

# MOTION OF A WAVE PACKET IN AN EXTERNAL YANG-MILLS FIELD\*

BY H. ARODŹ

Institute of Physics, Jagellonian University, Cracow\*\*

(Received April 18, 1983)

Two further examples of motion of a wave packet in external Yang-Mills fields are investigated. We find that because of the interaction with the gauge field the wave packet splits into a number of parts which remain spatially separated in the  $\hbar \rightarrow 0$  limit. Therefore, these examples confirm our point of view that quantum mechanics of colored particles does not have simple classical limit.

PACS numbers: 11.15.-q

## 1. Introduction

Classical mechanics of point-like particles with certain internal degrees of freedom is a very interesting and lively subject, see, e.g., the recent papers [1], [2] and references therein.

In the papers [3] we have investigated the problem to what extent classical mechanics of colored particles [2] can be regarded as a classical limit of quantum mechanics of colored particles in the Ehrenfest's sense, [4]. That is, the problem is whether the classical equations of motion (Wong's equations) can be obtained from quantum mechanical equations of motion for expectation values of operators describing the basic dynamical quantities like position, momentum, spin and color spin. We have shown that Wong's equations can be recovered within this approach, provided that the time evolution of a wave packet can approximately be described as a motion along a definite classical trajectory  $\vec{x}(t)$ . However, in [5] we have presented two examples of external Yang-Mills fields such that the wave packet, whose time evolution is governed by Schrödinger equation, does not follow a definite classical trajectory even in the  $\hbar \rightarrow 0$  limit. In this paper we continue investigations which have been started in [5].

---

\* Paper supported in part by the Ministry of Higher Education, Science and Technology, Project MR.I.7.

\*\* Address: Zakład Teorii Pola, Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

In [5] we have explicitly calculated the motion of the wave packet in rather simple SU(2) gauge potentials of two kinds: the Abelian type

$$A_\mu^a(\vec{x}, t) = e^a a_\mu(\vec{x}, t), \quad e^a = \text{const.},$$

and the constant potentials

$$A_\mu^a(\vec{x}, t) = \text{const.}$$

In this paper we present an analysis of the motion of the wave packet in more complicated SU(2) gauge potentials, namely in the spherically symmetrical one,

$$A_i^a = \varepsilon_{ais} a(r) \frac{x^s}{r}, \quad (1)$$

where  $a(r) = \alpha r$ ,  $\alpha = \text{constant}$ , and in the “color-tilt” gauge potential

$$\hat{A}_0 = \begin{cases} \alpha g \sigma^3 & \text{for } z \leq 0, \\ \alpha g \vec{n} \vec{\sigma} & \text{for } z > 0, \end{cases} \quad (2)$$

where  $\alpha = \text{constant}$ ,  $\vec{n}$  is a constant unit vector,  $\vec{n}^2 = 1$ , and we use the matrix notation  $\hat{A}_\mu = A_\mu^a \sigma^a / 2$ . We find that the wave packet placed in these more complicated potentials splits into a number of spatially separated parts, similarly as it does in the potentials considered in [5]. The number of parts depends on details of the gauge potential, on the gauge group and on the representation of the gauge group to which the colored particle belongs, see paragraph 2 of Section 4 of this paper. The difference between the trajectories of these parts can not be reduced to zero by taking the  $\hbar \rightarrow 0$  limit. For the potential (1), considered in Section 2, we construct wave packets moving along a circular orbit. For the potential (2) we consider a scattering of a wave packet, Section 3. Remarks are collected in Section 4. The Appendix contains an auxiliary construction of a wave packet moving along the equatorial circle in the potential of the ordinary, 3-dimensional, isotropic harmonic oscillator.

Results of [5] and of this paper indicate that quantum mechanics of colored particles in external Yang-Mills fields does not have its classical counterpart in the form of a classical mechanics of a single, point-like particle moving along a definite trajectory. We think that this is an interesting result because it shows that the existence of classical quark fields (regarded as wave functions on the level of quantum mechanics) does not lead to the notion of a classical, point-like quark. This corresponds rather well with the common expectation that quarks are not present in the particle spectrum of quantum chromodynamics [6].

## 2. The spherically symmetrical external field

In this Section we shall consider the wave packet in the spherically symmetrical potential (1). This potential can be regarded as the approximate form of the potential of the t'Hooft-Polyakov magnetic monopole in the Prasad-Sommerfeld limit for  $r^2 \mu^2 \ll 1$ , where  $\mu^2 \equiv -6g\alpha > 0$ . The potential (1) gives the following field-strengths:

$$F_{0i}^a = 0, \quad F_{ik}^a = 2\alpha \varepsilon_{aki} + g\alpha^2 \varepsilon_{rki} x^r.$$

The Schrödinger equation has the form

$$i\hbar \frac{\partial}{\partial t} \vec{\psi} = H \vec{\psi}, \quad (3)$$

where

$$H = (2\mu)^{-1} \left( \hat{p} - \frac{1}{2} \frac{g}{c} \sigma^a \vec{A}^a \right)^2 = (2\mu)^{-1} \left( \hat{p}^2 + \frac{1}{2} \frac{g^2}{c^2} a^2(r) + \frac{g}{c} \frac{a(r)}{r} \sigma^a \hat{L}^a \right), \quad (4)$$

and  $\hat{L}^a = \varepsilon_{ais} \hat{x}^i \hat{p}^s$ .

The following Hermitean operators commute with  $H$  and with themselves

$$\hat{K} = \vec{\sigma} \hat{L}, \quad \hat{J}^2 = \left( \frac{\hbar}{2} \vec{\sigma} + \hat{L} \right)^2, \quad \hat{J}_z = \frac{\hbar}{2} \sigma^3 + \hat{L}_z. \quad (5)$$

The common eigenfunctions of these three operators we denote by  $\vec{\Omega}_{\varepsilon lm}(\vartheta, \varphi)$ , where  $\vartheta, \varphi$  are the spherical angles,  $\varepsilon = \pm 1$ . The corresponding eigenvalues are

$$(\vec{\sigma} \hat{L}) \vec{\Omega}_{\varepsilon lm} = \hbar \kappa(\varepsilon, l) \vec{\Omega}_{\varepsilon lm}, \quad (6)$$

where  $\kappa(+1, l) = l$ ,  $\kappa(-1, l) = -l-1$ .

$$\hat{J}^2 \vec{\Omega}_{\varepsilon lm}(\vartheta, \varphi) = \hbar^2 j(\varepsilon)(j(\varepsilon)+1) \vec{\Omega}_{\varepsilon lm}(\vartheta, \varphi), \quad (7)$$

where  $j(+1) = l + \frac{1}{2}$ ,  $l = 0, 1, 2, \dots$ ,  $j(-1) = l - \frac{1}{2}$ ,  $l = 1, 2, \dots$ , and

$$\hat{J}_z \vec{\Omega}_{\varepsilon lm(\varepsilon)} = \hbar m(\varepsilon) \vec{\Omega}_{\varepsilon lm(\varepsilon)}, \quad (8)$$

where  $m(\varepsilon) = -j(\varepsilon), -j(\varepsilon)+1, \dots, j(\varepsilon)$ . The number  $l = 0, 1, \dots$  is introduced by

$$\hat{L}^2 \vec{\Omega}_{\varepsilon lm} = \hbar^2 l(l+1) \vec{\Omega}_{\varepsilon lm}.$$

