NEW GRAVITATIONAL INSTANTON SOLUTIONS IN EUCLIDEAN GRAVITY

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A new class of gravitational instanton solutions with (and without) Λ-term is given. The solutions are Euclidean generalizations of various Bianchi types and the Kantowski-Sachs model.

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1. Introduction

There has recently been considerable interest in gravitational instantons (Eguchi et al. 1980; Pope 1981; Perry 1982). These may be defined as complete positive metrics which are solutions of Einstein's field equations

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{1}$$

There are many well known mathematical results which have applications to gravitational instantons (Boyer 1981; Catenacci and Reina 1982, Pope 1982). For example, a gravitational instanton with $\Lambda > 0$ is necessarily compact (Milnor 1963). Compact gravitational instantons are of much interest in the space-time foam description of gravitational physics on very small scales (Hawking 1978; 1979a, b). If $\Lambda < 0$ there are many implicit compact examples given by the Calabi-Aubin-Yau theorem (Boyer 1981; Catenacci and Reina 1982) recently proven by Yau (1977, 1978). However, only a few are known explicitly. If $\Lambda = 0$, then only two explicit examples are known (Pope 1981; Perry 1982). Thus we are encouraged to construct further exact instanton solutions.

2. Metrics, field equations and solutions

In this paper we consider the Euclidean metrics

$$ds^2 = dt^2 + R_1^2[dx + 2lh(y)dz]^2 + R_2^2[dy^2 + h'^2(y)dz^2], \tag{2}$$

where $R_i = R_i(t)$ and $h(y)$ is $\cos y, y, \cosh y$, respectively, when $k = 1, 0, -1$. If $l = 0$ one obtains the Euclidean Bianchi types I ($k = 0$), III ($k = -1$) and the Kantowski-Sachs

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space-time in case of $k = 1$. With $l \neq 0$ (NUT-parameter), the metrics (2) are the Euclidean versions of the Taub-NUT-de Sitter space-times of Bianchi types II ($k = 0$), VIII ($k = -1$) and IX ($k = 1$). The corresponding field equations (1) can be reduced to

$$2\frac{\dot{R}}{R}\frac{\dot{S}}{S} + \left(\frac{\dot{R}}{R}\right)^2 + l^2\left(\frac{S}{R^2}\right)^2 - \frac{k}{R^2} = -\Lambda \tag{3a}$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + 3l^2\left(\frac{S}{R^2}\right)^2 - \frac{k}{R^2} = -\Lambda \tag{3b}$$

$$\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R}}{R}\frac{\dot{S}}{S} - l^2\left(\frac{S}{R^2}\right)^2 = -\Lambda, \tag{3c}$$

where $S := R_1$, $R := R_2$ and $(\,)^{\cdot} = d/dt$. With the aid of the "Ansatz"

$$S = \dot{R}f(R), \qquad \dot{R}^2 = -\left[\frac{g(R)}{R} + \frac{\Lambda}{3}R^2 - k\right], \tag{4}$$

we obtain the uncoupled system for f and g

$$f' - l^2\left(\frac{f}{R}\right)^3 = 0, \qquad g' - 3l^2\left(\frac{S}{R}\right)^2 = 0, \tag{5}$$

where $(\,)' := d/dR$ and which has the exact solutions

$$f = \left[1 + \left(\frac{l}{R}\right)^2\right]^{-1/2} \qquad g = 2m\left[1 + \left(\frac{l}{R}\right)^2\right]^{3/2} + k\frac{l^4}{R^3}$$
$$-3k\frac{l^2}{R}\left[1 + \left(\frac{l}{R}\right)^2\right] - \frac{\Lambda l^2}{R^3}\left[(R^2 + l^2)(R^2 + 3l^2) - \frac{l^4}{3}\right], \tag{6}$$

where $2m = \text{const}$. By transforming (2) to a canonical r system

$$r^2 = R^2 + l^2, \qquad u(r) = S^2, \tag{7}$$

we can express (2) and (6) in closed form

$$ds^2 = \frac{dr^2}{u(r)} + u(r)\,[dx + 2lh(y)dz]^2 + (r^2 - l^2)\,[dy^2 + h'^2(y)dz^2], \tag{8a}$$

where

$$u(r)\,(r^2 - l^2) = k(r^2 + l^2) - 2mr + \Lambda\left(l^4 + 2l^2r^2 - \frac{r^4}{3}\right). \tag{8b}$$

