THE PERTURBED SINE-GORDON EQUATION II: ANALYTIC BUILD-UP OF THE ONE-KINK SECTOR FOR THE MULTIPLE SINE-GORDON EQUATIONS

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We show that for a class of the multiple sine-Gordon equations, the form of the low lying excitations of a kink coincides with the recently introduced matrix generalization for the Gauss hypergeometric function.

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1. Introduction

The multiple sine-Gordon (MSG) equations

$$-\partial_t^2\varphi(x,t)+\partial_x^2\varphi(x,t) = \sum_{k=1}^{p} g_k \sin k\varphi(x,t) \tag{1.1}$$

were introduced in the nonlinear optics context [1]. We may also contemplate them as a class of the field-theoretical models with the attractive features of:
— Lorentz covariance;
— coincidence with the standard Klein-Gordon equation in the small-amplitude limit;
— universality of the large-amplitude Fourier-type nonlinearity $\Phi(\varphi)$;
— periodic vacuum degeneracy implying the existence of the kink (soliton-like) solutions;
— various physical pretensions, e.g., to describe the phenomena of the type of the pion condensate.

The best known example is the famous sine-Gordon (SG) one ($p = 1$) solvable in terms of the elementary functions. Unfortunately, it is rather exceptional in this respect — the overall properties of the next ($p > 1$) members of the MSG family (e.g., the presence of the annihilation component in the double SG (DSG) breather etc.) are known mostly from the numerically oriented studies. Our present purpose is to show that, rather sur-

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prisingly, an important class of exact solutions may be obtained by the purely analytic means in the vicinity of the one-kink state for an arbitrary multiplicity index p.

The material is organized as follows. In Section 2 we summarize some elementary properties of the kink (soliton-like) solutions and give their explicit form for a broad class of the MSG equations. In Section 3 we consider the lowest excitations of these kinks and show that the linearized form of the considered MSG equations coincides with the Gauss hypergeometric equation when generalized in the sense of Ref. [2]. In Section 4 we discuss some examples in more detail and emphasize the various physical aspects of the MSG dynamical picture, especially the SG-like interpretation of some limiting cases, the A-body binding role of the sin $A\varphi$ term in Eq. (1.1) and also the possible mass-spectrum generation mechanism connected with the nonlinearities of the strongly non-SG type.

2. *Kink solutions and their vicinity*

In the Klein-Gordon Lagrangian of the nonlinear type, the mass term $\frac{1}{2}\varphi^2$ is in general being replaced by a function $\frac{1}{2}f^2(\varphi) = \frac{1}{2}\varphi^2 + O(\varphi^3)$ with the minima $\frac{1}{2}f^2(\varphi) = C$ at the constant vacuum values $\varphi = \varphi^{(\mathrm{vac})}$. Although the corresponding differential equation

$$-\partial_t^2\varphi + \partial_x^2\varphi = f(\varphi)f'(\varphi), \quad f'(\varphi) = \partial_\varphi f(\varphi) \tag{2.1}$$

is of course independent of C, cerain technical complications appear in the $C \neq 0$ cases (an important DSG example is discussed in Ref. [3]). We shall therefore consider the $C = 0$ cases only.

The one-kink solution may be considered time-independent in the appropriate Lorentz frame, $\varphi(x, t) = y(x)$. By definition, it connects the neighbouring vacua — let us put

$$\lim_{x\to-\infty} y(x) = \varphi_0^{(\mathrm{vac})} = 0 < \lim_{x\to+\infty} y(x) = \varphi_1^{(\mathrm{vac})}. \tag{2.2}$$

In accord with these boundary conditions, Eq. (2.1) multiplied by the integration factor $\partial_x y$ gives an identity

$$\partial_x y(x) = f(y(x)). \tag{2.3}$$

Hence, the shape (energy distribution) and the finite complete energy

$$m = \int_{-\infty}^{\infty} dx(\tfrac{1}{2}(\partial_x y)^2 + \tfrac{1}{2}f^2(y)) = \int_{y(-\infty)}^{y(+\infty)} dy f(y) \tag{2.4}$$

of a kink are determined by the appropriate choice of $f(y)$.