$\hat{L}^2$  is not included into (5) because it is a linear function of  $\vec{\sigma} \hat{L}$  and  $\hat{J}^2$ . The explicit form of  $\vec{\Omega}_{\varepsilon lm}(\vartheta, \varphi)$  is the following

$$\vec{\Omega}_{\varepsilon lm(\varepsilon)}(\vartheta, \varphi) = \begin{pmatrix} \sqrt{\frac{1}{2} + l + \varepsilon m} Y_{m(\varepsilon) - \frac{1}{2}}^l(\vartheta, \varphi) \\ \varepsilon \sqrt{\frac{1}{2} + l - \varepsilon m} Y_{m(\varepsilon) + \frac{1}{2}}^l(\vartheta, \varphi) \end{pmatrix}, \quad (9)$$

where  $Y_n^l(\vartheta, \varphi)$  are the spherical functions. Thus,  $\vec{\Omega}_{\varepsilon lm(\varepsilon)}$  are the well-known [7] spherical spinors. The normalization of  $Y_n^l$  is such that  $\langle \vec{\Omega}_{+1lm} | \vec{\Omega}_{-1lm} \rangle = 0$  and  $\vec{\Omega}_{\varepsilon lm}$  are normalized to 1.

Generic wave function  $\vec{\psi}(\vec{x}, t)$  can be written as

$$\vec{\psi}(\vec{x}, t) = \sum_{\varepsilon lm} f_{\varepsilon lm}(r, t) \vec{\Omega}_{\varepsilon lm}(\vartheta, \varphi). \quad (10)$$

From (3), (4) it follows that

$$i\hbar \frac{\partial}{\partial t} f_{\varepsilon lm}(r, t) = (2\mu)^{-1} \left( -\hbar^2 \frac{\partial^2}{\partial r^2} + \frac{\hbar^2 l(l+1)}{r^2} + \frac{1}{2} \frac{g^2}{c^2} a^2(r) + \frac{g}{c} \frac{a(r)}{r} \kappa(\varepsilon, l) \right) f_{\varepsilon lm}(r, t). \quad (11)$$

Observe that the operator on the r.h.s. of (11) does not depend on  $m$ . Substituting  $a(r) = \alpha r$  we obtain from (11) that

$$f_{elm}(r, t) = \exp\left(-\frac{i}{h} \frac{g\alpha}{2\mu c} \kappa(\varepsilon, l)t\right) \tilde{f}_{elm}(r, t), \tag{12}$$

where  $\tilde{f}_{elm}(r, t)$  obeys the equation

$$i\hbar \frac{\partial}{\partial t} \tilde{f}_{elm}(r, t) = \left(-\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + \frac{1}{4\mu} \frac{g^2 \alpha^2}{c^2} r^2\right) \tilde{f}_{elm}(r, t),$$

which is identical in its form with the radial equation for the isotropic, 3-dimensional harmonic oscillator. Thus, looking at, e.g., [8], we find that

$$\tilde{f}_{elm}(r, t) = \sum_{n=0}^{\infty} c_{elmn} \exp\left(-\frac{i}{h} E_{nl}t\right) \chi_{nl}(r), \tag{13}$$

where

$$E_{nl} = \hbar\omega_0(2n+l+\tfrac{3}{2}), \quad \omega_0^2 = \tfrac{1}{2} \frac{g^2 \alpha^2}{\mu^2 c^2}, \quad n = 0, 1, 2, \dots, \tag{14}$$

and

$$\chi_{nl}(r) = r^{l+1} \exp\left(-\frac{\lambda}{2} r^2\right) F(-n, l+\tfrac{3}{2}; \lambda r^2), \tag{15}$$

where  $\lambda^2 = \frac{1}{2} g^2 \alpha^2 c^{-2} \hbar^{-2}$ . Substituting (12), (13) into (10) we obtain that

$$\vec{\psi}(\vec{x}, t) = \vec{\psi}_+(\vec{x}, t) + \vec{\psi}_-(\vec{x}, t),$$

where

$$\vec{\psi}_+(\vec{x}, t) = \sum_{\substack{n,l,m \\ l \geq 0}} c_{+nlm} \exp\left[-\frac{it}{h} \left(E_{nl} + \frac{\alpha g}{2\mu c} \hbar l\right)\right] \chi_{nl}(r) \vec{\Omega}_{+lm}(\vartheta, \varphi), \tag{16}$$

and

$$\vec{\psi}_-(\vec{x}, t) = \sum_{\substack{n,l,m \\ l \geq 1}} c_{-nlm} \exp\left[-\frac{it}{h} \left(E_{nl} - \frac{\alpha g}{2\mu c} \hbar(l+1)\right)\right] \chi_{nl}(r) \vec{\Omega}_{-lm}(\vartheta, \varphi). \tag{17}$$

Next, we apply to the components of  $\vec{\psi}_+$  and  $\vec{\psi}_-$  the reasoning presented in the Appendix. To this end, we assume that  $c_{+nlm}$  in (16) are different from zero only when  $m = l + \frac{1}{2}$  or  $m = -l - \frac{1}{2}$ . Correspondingly, we have

$$\vec{\psi}_+(x, t) = \begin{pmatrix} \psi_1(\vec{x}, t) \\ 0 \end{pmatrix} \quad \text{or} \quad \vec{\psi}_-(\vec{x}, t) = \begin{pmatrix} 0 \\ \psi_2(\vec{x}, t) \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned}\psi_1(\vec{x}, t) &= \sum_{n,l=0}^{\infty} c_{nl} \exp \left[ -\frac{it}{\hbar} \left( E_{nl} + \frac{\alpha g}{2\mu c} \hbar l \right) \right] \chi_{nl}(r) Y_l^l(\vartheta, \varphi), \\ \psi_2(\vec{x}, t) &= \sum_{n,l=0}^{\infty} d_{nl} \exp \left[ -\frac{it}{\hbar} \left( E_{nl} + \frac{\alpha g}{2\mu c} \hbar l \right) \right] \chi_{nl}(r) Y_{-l}^l(\vartheta, \varphi),\end{aligned}\quad (18')$$

as it follows from (9). Similarly, we assume that  $c_{-nlm} \neq 0$  only when  $m = l - \frac{1}{2}$  or  $m = -l + \frac{1}{2}$ . Then, respectively,

$$\vec{\psi}_-(\vec{x}, t) = \begin{pmatrix} \exp(-i\varphi) \operatorname{ctg} \vartheta \tilde{\psi}_1(\vec{x}, t) \\ \tilde{\psi}_1(\vec{x}, t) \end{pmatrix}, \quad \vec{\psi}_-(\vec{x}, t) = \begin{pmatrix} \tilde{\psi}_2(\vec{x}, t) \\ \exp(i\varphi) \operatorname{ctg} \vartheta \tilde{\psi}_2(\vec{x}, t) \end{pmatrix}, \quad (19)$$

where

$$\begin{aligned}\tilde{\psi}_1(\vec{x}, t) &= \sum_{n=0, l=1}^{\infty} \tilde{c}_{nl} \exp \left[ -\frac{it}{\hbar} \left( E_{nl} - \frac{\alpha g}{2\mu c} \hbar(l+1) \right) \right] \chi_{nl}(r) Y_l^l(\vartheta, \varphi), \\ \tilde{\psi}_2(\vec{x}, t) &= \sum_{n=0, l=1}^{\infty} \tilde{d}_{nl} \exp \left[ -\frac{it}{\hbar} \left( E_{nl} - \frac{\alpha g}{2\mu c} \hbar(l+1) \right) \right] \chi_{nl}(r) Y_{-l}^l(\vartheta, \varphi),\end{aligned}\quad (19')$$

as it follows from (9) and from the fact that for  $l \geq 1$

$$Y_{\pm l \mp 1}^l(\vartheta, \varphi) = -\sqrt{2l} Y_{\pm l}^l(\vartheta, \varphi) \operatorname{ctg} \vartheta \exp(\mp i\varphi).$$

The role of the above assumptions is to eliminate from (16), (17) spherical functions with  $m \pm 1/2 \neq \pm l$ . Then, we can apply the reasoning presented in the Appendix.

