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QUANTUM MECHANICS OF PHASE

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It is shown that the previously proposed gauge invariant theory of gradient current, whose essential part is the action of longitudinal field, leads, in the case of vanishing coupling constant, to very natural canonical commutation relations for the free phase.

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1. Introduction

In the previous paper [1] I obtained the action

$$-\frac{1}{16\pi}\int d^4x\left\{F^{\mu\nu}F_{\mu\nu}+\frac{1}{2\pi}(e\partial^\mu A_\mu+\Box S)^2\right\}.$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field, S is a scalar field called phase and e is the elementary charge. The action is invariant under gauge transformations

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu f,$$

$$S \to S' = S - ef,$$

where f is an arbitrary function.

The rather tortuous way leading to the above expression is described in Refs [1, 2]. In this note I wish to show that the above action leads to very natural commutation relations for the free phase and that the whole scheme has a certain unexpected consistency.

2. The Ostrogradski method

Let us put $e = 0$. Then the action is divided into two parts, the action of free electromagnetic field and the action of free phase. Such a division is not gauge invariant, of course, but it is always done for the Klein-Gordon and the Dirac equations. The action

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of free phase has the form

$$-\frac{1}{32\pi^2}\int d^4x(\Box S)^2.$$

It contains second derivatives and therefore it is not immediately obvious how to write down the canonical commutation relations. A way to do it, due to Ostrogradski, is described by Whittaker [3].

For the action principle

$$\delta\int L(q,\dot{q})dt = 0$$

the Euler-Lagrange equations have the form

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \equiv \frac{\partial L}{\partial q} - \frac{dp}{dt} = 0.$$

For the action principle

$$\delta\int L(q,\dot{q},\ddot{q})dt = 0$$

the Euler-Lagrange equations have the form

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0.$$

Whittaker writes this as

$$\frac{\partial L}{\partial q} - \frac{dp}{dt} = 0,$$

which allows to identify the momentum.

Applying this method to the field theory in question we have

$$\left[S(x), \frac{1}{16\pi^2}\partial_0\Box S(y)\right]_{x^0=y^0} = i\delta(\boldsymbol{x}-\boldsymbol{y}).$$

This equation can be integrated with the result

$$[S(x), S(y)] = \frac{2\pi}{i}\,\mathrm{sign}\,(x^0-y^0)\Theta[(x-y)(x-y)],$$

Θ being the Heaviside step function. Using the well known operator identity one has

$$e^{iS(x)}e^{iS(y)} = e^{iS(y)}e^{iS(x)}e^{-[S(x),S(y)]}$$

which means that exp (iS) is a classical Bose field.

3. *The theory of the total charge*

For $e \to 0$ the electric current given in [1] takes the form

$$j_\mu = -\frac{e}{16\pi^2}\partial_\mu \Box S.$$

Differentiating in the appropriate way the canonical commutation relations one finds

$$[j_\mu(x), S(y)] = ie\partial_\mu D(x-y),$$

where D is the Pauli-Jordan function. Integrating the last equation over the volume $x^0 = 0$ one has

$$[Q, S(y)] = ie,$$

where

$$Q = \int_{x^0=\mathrm{const}} d^3x j_0$$

is the total charge.

I assume now that there is a state $|0\rangle$ such that

$$Q|0\rangle = 0.$$

One has then

$$Qe^{iS}|0\rangle = [Q, e^{iS}]\,|0\rangle = -ee^{iS}|0\rangle$$

so that the state $\exp(iS)\,|0\rangle$ has the total charge $-e$; the state $\exp(2iS)\,|0\rangle$ has the total charge $-2e$ and so on. This seems to be the quantum mechanical counterpart of the classical result given in [1].

REFERENCES

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