The case $l \neq 0$, $k = 1$ has been already obtained by Gibbons and Pope (1978a), Tseytlin (1980) and Boutaleb-Joutei (1980) by completely different methods. By taking appropriate limits one obtains also the solutions of Eguchi and Hanson (1978), Hawking (1977) and the Fubini-Study metric (see Gibbons and Pope 1978a). The Weyl curvature tensor for metrics (8) is of algebraic type D and has two invariants, which may be combined into the form

$$\psi_2 = (m+kl)/(r+l)^3. \tag{9}$$

These invariants and the scalars formed from the Riemann tensor are regular functions throughout the coordinate region ($-\infty < r < \infty$). It follows that the Euclidean Taub--NUT-de Sitter metrics with $l \neq 0$ have no local singularities, although with $l = 0$ there is a $r = 0$ singularity. In the limit $l = 0$, the metrics (8) converts into the Euclidean "A--metrics" (Kramer et al. 1980), which includes the Kottler metric ($k = 1$) (see Gibbons and Pope 1978a), Schwarzschild ($k = 1$, $\Lambda = 0$), de Sitter ($k = 1, m = 0$) and the corresponding analog with planar ($k = 0$) and pseudospherical ($k = -1$) symmetries. The most general type D solution has been found by Lapedes and Perry (1981).

We now derive (anti)-self-dual solutions of (1). By requiring that the Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$ is (anti)-self-dual

$$R^{\mu}{}_{\nu\alpha\beta} = \frac{\delta}{2}\,\varepsilon_{\alpha\beta\gamma\varrho}R^{\mu}{}_{\nu}{}^{\gamma\varrho}, \tag{10}$$

where $\delta = 1$ for self-dual and $\delta = -1$ for anti-self-dual solutions, we obtain the following vacuum ($\Lambda = 0$) field equations

$$\frac{\ddot{S}}{S} = 2\delta l\,\frac{S}{R^2}\left(\frac{\dot{S}}{S} - \frac{\dot{R}}{R}\right) = -2\,\frac{\ddot{R}}{R}\,. \tag{11}$$

Introducing the new time variable η by $dt := SR^2 d\eta$ we obtain after a single integration

$$(\ln S^2)' = 2[R^2(1-\lambda_1)+l\delta S^2], \tag{12a}$$

$$(\ln R^2)' = -2S[l\delta S+\lambda_2 R], \tag{12b}$$

where $(\;)' := d/d\eta$. The λ_1 are constants obeying

$$\lambda_2 = \lambda_1\lambda_2, \quad \lambda_2^2 = 2(1-\lambda_1)l\delta+k. \tag{13}$$

The corresponding (anti)-self-dual solutions are classified according to the values of the parameters $(l, k, \lambda_1, \lambda_2^2)$:

(*i*) $(0, 0, 1, 0)$

$$S = a,\ R = b,$$

(*ii*) $(0, 0, \lambda_1 \neq 1, 0)$

$$R = b,\ S = \exp\,[b^2(1-\lambda_1)\,(\eta-\eta_0)],$$

(*iii*) $(0, k, 1, k)$

$$S = a, \quad R = [a\lambda_2(\eta-\eta_0)]^{-1}, \quad k^2 = 1,$$

(*iv*) $(l, 0, 1, 0)$

$$S^2 = [-2l\delta(\eta-\eta_0)]^{-1}, \quad S^2 = -2bl\delta(\eta-\eta_0),$$

(*v*) $= (l, k, 0, 0)$

$$(SR)^2 = \frac{r^4}{2} + a, \quad R^2 = k\frac{r^4}{2} + b, \quad k^2 = 1$$

(*vi*) $(l, k, \lambda_1 \neq 1, 0)$

$$(SR)^2 = -\frac{k}{2l\delta}\tau^4 + b, \quad R^2 = \ln\left[a(k\tau^4+b)\right]^{1/2k}, \quad k^2 = 1 \tag{14}$$

(*vii*) $(l, k, 1, k)$

$$SR = [\lambda_2(\eta-\eta_0)]^{-1}, \quad S^2 = [-2l\delta(\eta-\eta_1)]^{-1}, \quad k^2 = 1 \tag{14}$$

where a, b, η_i are constants of integration, $dr = r^{-3}(SR^2)^2 d\eta$ and $d\tau = \tau^{-3}R^6S^4 d\eta$.

The solutions (*i*), (*ii*) give flat Euclidean space. Solutions (*iii*) with $k = 1$ has been obtained by us recently in a somewhat different form (Lorenz 1983) and represents a Kantowski-Sachs gravitational instanton. The case (*v*) has been first given by Eguchi and Hanson (1978) with $k = 1$. If $k = 1$ solution (*vii*) can be transformed into the gravitational instanton first given by Hawking (1977) (see also Gibbons and Pope 1978b). The remaining solutions are new.

We finally would like to point out that in general it is possible to include self-dual electromagnetic fields into the Bianchi space-times, since such fields have vanishing energy-momentum tensor.

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