The wave packet constructed in the Appendix is concentrated at the equatorial circle,  $\vartheta = \pi/2$ . Therefore, if  $\vec{\psi}_1(\vec{\psi}_2)$  in (19) is such a wave packet, then the upper (lower) component of  $\vec{\psi}_-$  approximately vanishes.

Finally, we assume that at the initial time  $t = 0$

$$\psi_i(\vec{x}, t = 0) = \tilde{\psi}_i(\vec{x}, t = 0), \quad i = 1, 2.$$

We also assume that for  $t = 0$   $\psi_i(\vec{x}, t = 0)$  have the form (A.4), (A.10) with the same  $p_0$ . See the first paragraph of Section 4 for a comment on this point. Then, applying the reasoning presented in the Appendix we obtain that for all  $t$ , in the limit  $\hbar \rightarrow 0$ ,

$$\begin{aligned}\psi_1(r, \vartheta, \varphi, t) &= \psi_1 \left( r, \vartheta, \varphi - \left( \omega_0 + \frac{g\alpha}{2\mu c} \right) t, \quad t = 0 \right), \\ \tilde{\psi}_1(r, \vartheta, \varphi, t) &= \psi_1 \left( r, \vartheta, \varphi - \left( \omega_0 - \frac{g\alpha}{2\mu c} \right) t, \quad t = 0 \right).\end{aligned}\quad (20)$$

Similarly,

$$\begin{aligned}\psi_2(r, \vartheta, \varphi, t) &= \psi_2\left(r, \vartheta, \varphi + \left(\omega_0 + \frac{g\alpha}{2\mu c}\right)t, \quad t = 0\right), \\ \psi_2(r, \vartheta, \varphi, t) &= \psi_2\left(r, \vartheta, \varphi + \left(\omega_0 - \frac{g\alpha}{2\mu c}\right)t, \quad t = 0\right).\end{aligned}\quad (21)$$

Thus, we see that the initial wave packet has dissociated into two wave packets moving with different frequencies  $\omega = \omega_0 \pm \frac{g\alpha}{2\mu c}$ ,  $\omega_0^2 = \frac{1}{2} g^2 \alpha^2 \mu^{-2} c^{-2}$  along the equatorial circle,  $\vartheta = \pi/2$ , with the radius  $r_0 = \langle E_{nl} \rangle / \mu \omega_0^2$ . The reason for the splitting is that the time dependent phase factors in  $\vec{\psi}_+$ ,  $\vec{\psi}_-$  are different.

### 3. The color-tilt potential

As the second example we would like to analyse scattering of a wave packet (in the limit  $\hbar \rightarrow 0$ ), on the color-tilt potential (2). In order to avoid the trivial case, we assume that  $\vec{n} \neq (\delta^{a3})$ . Thus, at  $z = 0$  the potential  $\vec{A}_0$  suffers a sharp change of its direction in color space. The potential (2) gives the following field strengths

$$F_{ik}^a = 0, \quad F_{0i}^a = -\delta^{a3} \alpha g (\vec{n} \vec{\sigma} - \sigma^3) \delta(z).$$

The Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t} \vec{\psi} = H \vec{\psi}, \quad H = (2\mu)^{-1} \hat{\vec{p}}^2 + g \vec{A}_0. \quad (22)$$

We would like to find the solution of (22) such that at the initial instant  $t_0 \rightarrow -\infty$  it has the form of a wave packet localized at  $\vec{x}_0 = (0, 0, z_0)$ ,  $z_0 \rightarrow -\infty$ , with the average momentum  $\vec{p}_0 = \hbar \vec{k}_0 = (0, 0, p_0)$ ,  $p_0 > 0$ . Of course, we consider the wave packet with small dispersions of position and momentum. The initial color direction we shall choose to be  $(1, 0)$ . More general  $\vec{x}_0$ ,  $\vec{p}_0$ , and the color direction can be trivially introduced with the help of translations, Galilean boosts, and  $SU(2)$  gauge transformations, respectively.

As the first step, let us consider the eigenstates of  $H$  which describe the scattering of a plane wave on the potential (22) the  $\langle \vec{x} | \vec{p} \rangle_{\text{in}}$  states. For  $z < 0$  the equation

$$H \vec{\psi} = E \vec{\psi} \quad (23)$$

has the solution

$$\begin{aligned}\vec{\psi}(\vec{x}) &= \exp(i\vec{k}_\perp \vec{x}_\perp) \left[ a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(ik_+ z) \right. \\ &\quad \left. + b \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-ik_+ z) + c \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(ik_- z) + d \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(-ik_- z) \right],\end{aligned}\quad (24)$$

where  $\vec{k}_\perp = (k_1, k_2, 0)$ ,  $\vec{x}_\perp = (x, y, 0)$ , and

$$k_\pm = \frac{1}{\hbar} [2\mu(E_z \mp g\alpha)]^{1/2} \equiv \frac{P_\pm}{\hbar}, \quad E_z = E - \frac{\hbar^2}{2\mu} \vec{k}_\perp^2. \quad (25)$$

In order to have the  $\langle \vec{x} | \vec{p} \rangle_{\text{in}}$  solution with the color direction  $(1, 0)$  and momentum  $\vec{p} = \hbar(\vec{k}_\perp, k_+)$ , we assume that

$$E_z > g\alpha, \quad a = 1, \quad c = 0. \quad (26)$$

For  $z > 0$  we have from (23) that

$$\begin{aligned} \vec{\psi}(\vec{x}) = & \exp(i\vec{k}'_\perp \vec{x}_\perp) [\alpha \vec{e}_+ \exp(ik'_+ z) \\ & + \beta \vec{e}_+ \exp(-ik'_+ z) + \gamma \vec{e}_- \exp(ik'_- z) + \delta \vec{e}_- \exp(-ik'_- z)], \end{aligned} \quad (27)$$

where

$$k'_\pm = \frac{1}{\hbar} [2\mu(E'_z \mp g\alpha)]^{1/2}, \quad E'_z = E - \frac{\hbar^2}{2\mu} k_\perp'^2, \quad (28)$$

and

$$\vec{e}_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm \frac{n_1 - in_2}{\sqrt{1 \mp n_3}} \\ \sqrt{1 \mp n_3} \end{pmatrix} \quad (29)$$

are the normalized eigenvectors of  $n\vec{\sigma}$ , i.e.

