case) we write

$$A_\mu(x) = \frac{1}{2\pi}\int \frac{d^3k}{k^0}\, a_\mu(k)e^{-ikx} + \text{c.c.}.$$

Calculating F and $F_{\varrho\lambda}$ and applying Lemma 2, we have

$$\lim_{k^0\to 0} k^\mu a_\mu(k) = \text{const},$$

$$\lim_{k^0\to 0} [k_\lambda a_\varrho(k) - k_\varrho a_\lambda(k)] = \text{const}.$$

It is algebraically impossible to fulfil these conditions for all pairs $(\varrho\lambda)$; we can, however, assume that

$$\lim_{k^0\to 0} k^\mu a_\mu(k) = \frac{e'}{2\pi} = \text{const},$$

$$\lim_{k^0\to 0} [k_2 a_1(k) - k_1 a_2(k)] \stackrel{\text{df}}{=} \frac{e_{21}}{2\pi} = \text{const},$$

$$\lim_{k^0\to 0} [k_3 a_0(k) - k_0 a_3(k)] \stackrel{\text{df}}{=} \frac{e_{30}}{2\pi} = \text{const};$$

then the zero frequency limit of $a_\mu(k)$ is determined up to a gauge transformation. In other words, the change of $M_{\varrho\lambda}$ is well determined only for M_{12} and M_{03}:

$$\Delta M_{12} = -e'e_{12},$$

$$\Delta M_{03} = -e'e_{03};$$

note a striking analogy with quantum mechanics which shows up here in a purely classical context.

Now, since angular momentum parallel to a distinguished axis should be a multiple of 1/2 (Dirac) or 1 (Schwinger), we postulate the charge quantization condition

$$e'e_{12} = n, \qquad n = 0, 1, \ldots.$$

Can one say something about the magnitude of e_{12}? Without additional assumptions obviously nothing can be said: since

$$e_{12} = -\frac{1}{4\pi}\int d^3x\, \frac{\partial F_{12}}{\partial x^0}$$

and $\partial_0 F_{12}$ is a part of Cauchy data, the integral e_{12} can be an arbitrary real number.

Let us calculate the field $A_\mu(x)$ with the zero frequency limit determined by the conditions

$$e' = 0, \quad e_{03} = 0, \quad e_{12} \neq 0.$$

In the Coulomb gauge ($a_0(k) = 0$) the zero frequency limit is easily found to be

$$a_0(k) = a_3(k) = 0,$$

$$a_1(k) = \frac{e_{12}}{2\pi} \frac{k^2}{(k^1)^2+(k^2)^2},$$

$$a_2(k) = \frac{e_{12}}{2\pi} \frac{-k^1}{(k^1)^2+(k^2)^2}.$$

Since the integral e_{12} is determined by the zero frequency limit of the Fourier transform, we assume that the above expressions are valid for all frequencies. Thus we have

$$A_0(x) = A_3(x) = 0,$$

$$\begin{Bmatrix} A_1(x) \\ A_2(x) \end{Bmatrix} = \frac{e_{12}}{2\pi^2} \int \frac{d^3k}{k^0} \cos kx \frac{1}{(k^1)^2+(k^2)^2} \begin{Bmatrix} -k^2 \\ k^1 \end{Bmatrix}$$

$$= e_{12} \operatorname{sign}(x^0)\theta[(x^0)^2-(x^3)^2]\theta(-xx) \frac{1}{(x^1)^2+(x^2)^2} \begin{Bmatrix} -x^2 \\ x^1 \end{Bmatrix}$$

or

$$A_\mu(x)dx^\mu = e_{12} \operatorname{sign}(x^0)\theta[(x^0)^2-(x^3)^2]\theta(-xx)d\varphi,$$

where $\varphi = \operatorname{arctg}(x^2/x^1)$ and θ is the unit step function. We see that in the domain in which $A_\mu \neq 0$, $A_\mu dx^\mu$ is a perfect differential.

Let us put forward the following condition: it should be possible to transform away the potential $A_\mu(x)$ by means of the gauge transformation

$$\psi \to \psi' = e^{iee_{12}\varphi}\psi,$$

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu e_{12}\varphi.$$

The same condition is put forward when one derives the Dirac relation from the postulated absence of the Aharonov-Bohm effect [8]. The gauge transformation above will preserve continuity of the wave function ψ if

$$ee_{12} = n, \quad n = 0, 1, \ldots .$$

Thus we have two conditions

$$e'e_{12} = n,$$

$$ee_{12} = n,$$

which imply that the charges e and e' are commensurate. In particular, if the charge e is elementary, then

$$e' = ne,$$

n being an integer.

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