The bell-shaped f's (cf. the SG choice of $f(\varphi) = 2\sin\frac{1}{2}\varphi$) enable us to interprete kinks as the particle-type (localised) states. Their number should not be restricted — hence, we shall use the periodic f's, fixing their periodicity, $\varphi \equiv \varphi + \pi$, by an appropriate transformation $\varphi \to \mu\varphi$, $t^2 \to \mu t^2$, $x^2 \to \mu x^2$. For simplicity, we put $\pi = \varphi_0{}^{(\mathrm{vac})} + \pi = \varphi_M^{(\mathrm{vac})}$, $M \geqslant 1$, and factor out the corresponding zero of f^2 in the form of ansatz

$$f(\varphi) = \sin\varphi Q(\cos\varphi), \tag{2.5}$$

where $Q(\pm 1) \neq 0$ (otherwise, the field near $\varphi_0^{(vac)}$ would be massless). Then, the second integration of Eq. (2.1) gives the implicit definition of the function $y(x)$,

$$T(y(x)) = e^{\gamma x},$$

$$T(u) = \exp \gamma \int_{u_0}^{u} \frac{dt}{f(t)} = \exp -\gamma \int_{\cos u_0}^{\cos u} \frac{dx}{(1-x^2)Q(x)}, \tag{2.6}$$

where $\cos u_0$ is an appropriate integration constant corresponding to the choice of $x = x_0$.

Obviously, for any polynomial $Q(z)$, the function $T(u)$ may be represented by the elementary functions — the complete discussion of the corresponding integration (partial--fraction technique) may be found in any standard text-book [4]. Between any pair of the neighbouring vacua, the inversion of T also exists and defines $y(x)$ in a unique way. Moreover, the polynomial choice of Q (of degree q) leads always to the MSG nonlinearity in Eq. (2.1). Since $p = 2q+2$ in this case, some functional shapes of $\Phi(\varphi)$ are lost corresponding to the more complicated forms of $Q(z)$. For example, the DSG shape of $\Phi(\varphi)$ gives the square-root form of Q — this will not be considered here.

For the excitations of a kink, we get the MSG equation in the form

$$-\partial_t^2 z(x, t) + \partial_x^2 z(x, t) = \Phi(y+z) - \Phi(y),$$

$$\Phi(y+z) - \Phi(y) = 2 \sum_{k=1}^{s} g_k \sin \frac{k}{2} z \cos k\left(y + \frac{z}{2}\right),$$

$$z(x, t) = \varphi(x, t) - y(x). \tag{2.7}$$

In the linear approximation, i.e., in the vicinity of a kink, it simplifies considerably,

$$-\frac{d^2}{dx^2} \chi_k(x) + W(x)\chi_k(x) = k^2 \chi_k(x),$$

$$W(x) = f(y)f''(y) + f'(y) = \sum_{n=1}^{s} n g_n \cos ny(x),$$

$$z(x, t) = \sum_{j=1}^{j_{max}} \chi_{k_j}(x) e^{ik_j t} + \int_{k_0}^{\infty} dk \chi_k(x) e^{ikt}. \tag{2.8}$$

From Eq. (2.6), the shape of the potential $W(x)$ is known in principle. Our intention is to specify the analytic form of the solutions $\chi_k(x)$ as well.

3. *Exact solvability of the linearized MSG equations*

At zero energy, the explicit form of $\chi_0(x) \equiv f(y(x))$ is known. By its definition, it has no nodes on the real axis so that all the energies k^2 in Eq. (2.8) must indeed be positive (nonnegative). This implies that the one-kink state is stable. Reverting the argument, the appearance of an additional zero $\varphi_1^{(\mathrm{vac})} \in (0, \pi)$ in $Q(z)$ would force the halves of the nonstable configuration $y(x) = \varphi(x, t_0)$ to move apart.

Concerning the non-zero energies, the analytic form of the complete set of solutions $\chi_k(x)$ may be useful, e.g., in solving the class of the MSG initial value problems, etc. In the SG or DSG cases [3], $\chi_k(x)$ coincide respectively with the Gauss form, or the matrix generalization, of the hypergeometric functions [2]. In the present context, the DSG construction will be extended to any multiplicity p.