$$(\vec{n}\vec{\sigma})\vec{e}_\pm = \pm \vec{e}_\pm. \quad (30)$$

In order to have for  $z > 0$  the outgoing waves only, we assume that

$$\beta = \delta = 0. \quad (31)$$

The patching conditions at  $z = 0$  (they follow from (23) by integrations over  $z$  in the interval  $(-\varepsilon, \varepsilon)$ ),

$$\begin{aligned} \psi(\vec{x}_\perp, z = 0_+) &= \psi(\vec{x}_\perp, z = 0_-), \\ \partial_z \psi(\vec{x}_\perp, z = 0_+) &= \partial_z \psi(\vec{x}_\perp, z = 0_-), \end{aligned} \quad (32)$$

imply that

$$\vec{k}'_\perp = \vec{k}_\perp, \quad (33)$$

and

$$\gamma = -\sqrt{2} \frac{2k_+}{k_+ + k_-} \left( 1 + \frac{4k_+ k_-}{(k_+ + k_-)^2} \frac{1+n_3}{1-n_3} \right)^{-1} \frac{\sqrt{1+n_3}}{n_1 - in_2} a, \quad (34)$$

$$\alpha = -\frac{2k_-}{k_+ + k_-} \sqrt{\frac{1+n_3}{1-n_3}} \gamma, \quad (35)$$

$$d = \frac{1}{\sqrt{2}} \sqrt{1-n_3} \alpha + \frac{1}{\sqrt{2}} \sqrt{1+n_3} \gamma, \quad (36)$$

$$b = -1 + \frac{1}{\sqrt{2}} (n_1 - in_2) \left( \frac{\alpha}{\sqrt{1-n_3}} - \frac{\gamma}{\sqrt{1+n_3}} \right). \quad (37)$$

In order to investigate the scattering of the wave packet we consider

$$\vec{\psi}(\vec{x}, t) = \int d^3\vec{p} f(\vec{p}) \langle \vec{x} | \vec{p} \rangle_{in} \exp\left(-\frac{i}{\hbar} tE\right), \quad (38)$$

where

$$f(\vec{p}) = N \exp\left[-\frac{a}{2}(\vec{p}-\vec{p}_0)^2 - \frac{i}{\hbar} \vec{p}\vec{x}_0\right], \quad (39)$$

$N$  is a normalization constant. We shall assume that the constant  $a$ , which is related to the dispersion of momentum by

$$\sigma(\vec{p}) = \langle (\vec{p}-\vec{p}_0)^2 \rangle = \frac{3}{2} a^{-1},$$

depends on  $\hbar$  like  $\hbar^{-\alpha}$ ,  $0 < \alpha < 1$ . Then,

$$\sigma(\vec{x}) = \langle (\vec{x}-\vec{x}_0)^2 \rangle \sim \hbar a,$$

and in the limit  $\hbar \rightarrow 0$   $\sigma(\vec{p})$  and  $\sigma(\vec{x})$  vanish.

Let us calculate  $\vec{\psi}(\vec{x}, t)$  for  $t \rightarrow \pm\infty$ , in the limit  $\hbar \rightarrow 0$ . First let us consider  $\vec{\psi}(\vec{x}, t)$  for  $z > 0$ . Then, we have the following formula

$$\begin{aligned} z > 0: \quad \vec{\psi}(\vec{x}, t) &= \int d^3\vec{p} f(\vec{p}) \exp\left[\frac{i}{\hbar}(\vec{p}_\perp \vec{x}_\perp - Et)\right] \\ &\times \left(\alpha \vec{e}_+ \exp\left(\frac{i}{\hbar} p_+ z\right) + \gamma \vec{e}_- \exp\left(\frac{i}{\hbar} p_- z\right)\right), \end{aligned} \quad (40)$$

where

$$E = \frac{1}{2\mu}(\vec{p}_\perp^2 + p_+^2) + g\alpha, \quad p_- = \sqrt{p_+^2 + 4\mu g\alpha}, \quad \vec{p} = (\vec{p}_\perp, p_+). \quad (41)$$

In the limit  $\hbar \rightarrow 0$   $\vec{\psi}(\vec{x}, t)$  will not vanish only for such  $\vec{x}$  and  $t$  that the singular in  $\hbar$  part of the phase of the integrand in (40) is stationary for  $\vec{p} = \vec{p}_0$ . Thus, for the first term on the r.h.s. of (40) (that containing  $\vec{e}_+$ ) the condition is

$$\frac{\partial}{\partial \vec{p}}(p_+ z + \vec{p}_\perp \vec{x}_\perp - \vec{p}_0 \vec{x} - Et)|_{\vec{p}=\vec{p}_0} = 0. \quad (42)$$

It follows that this part of  $\vec{\psi}(\vec{x}, t)$  does not vanish only when

$$\begin{aligned} \vec{x}_\perp &\approx \mu^{-1} \vec{p}_{0\perp} t + \vec{x}_{0\perp} = \vec{x}_{0\perp}, \quad (\text{because } \vec{p}_{0\perp} = 0) \\ z &\approx z_+ \equiv \mu^{-1} p_0 t + z_0. \end{aligned} \quad (43)$$

For the second term on the r.h.s. of (40), with the singular part of the phase being  $p_- z + \vec{p}_\perp \vec{x} - Et - \vec{x}_0 \vec{p}$ , we obtain

$$\begin{aligned} \vec{x}_\perp &\approx \vec{x}_{0\perp}, \\ z &\approx z_- \equiv (1 + 4\mu g\alpha p_0^{-2})^{1/2} (\mu^{-1} p_0 t + z_0). \end{aligned} \quad (44)$$



We see that  $z > 0$ ,  $t \rightarrow -\infty$  is not consistent with (43), (44) because  $p_0 > 0$ . Therefore,  $\vec{\psi}(\vec{x}, t) \approx 0$  for  $z > 0$ ,  $t \rightarrow -\infty$ . On the other hand, for  $t \rightarrow +\infty$ ,  $\vec{\psi}(\vec{x}, t)$  does not vanish for  $\vec{x}_\perp \approx \vec{x}_{0\perp}$ , and  $z = z_+$  or  $z = z_-$ . Thus, for  $t \rightarrow +\infty$  we have two wave packets, with different positions and velocities, as it follows from comparison of (43) with (44).

