MAXIMAL EXTENSION OF A STATIC SOLUTION OF THE EINSTEIN-STRAUS-KLOTZ UNIFIED FIELD THEORY

By E. J. Vlachynsky

Department of Applied Mathematics, University of Sydney*

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It is shown that it is possible to construct an analogue of the Kruskal-Szekeres coordinate system for the background spacetime of a static solution of the Einstein-Straus-Klotz unified field theory. This results in the maximal analytic extension of the metric under consideration.

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In this note we examine the nature of the singularities contained in the metric of the background spacetime of a static spherically symmetric solution of the Einstein-Straus-Klotz unified field theory. A comprehensive discussion of this theory (which attempts to unify gravity and electromagnetism on the macroscopic level) and the derivation of the solution under consideration is presented in a recently published book [1] (and references given therein). The line element of the background (pseudo-Riemannian) spacetime can be written down as follows:

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2m}{r}\right)\left(1+\frac{r^2}{r_0^2}\right)^2} + \frac{r_0^2 r^2}{r_0^2+r^2}(d\theta^2+\sin^2\theta d\psi^2), \quad (1)$$

where m and r_0 are parameters.

We observe that the above line element is singular for two values of r, namely $r = 2m$ and $r = 0$ (it should be noted, however, that the r-coordinate in (1) is not the standard Schwarzschild one). Furthermore, it has been observed [1, 2] that in the limit $r_0 \to \infty$, m fixed, (1) reduces to the line element of the exterior Schwarzschild solution and also that (1) reduces (after a simple transformation) to the standard line element of the static Einstein universe if $m \to 0$, r_0 fixed. Given the similarities between the line element (1) and that of the Schwarzschild black hole it would seem reasonable to conjecture that there exists

* Address: Department of Applied Mathematics, University of Sydney, Sydney, N.S.W. 2006, Australia.

an analogue of the Kruskal-Szekeres coordinate system in terms of which (1) becomes regular at $r = 2m$. This problem has been considered previously [1, 3], however, the metric derived there was found to be singular at $r = 2m$. On the basis of this result it has been suggested [4] that a Kruskal-Szekeres-like coodinate system may not exist for the metric (1). We shall show in what follows that such a system of coordinates can indeed be constructed.

Exploiting the similarity between (1) and the exterior Schwarzschild line element we introduce, following Kruskal [5] and Szekeres [6], new radial and time coordinates, u and v respectively, which are defined by

$$u = \beta f(r) \cosh(\alpha t), \qquad v = \beta f(r) \sinh(\alpha t), \tag{2}$$

where α and β are constants. The constant β is introduced in order to ensure that u and v reduce to the standard Kruskal-Szekeres coordinates in the limit $r_0 \to \infty$, m fixed. In terms of the new coordinates metric (1) becomes

$$ds^2 = R^2(u, v)(du^2 - dv^2) + S^2(u, v)(d\theta^2 + \sin^2\theta d\psi^2). \tag{3}$$

A comparison of (1) and (3) yields

$$\beta^2 R^2 f^2 \alpha^2 = \left(1 - \frac{2m}{r}\right), \tag{4}$$

$$\beta^2 R^2 (f')^2 = \left(1 - \frac{2m}{r}\right)^{-1} \left(1 + \frac{r^2}{r_0^2}\right)^{-2}. \tag{5}$$

Dividing (5) by (4) and taking square root of both sides we obtain

$$\frac{1}{\alpha}\frac{f'}{f} = \pm\left(1 - \frac{2m}{r}\right)^{-1} \left(1 + \frac{r^2}{r_0^2}\right)^{-1}. \tag{6}$$

Selection of the positive sign and integration gives

$$\frac{1}{\alpha}\ln f = \frac{2mr_0^2}{4m^2 + r_0^2}\ln(r - 2m) - \frac{mr_0^2}{4m^2 + r_0^2}\ln\left(1 + \frac{r^2}{r_0^2}\right) + \frac{r_0^3}{4m^2 + r_0^2}\tan^{-1}\left(\frac{r}{r_0}\right). \tag{7}$$

