AN ALGORITHM FOR CALCULATING MASSLESS FEYNMAN DIAGRAMS

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A simple method for calculating multiloop massless Feynman diagrams is presented.

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Perturbation theory calculations in quantum chromodynamics have stimulated a considerable interest concerning methods for calculating multiloop Feynman diagrams. Calculations of renormalization group (RG) functions in various quantum field models [1–5], critical exponents in quantum statistics [6, 7] and some other important quantities necessitated the developing of methods for multiloop diagrams calculations.

The method developed in the above mentioned papers made possible calculating of the RG quantities up to the 3- or 4-loop level. At the same time a great numbei of higher-order diagrams defies calculating with use of these methods.

Some simple arguments essentially enlarging modern calculational abilities are suggested. In this paper, we describe an algorithm and illustrate it by some examples. More complicated examples will be published elsewhere including calculation of some non-trivial diagrams in arbitrary order as well as calculation of RG quantities.

In order to calculate a Feynman diagram it is necessary at first to carry out algebraic manipulations with nominators. Computers are very helpful for these purposes in the case of multiloop diagrams [8]. After that the problem is reduced to the calculation of some scalar integrals.

In this paper we consider massless scalar integrals. Calculation of massless diagrams is more simple because of quite a simple form of Fourier-transformation:

$$F\left[\frac{1}{(p^2+i0)^a}\right] = \frac{(2\pi i)^{D/2}\Gamma\left(\frac{D}{2}-a\right)}{\Gamma(a)} \frac{1}{(x^2-i0)^{D/2-a}}; \quad a \neq \frac{D}{2}, \dots \tag{1}$$

D being the space-time dimension. That is why a free massless propagator in momentum and coordinate spaces has a power-like behavior.

Moreover, massless diagrams play a particular role in calculations of RG functions. This results from polynomial dependence of counterterms on masses in minimal subtraction (MS) schemes. The problem of calculating the counterterm of an arbitrary L-loop diagram

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with arbitrary masses and an arbitrary number of external momenta within MS-scheme can be reduced to the problem of calculating some $(L-1)$-loop massless integrals with only one external momentum [10, 1].

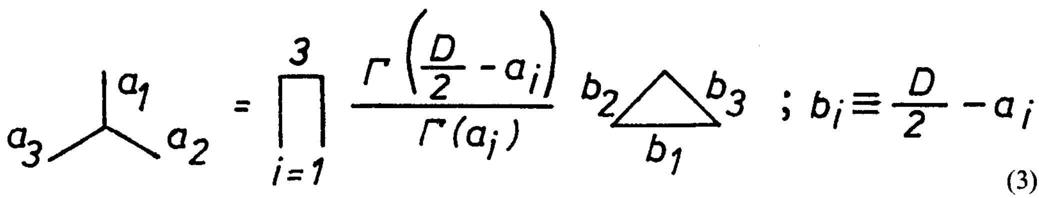
We work in coordinate space. Each line of a diagram carries a power-like factor $1/(x^2-i0)^a$, pictured as $\overset{a}{—}$. In this case the line is said to have an index a. To avoid complicating the formulae we omit henceforth the factors that are powers of 2, π, i. These factors can be easily restored in the final results.

The product and the convolution of lines with indices a_1 and a_2 in coordinate space have indices a_1+a_2 and $a_1+a_2-D/2$, respectively.

The idea of the method consists in reduction of a diagram into a linear combination of some other diagrams whose calculation is to a great extent simpler than that of the original one.

If a diagram contains a three-line vertex or a triangle with lines satisfying Eq. (2), (4) (see below) the diagram is reduced to a more simple one. The reduction can be done thanks to the following equation [11]. If

$$\sum a_i = D, \tag{2}$$



(3)