Now let us consider in the analogous manner  $\vec{\psi}(\vec{x}, t)$  for  $z < 0$ ,  $t \rightarrow \pm\infty$ , in the limit  $\hbar \rightarrow 0$ . We have

$$z < 0: \quad \vec{\psi}(\vec{x}, t) = \vec{\psi}_1(\vec{x}, t) + \vec{\psi}_2(\vec{x}, t) + \vec{\psi}_3(\vec{x}, t),$$

where

$$\begin{aligned} \vec{\psi}_i(\vec{x}, t) &= \int d^3\vec{p} f(\vec{p}) \exp \left[ \frac{i}{\hbar} (\vec{p}_\perp \vec{x}_\perp - Et) \right] \varphi_i, \\ \varphi_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp \left( \frac{i}{\hbar} p_+ z \right), \quad \varphi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp \left( -\frac{i}{\hbar} p_+ z \right), \\ \varphi_3 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp \left( -\frac{i}{\hbar} p_- z \right). \end{aligned}$$

We obtain that  $\vec{\psi}_1$  is different from zero only when

$$\vec{x}_\perp \approx \vec{x}_{0\perp},$$

$$z \approx z_1 \equiv \mu^{-1} p_0 t + z_0. \quad (45)$$

For  $\vec{\psi}_2$  we have

$$\vec{x}_\perp \approx \vec{x}_{0\perp}, \quad z \approx z_2 \equiv -\mu^{-1} p_0 t - z_0, \quad (46)$$

and for  $\vec{\psi}_3$

$$\vec{x}_\perp \approx \vec{x}_{0\perp}, \quad z \approx z_3 \equiv -(1 + 4\mu g \alpha p_0^{-2})^{1/2} (\mu^{-1} p_0 t + z_0). \quad (47)$$

Thus, we see that  $\vec{\psi}_2, \vec{\psi}_3$  vanish for  $t \rightarrow -\infty$ , because then (46), (47) are inconsistent with the assumption  $z < 0$ . On the other hand,  $\vec{\psi}_2, \vec{\psi}_3$  do not vanish for  $t \rightarrow +\infty$ . They represent two reflected wave packets, and they have different positions and velocities. For  $t \rightarrow -\infty$  the only nonvanishing contribution comes from  $\vec{\psi}_1$ .  $\vec{\psi}_1$  vanishes for  $t \rightarrow +\infty$ , because then (45) is not consistent with  $z < 0$ .

On the whole, the picture of the scattering is the following. For large, negative  $t$  we have the initial wave packet localized at  $z_1 = p_0 \mu^{-1} t + z_0 < 0$ . For  $t \rightarrow +\infty$  we have two reflected wave packets ( $\vec{\psi}_2, \vec{\psi}_3$  above) and two transmitted ones given by (40) (located at  $z_+, z_-$ ). The presence of the reflected wave packet is not a surprise, because the potential (2) is not smooth. The new feature is that there are two transmitted and two reflected wave packets in the limit  $\hbar \rightarrow 0$ .

The annoying fact that the reflected wave packets survive the limit  $\hbar \rightarrow 0$  we ascribe to the singular character of the potential. We expect that for a smooth potential the two reflected wave packets will vanish in the  $\hbar \rightarrow 0$  limit. Such a phenomenon is present in ordinary QM, [9]. Namely, for  $V(x) = -V_0 \theta(-x)$  the reflection coefficient for  $E < 0$ ,

$$R = V_0^2 (\sqrt{E+V_0} + \sqrt{E})^{-4}$$

does not vanish when  $\hbar \rightarrow 0$  because it does not contain  $\hbar$ . On the other hand, for the smoothened potential

$$V(x) = -V_0 \left(1 + \exp \frac{x}{a}\right)^{-1}$$

for  $E > 0$ ,

$$R = \sinh^2 [\pi a(\kappa - k)] \sinh^{-2} [\pi a(\kappa + k)],$$

where  $k = \hbar^{-1}(2\mu E)^{1/2}$ ,  $\kappa = \hbar^{-1}[2\mu(E + V_0)]^{1/2}$ . We see that this  $R$  vanishes when  $\hbar \rightarrow 0$ . Thus, the limits  $\hbar \rightarrow 0$ ,  $a \rightarrow 0$  are not interchangeable.

#### 4. Remarks

1°. It is necessary to comment on the choice of the initial data for the wave packets. We recall that we have explicitly specified the initial average position, the initial average momentum (with small dispersions) and the color direction of the wave packet. This corresponds to the fact that  $\hat{x}$ ,  $\hat{p}$ ,  $\hat{\sigma}/2$  form the complete set of quantum observables for the spinless colored particle, i.e. any other observable is a function of these three ones.

The canonical momentum  $\hat{\vec{p}}$  is not related to the velocity in a simple way. The velocity is given by

$$\dot{\vec{x}}(t) = \mu^{-1} \langle \psi | \hat{\vec{\Pi}} | \psi \rangle,$$

where  $\hat{\vec{\Pi}} = \hat{\vec{p}} - g/c\hat{\vec{A}}$  is the mechanical momentum, see, e.g., [3]. However, fixing  $\vec{p}$  at  $t = 0$  is equivalent to fixing  $\dot{\vec{x}}$  at  $t = 0$  because they differ by the constant  $g/c \langle \psi | \hat{\vec{A}} | \psi \rangle$  ( $t = 0$ ), where  $|\psi\rangle$  is the wave packet. The use of  $\vec{p} \equiv \langle \psi | \hat{\vec{p}} | \psi \rangle$  for specifying the initial data has the disadvantage that  $\vec{p}(t)$  is not gauge invariant, in contrary to  $\dot{\vec{x}}(t)$ . However, this drawback is not important for our investigations because the freedom of performing of gauge transformations is largely restricted by assuming the concrete form (1), (2) of the considered gauge potentials.

Observe that different components of  $\hat{\vec{\Pi}}$  do not commute,

$$[\hat{\Pi}^i, \hat{\Pi}^k] = -i\hbar \hat{F}_{ik}. \quad (48)$$

In the Abelian case the r.h.s. of (48) is of order  $\hbar$ . Therefore, one can fix large (macroscopic) average values of all components of  $\hat{\vec{\Pi}}$ ,  $\langle \hat{\Pi}^i \rangle = \mu \dot{x}^i$ , with small dispersions (of order  $\hbar$ ).

The situation is more complicated in the non-Abelian case. Namely, we have

$$\hat{F}_{ik} = \partial_i \hat{A}_k - \partial_k \hat{A}_i - \frac{ig}{\hbar c} [\hat{A}_i, \hat{A}_k]. \quad (49)$$

From (49) we see that the r.h.s. of (48) is not of order  $\hbar$ , in general. Therefore, we cannot specify  $\dot{\vec{x}}$  with small dispersions (of order  $\hbar$ ) for all three components. For noncommuting  $\hat{A}_i$  the dispersions  $\sigma(\hat{\Pi}^i)$  can be regarded as relatively small quantities only in the limit

of large velocities  $\dot{x}^i$ . Thus, in the non-Abelian case the use of the mechanical momentum is not so obvious as it is in the Abelian case.

Observe that the two components  $\vec{\psi}_+$ ,  $\vec{\psi}_-$  of the initial wave packet  $\vec{\psi} = \vec{\psi}_+ + \vec{\psi}_-$ , formulae (18), (19) have different average values of  $\hat{\Pi}^i$ . These values are consistent with the time development of  $\vec{\psi}_+$ ,  $\vec{\psi}_-$ , formulae (20), (21). Namely,

$$\dot{\vec{x}}_{\pm}|_{t=0} \equiv \mu^{-1} \langle \psi_{\pm} | \hat{\Pi} | \psi_{\pm} \rangle = \mu^{-1} \vec{p}_0 \pm \frac{\alpha g r_0}{2c\mu} \begin{pmatrix} -\sin \varphi_0 \\ \cos \varphi_0 \\ 0 \end{pmatrix},$$

where  $\vec{p}_0$  is given by (A.6). Therefore, it is tempting to try to modify slightly the wave packet  $\vec{\psi}_+$ , by taking  $p'_0$  instead of  $p_0$  in it, where  $p'_0 = p_0 + \alpha g r_0 / \mu$ . Then, the initial velocities of  $\vec{\psi}_+$ ,  $\vec{\psi}_-$  are identical, and one could expect that the wave packet would not split. However, this is not the case. The reason is that for the circular motion there is the relation between the modulus of momentum and the radius of the orbit,  $p_0 = \mu \omega r_0$ , derived in the Appendix. Therefore, the wave packet  $\vec{\psi}_-$ , with  $p_0$  replaced by  $p'_0$ , will move along a different circle. If we force that the two wave packets are localized at the same point  $(r_0, \vartheta_0, \varphi_0)$  at the initial time  $t = 0$ , then the motion of the  $\vec{\psi}_-$  component cannot be along a circular orbit.