We start by specifying the poles of the polynomial $Q(z)$,

$$Q(z) = \prod_{i=1}^{s_1} (1-\alpha_i^2 z^2) \prod_{j=1}^{s_2} (1+\beta_j^2 z^2), \qquad s_1+s_2 = s. \tag{3.1}$$

(Of course, the more general (zero mass ($\alpha_i^2 = \alpha_j^2$ or $\beta_i^2 = \beta_j^2$), or parity-violating ($Q(-z) \neq Q(z)$) cases could be considered along the same lines, although they seem to be physically less interesting.) As a consequence, we get

$$T(u) = \exp\left[-\operatorname{Arth} \cos u - Z(\cos u)\right], \tag{3.2}$$

where $\gamma = Q(1)$ and

$$Z(\cos u) = -\sum_{i=1}^{s_1} \alpha_i^{2s-1} \prod_{\substack{j=1\\ j\neq i}}^{s_1} \frac{1-\alpha_j^2}{\alpha_i^2-\alpha_j^2} \prod_{k=1}^{s_2} \frac{1+\beta_k^2}{\alpha_i^2+\beta_k^2} \operatorname{Arth} \alpha_i \cos u$$

$$+\sum_{i=1}^{s_2} \beta_i^{2s-1} \prod_{j=1}^{s_1} \frac{1-\alpha_j^2}{\beta_i^2+\alpha_j^2} \prod_{\substack{k=1\\ k\neq i}}^{s_2} \frac{1+\beta_k^2}{\beta_i^2-\beta_k^2} \operatorname{arctg} \beta_i \cos u. \tag{3.3}$$

Considering further the SG-type limit $\alpha_i^2 \to 0$, $\beta_j^2 \to 0$ as a guide, we get the two alternative exact formulas

$$\sin y(x) = 1/\operatorname{ch} \xi, \qquad \cos y(x) = -\operatorname{th} \xi,$$

$$\xi = \xi(x) = \gamma x + Z(\cos y(x)) \tag{3.4}$$

for the kink shape, with the iterative interpretation of the "distorted coordinate" ξ. In the non-SG-type regions, we may simply interprete Eqs (3.4) and

$$x = \frac{1}{\gamma}\left(\xi - Z(-\operatorname{th} \xi)\right) \tag{3.5}$$

as the ξ-parametric definitions of the dependent and independent variables y and x, respectively. The main idea of our paper is to transfer this parametric formulation to the linearized MSG Eq. (2.8) as well.

In the first step, we introduce the set of functions $\langle \xi | X(p, q) \rangle$,

$$\langle \xi | X \rangle = \operatorname{th}^p \xi / \operatorname{ch}^r \xi, \tag{3.6}$$

of the variable ξ defined by Eq. (3.5). Obviously, the kinetic energy part of the Hamiltonian $H = -d^2/dx^2 + W(x)$ may be transformed to the operator

$$-\frac{d^2}{dx^2} = -Q^2(-\text{th}\,\xi)\frac{d^2}{d\xi^2} + \frac{1}{\text{ch}^2\xi}Q(-\text{th}\,\xi)Q'(-\text{th}\,\xi)\frac{d}{d\xi}. \tag{3.7}$$

Its action on the functions $\langle\xi|X\rangle = X(p, r)$ of Eq. (3.6)

$$\frac{d^2}{dx^2}X(p, r) = Q^2(-\text{th}\,\xi)\,[p(p-1)X(p-2, r+4)-(2pr+2p+r) \times X(p, r+2)+r^2X(p+2, r)]-Q(-\text{th}\,\xi)Q'(-th\,\xi) \times [pX(p-1, r+4)-rX(p+1, r+2)]$$

does not change their form provided that we fix $p = 0$ or $p = 1$, and enhances merely the value of the exponent r since $\text{th}^2\xi = 1-\text{ch}^{-2}\xi$. The multiplication by the polynomial Q or its derivative Q' contributes in the same direction.