Note that incidentally

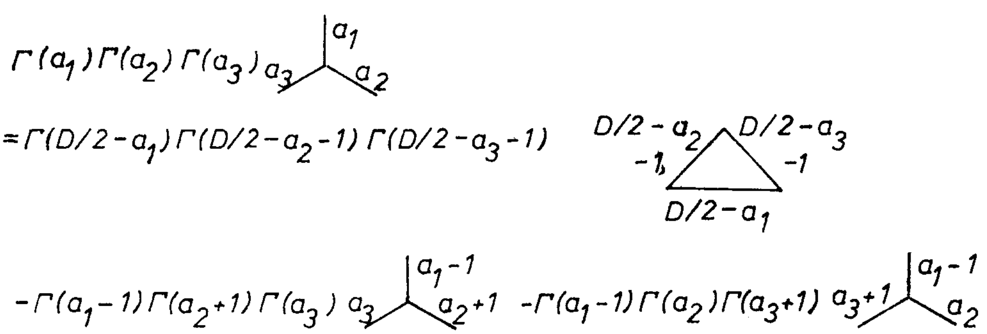
$$\sum b_i \equiv D/2. \tag{4}$$

The three-line vertex and the triangle having such lines are called the unique ones.

Unfortunately it happens very often that diagrams have no unique object. Normally every line of a diagram has an index $1+\alpha$. Here α is a parameter of some regularization that tends to zero with regularization being removed. So Eqs (2), (4) are not fulfilled.

However, even in that case, there is a possibility of a reduction of a diagram. It can be done with the help of the following equations [12]. If

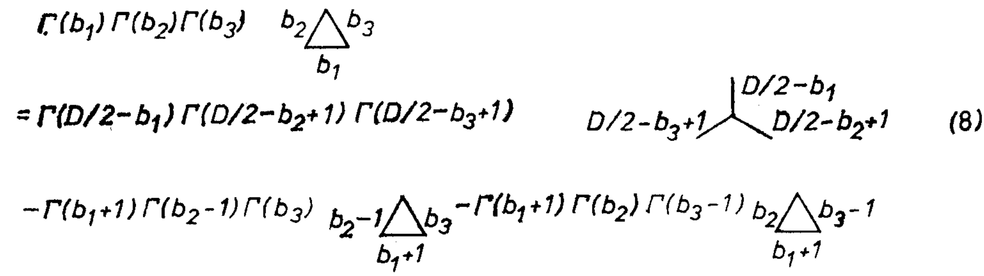
$$\sum a_i = D-1, \tag{5}$$



(6)

If

$$\sum b_i = D/2+1, \tag{7}$$



Eqs. (6), (8) can be obtained from Feynman parametrization of the corresponding graphs.

In Eqs. (6), (8) indices of some lines change by a unit. In such a way one can construct a unique óbject inside a diagram.

Before formulating the algorithm let us calculate one of the simplest nontrivial diagrams shown in Fig. 1a.

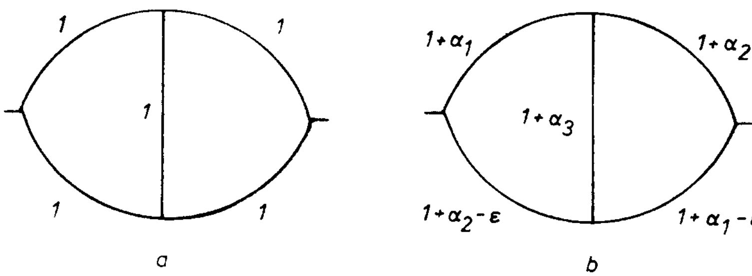


Fig. 1

It is convenient to set $D = 4-2\varepsilon$ and then to take a limit $\varepsilon \to 0$. From the formula (1) we see that every line has an index $1-\varepsilon$. Let us introduce an auxiliary analytic regularization, i.e. set an index of i-th line equal to $1+\alpha_i$. We choose parameters α_i as shown in Fig. 1b. Introduction of parameters α_i provides the fulfillment of Eqs. (5), (7) as well as regularization of all diagrams emerging during the calculation.

Now because of Eq. (6) the diagram of Fig. 1b can be reduced into the linear combinations of diagrams a, b, c of Fig 2 with coefficients $-(1+\alpha_1-\varepsilon)/\alpha_3$; $-(1+\alpha_2-\varepsilon)/\alpha_3$; $\Gamma(-\alpha_1)\Gamma(-\alpha_2)\Gamma(1-\varepsilon-\alpha_3)/\Gamma(1-\varepsilon+\alpha_1)\Gamma(1-\varepsilon+\alpha_2)\Gamma(1+\alpha_3)$, respectively.