Let us remark that for the color-tilt potential (2) we have  $\hat{\Pi}_i = p_i$ . Therefore  $\hat{\Pi}^i$  is of no use for understanding of the presence of the two transmitted wave packets in the potential (2).

The presence of  $\hbar$  in  $\hat{F}_{ik}$  was already noticed in [10]. It follows from a dimensional analysis based on the assumptions that the dimension of the action  $-1/4 \int d^4x F_{\mu\nu}^a F^{a\mu\nu}$  is  $\hbar$  and that  $g\hat{A}_0$  has the dimension of energy, as it follows from Eq. (22). Then, we obtain in the (cm, g, sec) units that

$$[g] = g^{1/2} \text{ sec}^{-3/2} \text{ cm}^2, \quad [\hat{A}_i] = [\hat{A}_0] = g^{1/2} \text{ sec}^{-1/2}.$$

In these units the gauge transformation has the form

$$\hat{A}'_{\mu} = \omega \hat{A}_{\mu} \omega^{-1} + i g^{-1} \hbar \partial_{\mu} \omega \omega^{-1}, \quad \omega \in \text{SU}(2).$$

The formal limit  $\hbar \rightarrow 0$  in (49) does not make sense for noncommuting  $\hat{A}_i$ . However, in our paper the  $\hbar \rightarrow 0$  limit is merely a formal trick. We use it in order to simplify reasonings. This trick is legitimate for investigations of the wave packets moving in fixed external Yang-Mills fields. In this problem the formal  $\hbar \rightarrow 0$  limit is equivalent to the true classical limit, that is the limit of large expectation values with small dispersions and  $\hbar$  fixed.

2°. The color-tilt example can be generalized by introducing many color-tilts placed at different  $z$ 's. Each such tilt will split the incoming wave packets, multiplying the number of them. Thus, the number of resulting wave packets depends on the details of the gauge potential.

It is easy to see that this number depends also on the gauge group, and on the representation of the gauge group to which the colored matter field (i.e. wave function) belongs.

3° One could wonder why the wave packet described in Section 2 splits into two

parts merely, in spite of the fact that the gauge potential (1) changes its direction in color space with  $\vec{x}$ . One could expect that each of the two parts should split further.

The reason that this does not happen is that the potential (1) possesses the integral of motion  $\hat{K} = \vec{\sigma} \hat{L}$ . This operator divides the Hilbert space of wave functions into two orthogonal parts,  $H = H_+ \oplus H_-$ , where  $H_-$  is spanned by the eigenvectors of  $\hat{K}$  with the corresponding eigenvalues  $\kappa = \hbar l$ ,  $l = 0, 1, 2, \dots$ , and  $H_+$  is spanned by the eigenvectors with the eigenvalues  $\kappa = -\hbar(l+1)$ ,  $l = 0, 1, 2, \dots$ . The crucial property of  $\hat{K}$  is that one can construct a localized wave packet using functions only from  $H_+$  or only from  $H_-$  (the  $\vec{\psi}_+$ ,  $\vec{\psi}_-$  wave packets from Section 2). Because  $\hat{K}$  is the integral of motion,  $\vec{\psi}_+$  ( $\vec{\psi}_-$ ) will remain in  $H_+$  ( $H_-$ ) for all  $t$ . This makes the further splitting impossible, because the color degrees of freedom are effectively eliminated within  $H_+$  ( $H_-$ ). Namely, the time-dependent phase factors in  $H_+$  and  $H_-$  (see (18') and (19')) do not depend on color degrees of freedom. The general time-dependent phase factor in  $H$  (see (10) and (12)) does depend on color degrees of freedom via the sign of  $\kappa$ .

For the color-tilt potential the situation is different. This potential also possesses the integral of motion  $U$ , defined as follows

$$U\vec{\psi}(x, y, z, t) = u(\pi)\vec{\psi}(x, y, -z, t),$$

where  $u(\pi)$  is the rotation in color space interchanging  $\sigma^3$  with  $\vec{n}\vec{\sigma}$ , i.e.,

$$u(\pi)\sigma^3 = \vec{n}\vec{\sigma}u(\pi), \quad u \in \text{SU}(2).$$

Because  $U^2 = 1$ , the eigenvalues of  $U$  are  $\pm 1$ . Thus, we can decompose  $H$  into two parts,  $H = H_+ \oplus H_-$ . However, the wave functions belonging to  $H_+$  ( $H_-$ ) are, apart from the color rotation given by  $u(\pi)$ , symmetric (antisymmetric) under  $z \rightarrow -z$ . Therefore, one cannot construct a single wave packet located at  $z < 0$  from functions belonging only to  $H_+$  or  $H_-$ , because such a wave packet is neither symmetric nor antisymmetric under  $z \rightarrow -z$ . The fact that  $U$  is not useful for an effective reduction of color degrees of freedom can also be seen from the fact that  $U$  is a nonlocal operator — it relates  $\psi_i$  at different points, while for the reduction of color degrees of freedom we need a relation between the two components of  $\vec{\psi} = (\psi_1, \psi_2)$  at each point  $\vec{x}$ . Thus, the color-tilt potential does not possess an integral of motion which could effect the reduction of color degrees of freedom. This is the reason why the wave packet, which preserves its identity for certain time, can split when it reaches the color-tilt at  $z = 0$ .

4° There exists the extensive paper by Boulware et al., [11], devoted to the quantum mechanical scattering of a color charged particle on the potential of the t'Hooft-Polyakov magnetic monopole. The  $\hbar \rightarrow 0$  limit in the non-Abelian case is not discussed in that paper. The authors discuss this limit only for the Abelian Dirac monopole (Sect. V of that paper), with the expected result that there exists a classical trajectory obeying the classical equations of motion. However, from formulae derived in that paper we deduce that the  $\hbar \rightarrow 0$  limit in the non-Abelian case will not lead to a definite classical trajectory. For example, it is shown in the Section IV of that paper that for  $\mu r \gg 1$  the non-Abelian problem can be reduced to a set of the Abelian ones, with the Abelian magnetic charge being  $g = n$ , where  $n$  can be any eigenvalue of  $T_3$ .  $T_3$  is the third generator of the representa-

tion to which the wave function of the particle belongs. Thus, for each  $n$  we obtain a different classical trajectory.

The potential of the t'Hooft-Polyakov magnetic monopole allows for the integral of motion  $\hat{K} = \vec{\sigma} \hat{L}$  (the one used for our potential (1)) which can be used to effectively reduce the color degrees of freedom, as discussed in the paragraph 3° of this Section. Therefore, we do not expect that for this potential the component wave packets  $\vec{\psi}_+$ ,  $\vec{\psi}_-$ , constructed as in Sect. 2 with the appropriately modified radial function  $\chi(r)$ , will split in the course of time.