In the second step, we conclude that the differential Eq. (2.8) has the form of the generalized hypergeometric equation $H\psi = E\psi$ as described in Ref. [2], since also the potential term

$$W = \left(1-\frac{2}{\text{ch}^2\,\xi}\right)Q^2+5\frac{\text{th}\,\xi}{\text{ch}^2\,\xi}QQ'+\frac{1}{\text{ch}^4\,\xi}(QQ''+Q'^2) \tag{3.8}$$

acts on $X(p, r)$ in the same way. We may therefore introduce the generalized Gauss series

$$\langle\xi|F\rangle = \sum_{n=1}^{\infty}\sum_{j=1}^{M_n}\langle\xi|X_n^j\rangle D_n^j,$$

$$D_n^i = -\sum_{m=1}^{M_n}\sum_{j=1}^{M_{n-1}} D_{n-1}^j B_{n-1}^{jm}(A_n^{-1})^{mi}, \quad i = 1, 2, \ldots, M_n, \quad n = 2, 3, \ldots \tag{3.9}$$

where the $M_n \times M_n$ matrices A_n and $M_n \times M_{n+1}$ matrices B_n are defined by the operator identity

$$H|X_n^m\rangle = \sum_{j=1}^{M_n}|X_n^j\rangle A_n^{mj} + \sum_{j=1}^{M_{n+1}}|X_{n+1}^j\rangle B_n^{mj},$$

$$m = 1, 2, \ldots, M_n, \quad n = 1, 2, \ldots \tag{3.10}$$

In principle, we would get various forms of Eq. (3.9) when starting from the various initial choices $|X_1^m\rangle$, $m = 1, 2, \ldots, M_1$ of the functions X in the recurrent Lanczos-type formula (3.10) specifying the values of M_n, A_n, B_n and $|X_{n+1}\rangle$ for $n = 1, 2, \ldots$ For each of these choices, the increase of the number of exponents r is at most $N = 2q+2$ in each step. For Q defined as an even polynomial, we get $N = q+1$ since the parity of the indices p and r is preserved in this case.

In the third step, we may show that the indexation

$$\langle\xi|X_n^m\rangle = \text{th}^p\,\xi/\text{ch}^{N(n-2)+m+\kappa}\xi,$$

$$p = 0, 1, \ldots, \quad N = M_n, \quad n = 1, 2, \ldots \tag{3.11}$$

with an arbitrary parameter κ is particularly suitable since the partition dimensions $M_n = N$ become minimized and constant. Furthermore, the series (3.9) appears to be convergent, at least in some vicinity of $\xi \to -\infty$, since the powers of $1/\text{ch}\,\xi$ decrease quickly with the increasing exponent.

The rigorous mathematical discussion of the analytic continuation of Eq. (3.9) depends on the singularity structure of the polynomials Q. Recalling Ref. [2] as a methodical guide to the general cases, we shall stress here only the following two most important points.

Remark 1. In accord with the known asymptotic behaviour of the potential $W(x) = Q^2(\mp 1)(1+O(1/\text{ch}\,\xi))$, $\xi \to \pm\infty$, the correct physical asymptotic behaviour of the series (3.9)

$$\langle \xi | F \rangle \sim \langle \xi | X_1 \rangle \sim e^{\pm \kappa \xi} \tag{3.12}$$

is specified by the value of the parameter κ. Depending on the value of energy $E = k^2$, it may be either real or purely imaginary. For example, in the asymptotic region $\xi \to -\infty$ we get the physical consistency requirement

$$\kappa = \pm (Q^2(1) - E)^{1/2}. \tag{3.13}$$

In the cases without singularities and $\xi \in (-\infty, \infty)$, the direct extension of the DSG case analysis as performed in Ref. [2] is possible so that the analytic continuation becomes equivalent to the matching of the logarithmic derivatives at $\xi = 0$.