Calculation of these diagrams is trivial as there are unique objects there. The result is

$$1-\alpha_3 \frac{\Gamma(\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \frac{1}{\alpha_1\alpha_2\alpha_3} \mathscr{T} \left\{ \alpha_2 \frac{\Gamma(1+\alpha_1)\Gamma(1-\varepsilon-\alpha_2)\Gamma(1+\alpha_3)}{\Gamma(1-\varepsilon+\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\varepsilon+\alpha_3)} \right.$$

$$\left. +\alpha_1 \frac{\Gamma(1-\varepsilon-\alpha_1)\Gamma(1+\alpha_2)\Gamma(1+\alpha_3)}{\Gamma(1-\alpha_1)\Gamma(1-\varepsilon+\alpha_2)\Gamma(1-\varepsilon+\alpha_3)} + \alpha_3 \frac{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)\Gamma(1-\varepsilon-\alpha_3)}{\Gamma(1-\varepsilon+\alpha_1)\Gamma(1-\varepsilon+\alpha_2)\Gamma(1-\alpha_3)} \right\}, \tag{9}$$

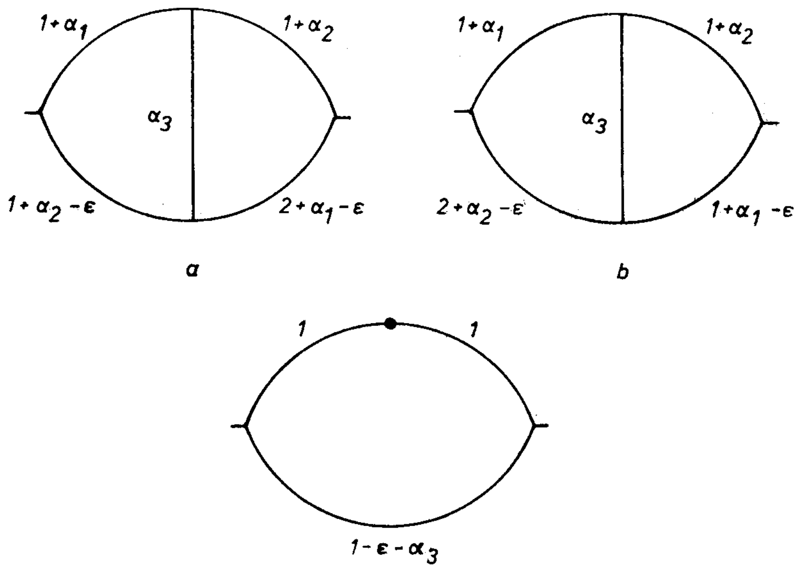


Fig. 2

where

$$\mathscr{T} \equiv \prod_{i=1}^{3} \Gamma(1-\alpha_i)/\Gamma(1+\alpha_i).$$

To obtain the final result let ε and α tend to zero. Note that there are poles in parameters of regularization in separate addends but they cancel in the sum. That yields

$$\frac{1}{K^2}(-3\psi''(1)) = 6\zeta(3)\frac{1}{K^2} \tag{10}$$

ζ is Riman ζ-function.

This example clarifies the algorithm for calculating an arbitrary diagram. We describe it for diagrams where every line has an index $1+\alpha$. For other cases the procedure is similar or even simpler.

1. To calculate a given diagram introduce an auxiliary analytic regularization, i.e. set an index of i-th line equal to $1+\alpha_i$. Because of the regularization all quantities obtained further on are well defined.

2. Choose a vertex (or a triangle) and get the indices of its lines to fulfill Eq. (5) (or Eq.(7)).

3. With the help of Eqs (6), (8) reduce the diagram into a linear combination of three other diagrams. Now consider each obtained diagram one by one. As indices of some lines change by a unit the reduction makes it possible to construct unique objects inside diagrams.

4. Have some parameters α_i fulfilling certain conditions to obtain unique vertices or unique triangles. If it is not possible go back to item 2.

5. Reduce the diagram with the help of Eq. (3). Further calculating is an iteration

of items 2-5. A system of linear algebraic equations for parameters α_i is obtained as a result of this procedure. Suppose this system to have a solution that does not reduce to zero the argument of any emerging Γ-function (henceforth we call such solutions permitted ones). We obtain the result by substituting this solution and removing the regularization.

As an example of applying the algorithm let us consider calculation of nonplanar diagram pictured in Fig. 3a. For the first time this diagram was calculated in Ref. [1] by a method which was rather complicated. It is convenient to set $D = 4-2\varepsilon$ and use an analytic regularization as shown in Fig. 3b.