5° Investigations carried out in this paper entirely lie within the framework of elementary QM. We have shown that the presence of non-Abelian gauge fields causes rather unexpected phenomena even on this elementary level. Examples of other interesting phenomena present on the level of elementary QM with external Yang-Mills fields can be found in, e.g., [11].

Of course, at the moment we can hardly hope to obtain from such simple investigations of classical chromodynamics (that is, classical quark fields interacting with classical gluon fields) realistic numerical values of parameters characterising hadrons. Such numbers have likely to be calculated within the framework of fully second-quantized theory. Pessimistic point of view is that classical chromodynamics should rather be regarded as a model, much in the same spirit as 2-dim. space-time quantum field models. Optimistic point of view is that classical chromodynamics should be completed with a merely rough estimate of quantum effects, in order to be able to provide the credible numbers. Anyway, we are convinced that thorough examination of classical chromodynamics will provide facilities for understanding the full theory of color interactions.

I would like to thank H. B. Nielsen from the Niels Bohr Institute for stimulating discussions which have induced me to continue investigations of the colored wave packets. Also, I would like to thank P. A. Horváthy from the Bielefeld University for calling my attention to the paper by D. G. Boulware et al.

## APPENDIX

We shall present here the construction of wave packet moving along the circular trajectory in the potential  $V = (1/2)\mu\omega^2 r^2$ .<sup>1</sup>

The unnormalized eigenfunctions of the Hamiltonian  $H = 2\mu^{-1}\hat{p}^2 + V$ , in the spherical coordinates, are [8]

$$\psi_{nlm}(r, \vartheta, \varphi) = r^{l+1} \exp\left(-\frac{\lambda}{2} r^2\right) F(-n, l + \frac{3}{2}; \lambda r^2) Y_m^l(\vartheta, \varphi), \quad (\text{A1})$$

where  $\lambda = \hbar^{-1}\mu\omega$ ,  $n = 0, 1, 2, \dots$ . The corresponding eigenvalues are

$$E_{nl} = \hbar\omega(2n + l + \frac{3}{2}). \quad (\text{A2})$$

---

<sup>1</sup> In spite of the fact that this is an exercise in elementary QM, we have not found it in literature — this is why we will present this calculation explicitly.

The time development of a wave function is given by

$$\psi(r, \vartheta, \varphi, t) = \sum_{n,l,m} \exp\left(-\frac{i}{\hbar} t E_{nl}\right) \psi_{nlm}(r, \vartheta, \varphi). \quad (\text{A3})$$

Now, let us prepare the initial wave packet  $\psi(r, \vartheta, \varphi, t = 0)$ . This packet is to be localized at  $(r_0, \vartheta_0, \varphi_0)$ , and it should have the initial average momentum  $\vec{p}_0$ , with a small dispersion. Then, we can expect that its time evolution will follow the corresponding classical trajectory defined by the initial position  $\vec{x}_0$  and the initial momentum  $\vec{p}_0$ . Such a wave packet is provided by

$$\psi(\vec{x}, t = 0) = \exp\left(\frac{i}{\hbar} \vec{p}_0 \vec{x}\right) f(\vec{x}), \quad (\text{A4})$$

where  $f(\vec{x})$  is concentrated at  $\vec{x} = \vec{x}_0$ , and each component  $\vec{p}_0$  is assumed to be large in comparison to

$$\hbar(\int (\nabla|f|)^2 d^3\vec{x})^{1/2} \quad \text{and} \quad \hbar(\int |f|^2 (\nabla\chi)^2 d^3\vec{x})^{1/2},$$

where  $\chi$  is the phase of  $f$ ,  $f = |f| \exp(i\chi)$ . Then it is easy to check that in the state (A4)

$$\langle \hat{\vec{p}} \rangle = \vec{p}_0, \quad \sigma^{1/2}(\vec{p}) = (\langle \hat{\vec{p}}^2 \rangle - \langle \hat{\vec{p}} \rangle^2)^{1/2} \ll |p_0^i|. \quad (\text{A5})$$

Observe that (A5) can be achieved in two ways. Formally, by putting  $\hbar \rightarrow 0$ , or more physically, by taking macroscopic, i.e., large value of  $\vec{p}_0$ . For our purposes, we find it convenient to use the formal  $\hbar \rightarrow 0$  procedure.

The next step is to expand (A4) into the eigenfunctions (A1) and to apply (A3). In general, this leads to rather cumbersome calculations. Therefore, we restrict ourselves to the simplest case, when the wave packet performs the circular motion along the equator  $\vartheta = \pi/2$ ,  $r = r_0$ . Then, the initial momentum is  $(\vec{p} \stackrel{\text{df}}{=} \langle \psi | \hat{\vec{p}} | \psi \rangle (t = 0))$

$$\vec{p} = \vec{p}_0 = p_0 \begin{pmatrix} -\sin \varphi_0 \\ \cos \varphi_0 \\ 0 \end{pmatrix}. \quad (\text{A6})$$

It follows from (A4) that in this case

$$\begin{aligned} \langle L_z \rangle &\approx p_0 r_0, \\ \langle L_x \rangle &\approx \langle L_y \rangle \approx 0, \\ \langle E \rangle &\approx (2\mu)^{-1} p_0^2 + \frac{1}{2} \mu \omega^2 r_0^2 + O(\hbar), \end{aligned} \quad (\text{A7})$$

with small dispersions and independently of details of the shape of  $f(\vec{x})$ . (A7) holds for all times because  $\hat{L}$ ,  $H$  are integrals of motion.

Now let us choose a convenient  $f(\vec{x})$ . Because the energy  $E$ , the angular momentum  $\hat{L} = (\hat{L}^2)^{1/2}$ , and  $\hat{L}_z$  are integrals of motion, we fix the average values of them. This gives

the average  $n$  and  $l$ ,

$$\langle l + \frac{1}{2} \rangle \approx \frac{1}{\hbar} \langle \hat{L} \rangle,$$

$$\langle n \rangle = (2\omega\hbar)^{-1} (\langle E \rangle - \omega\langle L \rangle) - \frac{3}{4}. \quad (\text{A8})$$

For the motion along the equator we have  $\vec{L} = (0, 0, L_z)$  at the classical level. Therefore, on the quantum level we assume that

$$\hbar^{-1} \langle L_z \rangle = \langle m \rangle = \langle l \rangle. \quad (\text{A9})$$