Remark 2. The detailed comparison of Eqs (3.9) and (3.10) shows that $\langle \xi | F \rangle$ satisfies in general only the nonhomogeneous equation

$$H\langle \xi | F \rangle - E\langle \xi | F \rangle = - \sum_{i,j=1}^{M_1} \langle \xi | X_1^j \rangle A_1^{ij} D_1^i. \tag{3.14}$$

Nevertheless, in full analogy with Ref. [2], we find that $A_1^{NN} = 0$ as a consequence of our choice of κ. Hence, the normalization $D_1^n \sim \delta_{nN}$ gives indeed the correct zero right-hand side in Eq. (3.14). In fact, Eq. (3.13) is given by the requirement $D_1 \neq 0$, $A_1 D_1 = 0$ rather than by the above argument. As a consequence, the generalized Gauss series Eq. (3.9) represents the general solution of the linearized MSG Eq. (2.8) which may be used with any type of the physical boundary conditions imposed on the general superposition of the $+\kappa$ and $-\kappa$ components $\langle \xi | F \rangle$.

4. *Example — triple sine-Gordon equations*

A. Perturbed sine-Gordon equations

The small-amplitude perturbations of the trivial vacuum $\varphi^{(\text{vac})} = \text{const.}$ are described by the ordinary Klein-Gordon equation. Physically, they may be interpreted as free "pions". This interpretation ceases to be consistent when we switch on a sufficiently strong external field. To stop an unlimited colaps — pionic condensation — we should use the nonlinearity, e.g., of the MSG type. The resulting stable and localized condensate could then be described by a kink solution $y(x)$ of the preceding sections, with the remaining free pions $z(x, t)$ interpreted as moving over or in the "natural relativistic" shallow potential well $W(x)$ of Eq. (2.8).

The "experimental" fit of the pion-condensate scattering of the free pions $z(x, t)$ may be achieved by the phenomenological choice of the appropriate MSG forces. Using one free parameter, we get the triple sine-Gordon (TSG) nonlinearities

$$f(u) = a \sin u + b \sin 3u, \quad Q(z) = d + 4bz^2, \quad d = a - b,$$

$$\Phi(\varphi) = \left(\frac{a^2}{2} - ab\right) \sin \varphi + 2ab \sin 2\varphi + \frac{3b^2}{2} \sin 3\varphi, \quad \varphi = 2u \tag{4.1}$$

which generalize the pure SG case $b = 0$. Its vicinity (with any fixed $b > 0$ and variable difference d) may be characterized either as a domain D1($8b \leqslant d$) or D5($d \leqslant -4b$) — the pionic functions $\chi_k(x)$ become modified but the shape of $y(x)$ is only insignificantly deformed. In accord with the formulas of Section 3 we get the function $Z(\cos y)$ as a small $O(4b/d)$ perturbation with either $s_1 = 0$, $s_2 = 1$ and $\beta_1^2 = \beta^2 = 4b/d < \frac{1}{2}$ or $s_1 = 1$, $s_2 = 0$ and $\alpha_1^2 = 4b/|d| < 1$ in D1 or D5, respectively, so that ξ is more or less proportional to the original SG coordinate x.

B. The deuteron-type configurations

In SG case, an interaction of the kink pairs is repulsive — we cannot obtain any bound particle-particle system. In the TSG model, such a system represented by the kink localized at the two separate points with the maximal energy density $f^2(y(x_0))$

$$x_0 = \pm \frac{1}{\gamma} (\beta \operatorname{arctg} \beta\delta + \operatorname{Arth} \delta), \quad \delta = \frac{8b - d}{12b} \tag{4.2}$$

exists in the domain of parameters D2($0 < d < 8b$). Obviously, the two centers move apart as $\beta^2 \to \infty$ — we are permitted to fix phenomenologically the strength of binding or the magnitude $2x_0$ of the two-body bound system $y(x)$. Its low-lying excitations are to be described by the correction term $z(x, t)$ and Eq. (2.8).

Similar picture was obtained in Ref. [3] for DSG system which decays in the limit $\beta \to \infty$ into two exact SG solitons. Here, the decay is not symmetric — the peculiar TSG space asymmetry characterizes also the TSG solutions in the limiting domain D3 ($d = 0$). As well as in the DSG case, the transition $d \to 0$ is singular since the range of the auxiliary parameter

$$\xi = \gamma x - 1/\cos y(x) \tag{4.3}$$

becomes restricted to the half-interval $\xi = \xi_1 \in (-\infty, 0)$ while x moves along the whole real axis.