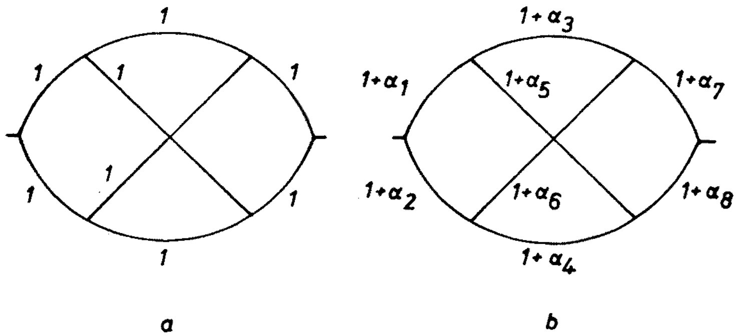


Fig. 3

If the indices of lines of the vertex A satisfy equation $\alpha_2+\alpha_4+\alpha_6 = -2\varepsilon$, then with the help of Eq. (6) the diagram can be reduced into the linear combination of diagrams a, b, c of Fig. 4. Reductions of these diagrams are shown in Figs. 5, 6, 7. The following

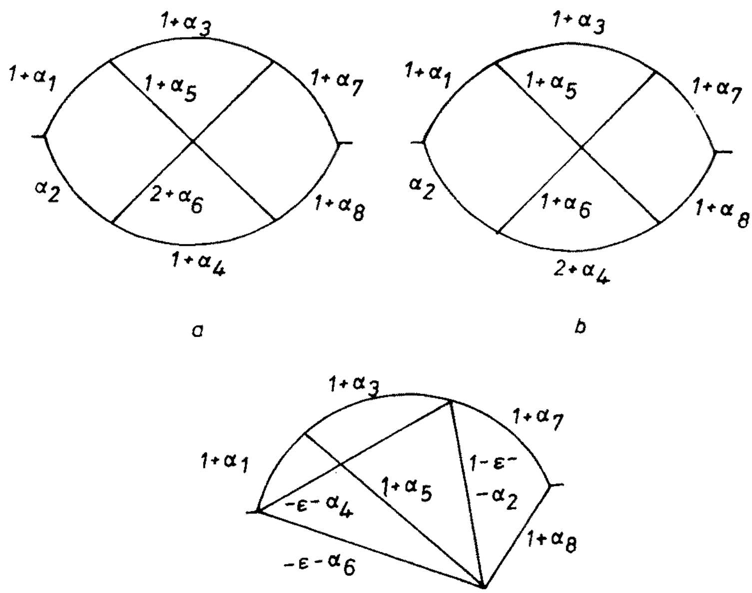


Fig. 4

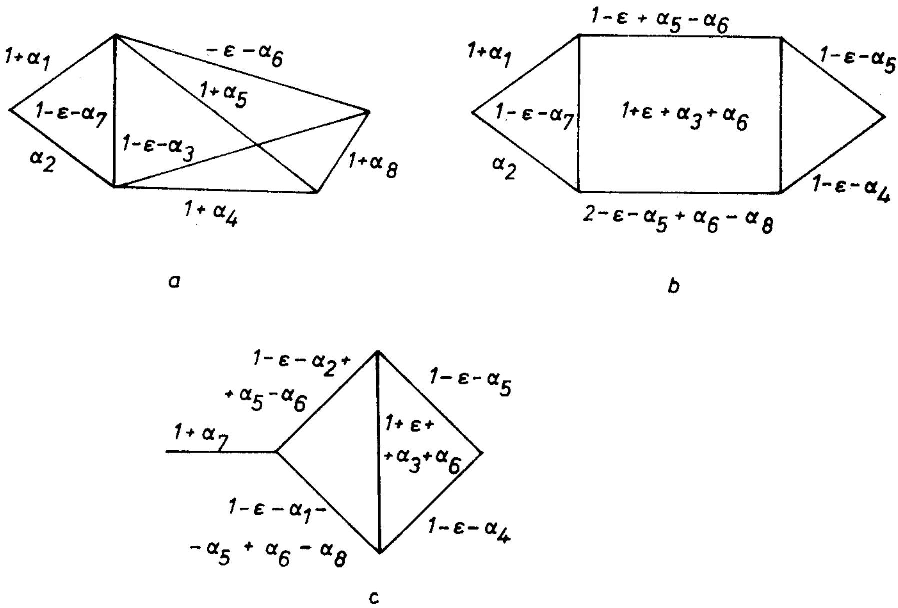

1+α1
1-ε-α7
α2
-ε-α6
1+α5