Then,

$$\langle l \rangle \approx p_0 r_0.$$

Now let us consider

$$f(r, \vartheta, \varphi) = \sum_{n,l} c_{ln} \chi_{nl}(r) P_l^l(\cos \vartheta) \exp [i(l - \langle l \rangle)\varphi], \quad (\text{A10})$$

where  $c_{ln}$  will be specified below. It is easy to check that for  $E_{ln} \gg \hbar\omega$ ,  $\chi_{nl}$  has a sharp maximum at

$$r_0^{n,l} = (\mu^{-1} \omega^{-2} E_{nl})^{1/2},$$

with the dispersion of order  $\hbar/\mu\omega$ . The approximate form of the dominant contribution to  $\chi_{nl}(r)$  is  $\exp [-\hbar^{-1} \mu\omega(r - r_0^{n,l})^2]$ . The subdominant contributions are of order  $E_{nl}^{-2}(\hbar\omega)^2$ . Observe that the average value of  $r_0^{n,l}$ , defined as  $r_0 = (\langle E_{nl} \rangle / \mu\omega^2)^{1/2}$  is in accordance with classical mechanics — the classical circular motion with the classical energy  $E$  requires the radius of the orbit to be just  $(E/\mu\omega^2)^{1/2}$ . We shall identify  $\tilde{r}_0$  with the  $r_0$  introduced earlier. Observe that this value of  $r_0$  gives  $p_0 = \mu\omega r_0$ . From (A7), (A8) it follows then that  $\langle n \rangle \approx 0$  for the circular orbit. Assuming that  $l$  is so large that  $E_{nl} \gg \hbar\omega$  (or that  $\hbar \rightarrow 0$ ), and neglecting the nondominant contributions to  $\chi_{nl}(r)$ , we may write

$$f(r, \vartheta, \varphi) \approx \exp \left( -\frac{\mu\omega}{2\hbar} r^2 \right) \sum_{n,l} c_{ln} \left( \frac{r}{a} \right)^{2n+l+1} (\sin \vartheta)^l \exp [i(l - \langle l \rangle)\varphi]. \quad (\text{A11})$$

In this formula the dominant contribution to  $\chi_{nl}(r)$  (that containing the highest power of  $\lambda r^2$  from  $F(-n, l + 3/2; \lambda r^2)$ ) is written in the precise form. Also, we have used the fact that

$$P_l^l(\cos \vartheta) \sim \sin^l \vartheta.$$

$a$  is a dimensional constant,  $[a] = [r]$ , introduced by a redefinition of  $c_{ln}$ . We shall take

$$c_{ln} = \delta_{n,0} \exp \left[ -\frac{b^2}{4} (l - \langle l \rangle)^2 - i(l - \langle l \rangle)\varphi_0 \right] \quad (\text{A12})$$

for

$$\langle l \rangle - N \leq l \leq \langle l \rangle + N, \quad N — \text{fixed}$$

and

$$c_{ln} = 0$$

for the other values of  $l$ . We assume that  $Nb \gg 1$ . Then, the sum over  $l$  in (A11) can be approximated by an integral yielding

$$f(r, \vartheta, \varphi) \approx r^{\langle l \rangle + 1} \sin^{\langle l \rangle} \vartheta \exp \left[ -\frac{1}{2\hbar} \mu \omega r^2 - \frac{4}{b^2} (\varphi - \varphi_0)^2 \right] \\ \times \exp \left[ \frac{4}{b^2} \ln^2 \frac{r}{a} \sin^2 \vartheta + \frac{8i}{b^2} (\varphi - \varphi_0) \sin \vartheta \ln \frac{r}{a} \right], \quad (\text{A13})$$

apart from unessential overall constant. In the limit  $\langle l \rangle \rightarrow \infty$  (equivalently,  $\hbar \rightarrow 0$ ),  $(\sin \vartheta)^{\langle l \rangle}$  behaves like  $\exp \left[ -\left( \vartheta - \frac{\pi}{2} \right)^2 \langle l \rangle \right]$ . The radial part of  $f(r, \vartheta, \varphi)$  has a sharp maximum at  $r_0$ . Taking sufficiently small  $b$  we can get also a very good localization in  $\varphi$ . Thus, the approximate expression (A13) shows that  $f(r, \vartheta, \varphi)$  defined by (A10) obeys all our requirements (specified below the formula (A4)) if  $b^2 \approx \hbar^\alpha$ , where  $0 < \alpha < 1$ .

Now, we can easily obtain the time development of the wave packet. At the initial time  $t = 0$  we have

$$\psi(r, \vartheta, \varphi, t = 0) = \exp(-i\langle l \rangle \varphi_0) \sum_{n,l} c_{ln} \chi_{nl}(r) P_l^l(\cos \vartheta) \exp(il\varphi), \quad (\text{A14})$$

because for the initial momentum (A6)

$$\psi(r, \vartheta, \varphi, t = 0) \approx \exp \left[ \frac{1}{\hbar} i p_0 r_0 \sin(\varphi - \varphi_0) \right] f(r, \vartheta, \varphi) \\ \approx \exp[i\langle l \rangle (\varphi - \varphi_0)] f(r, \vartheta, \varphi).$$

Here we have used (A8) and the fact that  $f(r, \vartheta, \varphi)$  is localized at  $r_0, \varphi_0$ . For  $t < 0$  we obtain from (A2), (A3), (A14), (A12) that

$$\psi(r, \vartheta, \varphi, t) = \exp(-i\langle l \rangle \varphi_0) \exp \left( -\frac{i}{\hbar} t \langle E \rangle \right) \\ \sum_l c_{ln} \exp[-it\omega(l - \langle l \rangle)] \chi_{\langle n \rangle l}(r) P_l^l(\cos \vartheta) \exp(il\varphi),$$

i.e.,

$$\psi(r, \vartheta, \varphi, t) = \exp \left[ -i\langle l \rangle \varphi_0 - \frac{i}{\hbar} t \langle E \rangle + it\omega \langle l \rangle \right] \psi(r, \vartheta, \varphi - \omega t). \quad (\text{A15})$$

Thus, the wave packet performs the motion along the equator  $r = r_0 = (\langle E \rangle / \mu \omega^2)^{1/2}$ , with the constant frequency  $\omega$ .

#### REFERENCES

- [1] Th. W. Ruijgrok, H.v.d. Vlist, *Physica* **101A**, 571 (1980); J. Azcarraga, J. Lukierski, *Phys. Lett.* **117B**, 51 (1982).
- [2] S. K. Wong, *Nuovo Cimento* **LXVA**, 689 (1970); A. Barducci, R. Casalbuoni, L. Lusanna,



- Nucl. Phys.* **B124**, 521 (1977); A. P. Balachandran, P. Salomonson, B. S. Skagerstam, J. O. Winnberg, *Phys. Rev.* **D15**, 2308 (1977).
- [3] H. Arodź, *Phys. Lett.* **116B**, 251 (1982); *Acta Phys. Pol.* **B13**, 519 (1982).
- [4] P. Ehrenfest, *Z. Phys.* **45**, 455 (1927).
- [5] H. Arodź, *Phys. Lett.* **116B**, 255 (1982); *Acta Phys. Pol.* **B14**, 13 (1983).
- [6] M. Bander, *Phys. Rep.* **75**, 205 (1981).
- [7] A. I. Akhiezer, V. B. Berestecki, *Quantum Electrodynamics*, Ch. 1, Sect. 4, Nauka, Moskva 1981.
- [8] S. Flügge, H. Marshal, *Problems in Quantum Mechanics*, Ch. 3, Ex. 35, Springer-Verlag, Berlin 1952.
- [9] I. I. Goldman, V. D., Krivchenkov, *Problems in Quantum Mechanics*, Section 2, Ex. 1, 2, Moscow 1957.
- [10] M. B. Halpern, W. Siegel, *Phys. Rev.* **D16**, 2486 (1977).
- [11] D. G. Boulware et al., *Phys. Rev.* **D14**, 2708 (1976). P. A. Horváthy, *The non-Abelian Bohm-Aharonov Effect*, Universität Bielefeld preprint, BITP 82/14, May 1982; R. Jackiv, C. Rebbi, *Phys. Rev.* **D13**, 3398 (1976).