EMBEDDING OF THE BACKGROUND SPACETIME OF A SOLUTION OF THE EINSTEIN-STRAUS-KLOTZ UNIFIED FIELD THEORY

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We obtain the complete analytic extension of the background spacetime of a static spherically symmetric solution of the Einstein-Straus-Klotz unified field theory by embedding it in a six-dimensional pseudo-Euclidean space. This throws some light on the nature of the background spacetime.

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1. Introduction

The spacetime to be considered in this paper is the background spacetime of a static spherically symmetric solution of the Einstein-Straus-Klotz unified field theory. This theory, which attempts to unify macroscopic gravitational and electromagnetic fields, is based on the weak field equations of Einstein and Straus [1] supplemented with the metric hypothesis introduced by Klotz [2]. For comprehensive discussion of the theory and the derivation of the solution under consideration see Klotz [3, and references given therein].

The line element of the (pseudo-Riemannian) background spacetime is of the form

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)^{-1}\frac{r_0^4dr^2}{(r_0^2+r^2)^2}-\frac{r_0^2r^2}{r_0^2+r^2}(d\theta^2+\sin^2\theta d\varphi^2), \tag{1}$$

where m and r_0 are parameters. Firstly, we observe that this line element is singular on the surface $r = 2m$ and at the point $r = 0$. On the other hand the radial geodesic equation [4] is given by

$$\ddot{r}+\frac{m}{r^2}\left(1+\frac{r^2}{r_0^2}\right)^2-\frac{2r}{r_0^2}\left(k+\frac{2m}{r}\right)\left(1+\frac{r^2}{r_0^2}\right) = 0, \tag{2}$$

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where k is a constant and dots denote differentiation with respect to proper time. This equation is singular at $r = 0$ only. The regular behaviour of equation (2) at $r = 2m$ tends to suggest that the singularity of metric (1) on this surface is a manifestation of some pathology of the coordinate system used. This implies that the coordinate system (t, r, θ, ψ) is incomplete, i.e. the manifold of the background spacetime is regular for $0 < r < \infty$, but the coordinate system (t, r, θ, ψ) covers the region with $2m < r < \infty$ only.

In what follows we shall show that it is possible to complete the manifold defined by (1) for $r > 2m$. This demonstration will be accomplished by embedding the background spacetime in a pseudo-Euclidean space of six dimensions. In order to do this we firstly observe that (1) can be approximated by the Schwarzschild line element provided that $r \ll r_0$. This suggests the use of embedding transformations similar to those used in the case of the Schwarzschild line element. It has been shown by Kasner [5, 6] that in this case the embedding space has to be at least six dimensional and that furthermore an embedding in such a space (with signature-2) can be explicitly constructed (a Kasner-like embedding of the spacetime represented by (1) has been obtained previously [7]). The construction presented by Kasner has two shortcomings. Firstly the embedding cannot be extended to the whole manifold and secondly periodic functions of the Schwarzschild time coordinate are used so that distinct points of the Schwarzschild manifold are identified. These problems have been avoided by Fronsdal [8] who constructed a different embedding in a six dimensional space of signature-4. We shall follow the procedure outlined in Fronsdal's paper.

2. *Embedding*

Let us introduce six dimensional pseudo-Euclidean space covered by the coordinate system $(w_1, w_2, w_3, w_4, w_5, w_6)$. We seek a set of transformations between the old coordinates (t, r, θ, ψ) and the new in terms of which the line element (1) becomes

$$ds^2 = dw_1^2 - dw_2^2 - dw_3^2 - dw_4^2 - dw_5^2 - dw_6^2. \tag{3}$$

Generalising the transformations used in [8] we set

$$\begin{aligned} w_1 &= h(r) \sinh (\alpha t), & w_2 &= h(r) \cosh (\alpha t), \\ w_3 &= g(r), & w_4 &= f(r) \sin \theta \sin \psi, \\ w_5 &= f(r) \sin \theta \cos \psi, & w_6 &= f(r) \cos \theta, \end{aligned} \tag{4}$$

where α is a non-zero constant. In order to determine the form of the three functions ($f(r)$, $g(r)$ and $h(r)$) we substitute the ansatz (4) into the line element (3). This results in

$$ds^2 = \alpha^2 h^2 dt^2 - \{(h')^2 + (g')^2 + (f')^2\} \, dr^2 - f^2(d\theta^2 + \sin^2 \theta d\psi^2), \tag{5}$$

where the dashes denote differentiation with respect to r. Comparing this expression with the line element (1) we obtain

$$h(r) = \pm \frac{1}{\alpha}\left(1-\frac{2m}{r}\right)^{1/2}, \quad f(r) = \pm \frac{r_0 r}{(r_0^2+r^2)^{1/2}},$$

$$g(r) = \pm \int \left\{\frac{(\alpha^2 r_0^4 - m^2)r^6 - 3m^2 r_0^2 r^4 + 2m\alpha^2 r_0^6 r^3 - 3m^2 r_0^4 r^2 - m^2 r_0^6}{\alpha^2 r^3 (r-2m)(r_0^2+r^2)^3}\right\}^{1/2} dr. \tag{6}$$