If we now let

$$\alpha = \frac{4m^2 + r_0^2}{4mr_0^2}, \tag{8}$$

and substitute this choice of α into (7), then f is of the form

$$f(r) = \left(\frac{r - 2m}{\sqrt{1 + r^2/r_0^2}}\right)^{1/2} \exp\left\{\frac{r_0}{4m}\tan^{-1}\left(\frac{r}{r_0}\right)\right\}. \tag{9}$$

In order to obtain the coordinate system (v, u, θ, ψ) which goes over to the Kruskal-Szekeres one in the aforementioned limit we must set

$$\beta = \frac{1}{\sqrt{2m}}. \tag{10}$$

Then

$$u = \left(\frac{r/2m-1}{\sqrt{1+r^2/r_0^2}}\right)^{1/2} \exp\left\{\frac{r_0}{4m}\tan^{-1}\left(\frac{r}{r_0}\right)\right\} \cosh\left\{\left(\frac{4m^2+r_0^2}{4mr_0^2}\right)t\right\}, \tag{11a}$$

and

$$v = \left(\frac{r/2m-1}{\sqrt{1+r^2/r_0^2}}\right)^{1/2} \exp\left\{\frac{r_0}{4m}\tan^{-1}\left(\frac{r}{r_0}\right)\right\} \sinh\left\{\left(\frac{4m^2+r_0^2}{4mr_0^2}\right)t\right\}. \tag{11b}$$

Also

$$R^2 = 2m\left(\frac{4mr_0^2}{r_0^2+4m^2}\right)^2 \frac{\sqrt{1+r^2/r_0^2}}{r} \exp\left\{-\frac{r_0}{2m}\tan^{-1}\left(\frac{r}{r_0}\right)\right\}. \tag{12}$$

Thus the line element (1) can be rewritten, in terms of the coordinates (2), as

$$ds^2 = 2m\left(\frac{4mr_0^2}{r_0^2+4m^2}\right)^2 \frac{\sqrt{1+r^2/r_0^2}}{r} \exp\left\{\frac{-r_0}{2m}\tan^{-1}\left(\frac{r}{r_0}\right)\right\}(du^2-dv^2)$$

$$+\frac{r_0^2r^2}{r_0^2+r^2}(d\theta^2+\sin^2\theta d\psi^2). \tag{13}$$

This line element is clearly singular at $r = 0$ only. We have, therefore, been able to construct a coordinate system which is analogous to the Kruskal-Szekeres one. As in the Schwarzschild case the transformation (11) becomes meaningless when one moves from $r > 2m$ to $r < 2m$. This is due to the pathologies of the original coordinate system (t, r, θ, ψ). There are two exterior regions, one covered by (11), the other by coordinates obtained from (11) through multiplication of the right hand side by -1, and two interior regions the coordinates of which are obtained from the exterior ones by replacing $(r/2m-1)$ with $(1-r/2m)$ and interchanging the sinh and the cosh terms. There are also two singularities at $r = 0$ (as in the Schwarzschild case).

On the basis of the above analysis we conclude that the line element (1) can be extended so as to be analytic on the manifold covered by the coordinate system (v, u, θ, ψ) where $0 < u < \infty$. The singularity of the metric (1) at $r = 0$ is physical as can be shown by calculating the physical components of the curvature tensor (constructed from the symmetric connection since we are considering the background spacetime only) and observing that some of these diverge [7]. Thus the background geometry based on (1) seems to be equivalent to that generated by a static black hole in a static Einstein universe. This interpretation would appear to contradict the original interpretation of (1) as representing the spacetime of an expanding universe [1, 8]. This, however, is not necessarily so. It has been pointed out by Ellis [9] that a class of static spherically symmetric space-times reproduces current cosmological observations. Each of these spacetimes has two centres, one of which coincides with a singularity. Our galaxy is situated near the other centre. The galactic redshifts are thus of gravitational origin. Now the spacetime described by (1) is spherically symmetric about two points only, one of these being at the centre of a static singularity [7]. It would therefore appear to be a member of the Ellis class. Thus if it can be shown that (1) transforms to the Ellis line element [9] then the 'expanding universe' interpretation of metric (1) can be retained, with the understanding that the dispersion of galaxies is caused by the gravitational field of the singularity rather than by the expansion of the spacetime (as in the Robertson-Walker models).

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