1-ε-α3
1+α8
1+α4
a
1-ε+α5-α6
1+α1
1-ε-α7
α2
1+ε+α3+α6
1-ε-α5
1-ε-α4
2-ε-α5+α6-α8
b
1-ε-α2+
+α5-α6
1+α7
1+ε+
+α3+α6
1-ε-α5
1-ε-α1-
-α5+α6-α8
1-ε-α4
c

Fig. 5

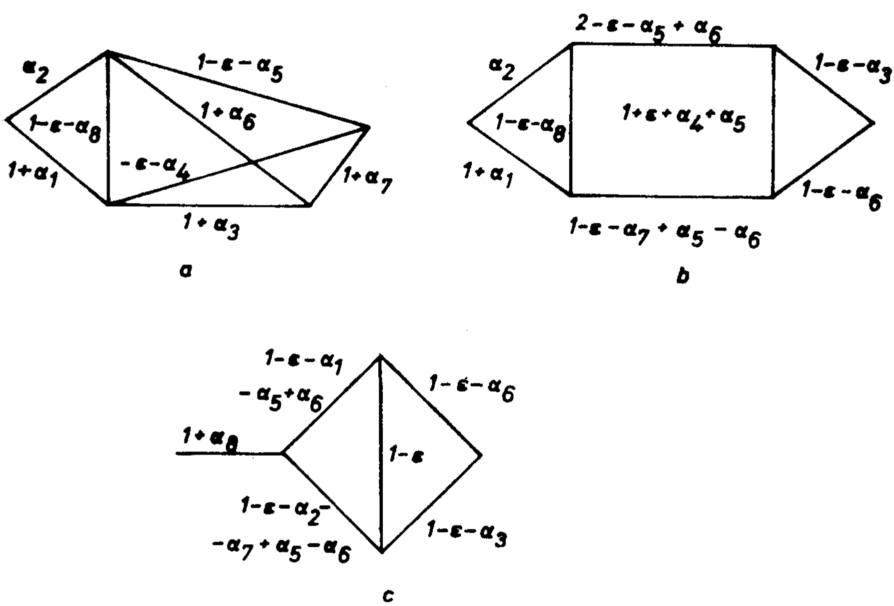

α2
1-ε-α8
1+α1
1-ε-α5
1+α6
-ε-α4
1+α7
1+α3
a
2-ε-α5+α6
α2
1-ε-α8
1+α1
1+ε+α4+α5
1-ε-α3
1-ε-α6
1-ε-α7+α5-α6
b
1-ε-α1
-α5+α6
1+α8
1-ε
1-ε-α6
1-ε-α2-
-α7+α5-α6
1-ε-α3
c

Fig. 6

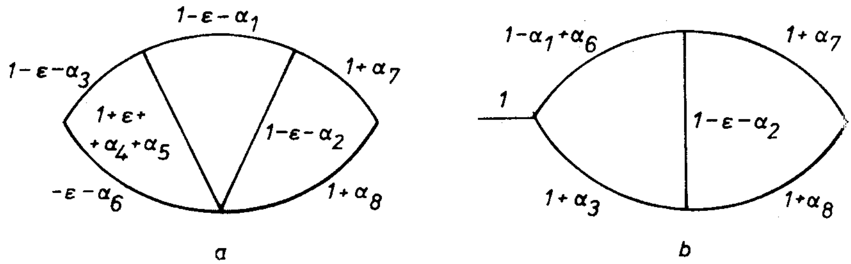


Fig. 7

equations are necessary for that:

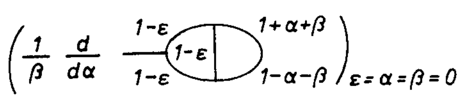
$$\begin{gathered}\alpha_2+\alpha_4+\alpha_6 = -2\varepsilon, \quad \alpha_4+\alpha_5+\alpha_8 = -2\varepsilon, \quad \alpha_3+\alpha_6+\alpha_7 = -2\varepsilon,\\ \alpha_3-\alpha_4+\alpha_7-\alpha_5+\alpha_6 = 0, \quad \alpha_1+\alpha_2-\alpha_8 = 0, \quad \alpha_1+\alpha_3-\alpha_4 = 0,\\ \alpha_5-\alpha_6+\alpha_8-\alpha_3+\alpha_4 = 0, \quad \alpha_1+\alpha_2-\alpha_7 = 0.\end{gathered} \tag{11}$$

It is evident that some of these equations are consequences of the other ones. A solution of the system (11) can be chosen in the form

$$\begin{gathered}\alpha_1 = -\alpha_2 = \alpha, \quad \alpha_3 = -\varepsilon-\alpha-\beta, \quad \alpha_4 = -\varepsilon-\beta,\\ \alpha_5 = -\varepsilon+\beta, \quad \alpha_6 = -\varepsilon+\alpha+\beta, \quad \alpha_7 = \alpha_8 = 0.\end{gathered} \tag{12}$$

α, β being arbitrary parameters.

Taking into account all the factors appearing during the reduction account we obtain the following expression:



(13)

The diagram entering Eq. (13) can be calculated as shown above. Eq. (13) leads to

$$\frac{1}{(K^2)^2}\left(\frac{\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\,\Gamma(1-\varepsilon)\left\{-\tfrac{5}{6}\left(\psi'''(1)-\psi'''(1-\varepsilon)\right)+\left(\psi'(1)-\psi'(1-\varepsilon)\right)^2\right\}\right)_{\varepsilon=0}, \tag{14}$$

that with $\varepsilon \to 0$ yields

$$-\tfrac{5}{6}\,\psi^{(4)}(1)\,\frac{1}{(K^2)^2} = 20\zeta(5)\,\frac{1}{(K^2)^2}\,. \tag{15}$$

Note that the obtained system of linear algebraic equations for parameters of an auxiliary regularization depends on the way of performing the reduction. If a system having no permitted solutions is obtained it does not mean that the diagram cannot be calculated by the described method. It might result from an unsuccessful choice of the way of reduction.

As the ways of reduction are unnumerous for multiloop diagrams there will be an element of art in choosing a most convenient way of reduction until it is done with the help of a computer.

We have considered so far only convergent diagrams. If there are ultraviolet or infrared divergences in a diagram they manifest themselves as poles in parameters of regularization in the final expression. Note that in the case of convergent diagrams all poles appearing during the reduction are cancelled in the final results (see Eqs. (9), (10)).

As an example of calculating of divergent diagrams let us consider calculation of scalar "Mercedes" diagram (Fig. 8a). There is an infrared divergence due to the integration at large x.

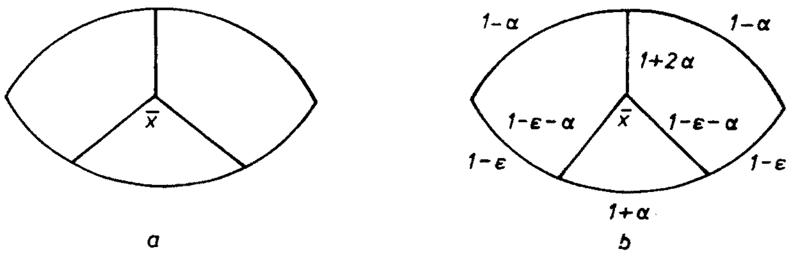


Fig. 8

Let $D = 4-2\varepsilon$. One of possible choices of an auxiliary regularization parameters (i.e. a permitted solution of a system of linear algebraic equations) is shown in Fig. 8b. It is convenient to begin reduction from the vertex x. Reducing the diagram and taking a limit $\alpha \to 0$ one can obtain the result

$$\frac{1}{(K^2)^2}\left\{-\frac{4\zeta(3)}{\varepsilon}+0(\varepsilon^0)\right\}. \tag{16}$$

The pole term corresponds to the infrared divergency. The coefficient does not depend on the choice of parameters of regularization.

We have only considered above the calculation of propagator diagrams. However the method is literally applicable as well to calculating diagrams with an arbitrary number of external momenta. For instance arbitrary-order ladder vertex diagrams can be reduced by this method to a one-loop vertex.

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