It may appear that the integrand in (6) is singular for both $r = 0$ and $r = 2m$, however, we are free to choose the constant α in such a way as to eliminate the singularity at $r = 2m$. There are two values of α which remove this singularity, namely,

$$\alpha = \pm \left\{\frac{64m^6 + 48m^4 r_0^2 + 12m^2 r_0^4 + r_0^6}{64m^4 r_0^4 + 16m^2 r_0^6}\right\}^{1/2}. \tag{7}$$

In what follows we shall take the positive value. The integral in (6) now becomes

$$g(r) = \pm \int \left\{\frac{a_1 r^5 + 2ma_1 r^4 + a_2 r^3 + a_3 r^2 + a_4 r + 2ma_4}{\alpha^2 r^3 (r_0^2+r^2)^3}\right\}^{1/2} dr, \tag{8}$$

where

$$a_1 = \alpha^2 r_0^4 - m^2, \quad a_2 = 4m^2 a_1 - 3m^2 r_0^2,$$

$$a_3 = 2ma_2 + 2m\alpha^2 r_0^6, \quad a_4 = 2ma_3 - 3m^2 r_0^4. \tag{9}$$

The original manifold (defined by (1) for $r > 2m$) can now be viewed as a hypersurface in the six dimensional pseudo-Euclidean space with coordinates (w_1, w_2, w_3, w_4, w_5, w_6). The equations defining this hypersurface are given by

$$w_4^2 + w_5^2 + w_6^2 = \frac{r_0^2 r^2}{r_0^2 + r^2}, \quad w_2^2 - w_1^2 = \frac{1}{\alpha^2}\left(1-\frac{2m}{r}\right),$$

$$w_3 = \pm \int \left\{\frac{a_1 r^5 + 2ma_1 r^4 + a_2 r^3 + a_3 r^2 + a_4 r + 2ma_4}{\alpha^2 r^3 (r^2 + r_0^2)^3}\right\}^{1/2} dr. \tag{10}$$

We note that all the coefficients in the numerator of the integrand in (10) are greater than zero for all values of m and r_0 such that $r_0 > 2m$. Thus w_3 is a monotonic function of r which can be left as a parameter in (10), provided that $r_0 > 2m$. The hypersurface (10) is analytic for all values of r in the range $0 < r < \infty$ and so it is an analytic continuation of the manifold defined by the line element (1). It is also complete. We have, therefore, been able to construct a manifold which is analytic and complete everywhere except at $r = 0$.

It should be pointed out that in the limit $r_0 \to \infty$ the line element (1) reduces to the Schwarzschild line element and all the above equations reduce to the respective equations derived for the Schwarzschild metric by Fronsdal [8]. Furthermore, the hypersurface (10) is invariant under the same set of transformations as its Schwarzschild counterpart. This set consists of a group of motions isomorphic to the real rotation group, a one-parameter translation group, space reflections, time reflections and the reflections in the w_2 coordinate [8]. Finally, we observe that as in the Schwarzschild case equation

$$w_2^2 - w_1^2 = \frac{1}{\alpha^2}\left(1 - \frac{2m}{r}\right), \tag{11}$$

can be used to partition the hypersurface (10) into two exterior ($r > 2m$) and two interior, ($r < 2m$) regions.

3. Conclusion

We have been able to show, by embedding the background spacetime defined by (1) in six dimensional pseudo-Euclidean space, that the singularity of metric (1) on the surface $r = 2m$ is removable (i.e. coordinate system dependent). In other words there exists at least one coordinate system (similar to the Kruskal-Szekeres one [9]) in terms of which the metric (1) becomes regular on $r = 2m$. The explicit construction of such a coordinate system will be carried out elsewhere.

It is of some inteiest to know what the embedded surface, represented (in terms of the original radial coordinate) by the set of equations (10), looks like. For $r_0 \gg 2m$ the region near $r = 0$ can be approximated very well by the (embedded) Schwarzschild hypersurface. Thus we get a 'throat' connecting two universes. Numerical investigation shows, however, that these two spacetimes are not asymptotically flat (as in the Schwarzschild case). In fact each of these two universes can be approximated by the static Einstein spacetime when $r \gg 0$. We are led to the conclusion that the background spacetime represented by (1) corresponds to that generated by a static black hole in a static spherically symmetric universe.

The above conclusion seems to contradict the original interpretation of metric (1) as a metric of an expanding universe [3]. This does not mean, however, that the metric (1) cannot be used to model the spacetime of our universe. The reason for this assertion is that there exists a class of static cosmological models [10] which possess many of the features of our universe. In these models the apparent expansion of the universe arises due to the force exerted by the cosmological singularity (at $r = 0$) on galaxies rather than due to the expansion of the spacetime itself. The fact that these models imply that the galactic redshifts are of gravitational origin is of no practical importance since we cannot distinguish experimentally between gravitational and cosmological redshifts. Furthermore, the spacetime of these models is spherically symmetric with respect to two points only, one of these being occupied by the cosmological singularity.

The spacetime described by metric (1) is spherically symmetric about two points, one of which ($r = 0$) is occupied by a singularity. Thus if it proves possible to transform (1) into the metric of the above mentioned class, then (1) still remains a candidate for the model describing our universe. We should point out, however, that the static cosmological models require that the earth (about which the universe seems to be approximately spherically symmetric) be situated in the neighbourhood of the special point not occupied by the cosmological singularity. This restores the privileged position of man in the cosmos which may or may not be philosophically attractive, depending on one's point of view.

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