An interesting phenomenon is also the $x \leftrightarrow -x$ asymmetry of the corresponding pionic potential $W_1(x)$: The lowest excitations $z(x, t)$ are allowed to radiate in one direction only since $W_1(-\infty) = 16b^2 > 0$ while $W_1(+\infty) = 0$.

The reversed situation corresponds to the second, nonequivalent kink $y_{II}(x)$ parametrized by $\xi_{II} \in (0, \infty)$. The possible physical interpretation of such exceptional oriented states with $d = 0$, $M = 2$ and $\varphi_1^{(vac)} = \pi/2$ is not clear at present because of the character of the "domain" D3 (= one point only).

C. Model of the dynamically generated mass spectrum

In the remaining domain D4$(-4b < d < 0)$ of the TSG parameters, we get $M = 3$ with the two additional vacua $\varphi_1^{(vac)} = \frac{\pi}{2} - \varepsilon = V_-$ and $\varphi_2^{(vac)} = \frac{\pi}{2} + \varepsilon = V_+$ where $\cos V_\pm = \mp(-d/4b)^{1/2}$. Formally, we get three possible ranges of the parameter ξ, $\xi_I \in (-\infty, \xi_-)$, $\xi_{II} \in (\xi_-, \xi_+)$ and $\xi_{III} \in (\xi_+, \infty)$ with th $\xi_\pm = -\cos V_\pm$, correspondig to the three different "one-particle" kinks y_I, y_{II} and y_{III}, respectively. The mass of the second one is different — this is a new phenomenon not encountered in the simple DSG model. We may extend simply the discussion to any ansatz $Q(z) = \sum_{k=0}^{s} a_k z^k$ with the zeros $z_i \in (-1, 1)$ which corresponds to the admitted additional vacuum states $\varphi_i^{(vac)}$ lying in the interval $(0, \pi)$, and found that the masses

$$m_i = \left| \int_{\varphi_i^{(vac)}}^{\varphi_{i+1}^{(vac)}} f(y)dy \right| = |\bar{Q}(z_{i+1}) - \bar{Q}(z_i)|, \quad 0 \leqslant i \leqslant M-1$$

$$\bar{Q}(z) = \sum_{k=0}^{s} \frac{a_k}{k+1} z^{k+1} \tag{4.4}$$

form a multiplet generated dynamically by the mere choice of the functions $f(u)$ in the Lagrangian. Of course, the practical use of this procedure is again strongly restricted by the oversimplified one-dimensional character of our model.

5. *Conclusions*

There is a natural hierarchy in the relativistic particle models:

— The group-theoretical and linear field equations are able to substantiate or explain the existence of the quantum numbers like spin but fail to suppress the dispersion of the wave packets.

— The soliton-possessing nonlinear equations provide the localized and stable solutions but suffer from the technical oversimplifications: On the physical grounds, both the absence of singularities and elimination of the radiative components seem to be the too restrictive and superfluous mathematical rather than physical assumptions.

— The simplest "non-solvable" MSG models provide the far more flexible dynamics (masses, interactions, etc.). Moreover, they may become a useful technical and methodical laboratory for the transition to the more realistic models the structure of which is even far less transparent, especially in more dimensions. In this spirit, the present paper tried to present the MSG class of models as enabling us to

1) investigate the general dynamical assumptions (shape of nonlinearity and its various physical consequences) in a systematic way, via the Fourier-series approximations;
2) work. out new methods adequate to the nonlinear equations (perturbation treatment preceded by the algebraic, correctly formulated linearization);
3) visualize the possible qualitative features of the models via the nonnumerical methods (linearized description of the "wobbling" motion of the many-body-like kinks, analytic fit of masses, understanding the character of singular cases etc.);
4) investigate the open problems, (let us emphasize that our construction of the one--kink sector (space spanned by the solutions of the linearized MSG equation) represents one of the technicalities essential in the Feynman path-integral quantization procedure [5]).

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