GRAVITATIONAL INSTANTON SOLUTIONS FOR BIANCHI TYPES I-IX

By D. Lorenz

Fachbereich für Physik, Universität Konstanz*

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We investigate Euclidean solutions of the vacuum Einstein equations for diagonal Bianchi types I–IX. Explicit solutions are given for all types which in some cases may be considered as gravitational instanton solutions.

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1. *Introduction*

The discovery of pseudoparticle (instanton) solutions to the Euclidean SU(2) Yang-Mills theory (Belavin et al. 1975) has suggested the possibility that analogous solutions might occur in Einstein's theory of gravitation. Since the Yang-Mills instantons posses self-dual field strength, one likely possibility is that gravitational instantons are characterized by self-dual curvature. However, there are also some gravitational instantons which are not characterized by self-dual curvature (Perry 1982).

In general a gravitational instanton can be defined as a complete nonsingular Riemannian manifold (with Euclidean signature) (M, g) of dimension four with finite action satisfying the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$, where $R_{\mu\nu}$ denotes the Ricci tensor and Λ the cosmological constant. One interesting class is compact instantons. These have to do with the space-time foam description of gravitational physics (Hawking 1978, 1979). A gravitational instanton with $\Lambda > 0$ is necessarily compact (Boyer 1981). The other interesting class comprise non-compact instantons. Starting from the idea that since the Yang-Mills potential is asymptotically a pure gauge (see Actor 1979; Rajamaran 1982 for a detailed description of Yang-Mills instantons), a gravitational instanton should have an asymptotically flat metric. Four types of such gravitational instantons have been discovered (for recent reviews see Eguchi et al. 1980; Pope 1981; Perry 1982; Gibbons 1980): asymptotically Euclidean (AE), asymptotically locally Euclidean (ALE), asymptotically

* Address: Fachbereich für Physik, Universität Konstanz, Bücklestr. 13, D-7750 Konstanz, West Germany.

flat (AF) and asymptotically locally flat (ALF) (Gibbons and Hawking 1979). The first two types corresponds to vacuum or zero temperature physics, the second ones to finite temperature physics (Gibbons 1980).

A variety of solutions of Einstein's field equations with instanton-like properties have been discovered (Eguchi et al. 1980; Pope 1981; Perry 1982; Gibbons 1980). The corresponding metrics of many of these solutions may be cast into the Euclidean equivalent of the Bianchi type-IX space-time (see Ryan and Shepley 1975; MacCallum 1979; Kramer et al. 1980 for a detailed description of the Bianchi classification scheme). Thus one is forced to consider the complete set of Bianchi types I–IX. The Bianchi type-IX solution has been first discovered by Belinskii et al. (1978) (see also Gibbons and Pope 1979). It includes as special cases the Eguchi-Hanson metrics (Eguchi and Hanson 1978) and the self-dual solution given by Hawking (1977).

In this paper we consider (anti)-self-dual Euclidean spacetimes for diagonal Bianchi types I–IX. Explicit solutions are given for all types including non-self-dual vacuum solutions (with and without) Λ-term. In Section 2 we set up the general Bianchi formalism and give a brief outline of the method for obtaining (anti)-self-dual solutions. In Section 3 we present new exact (anti)-self-dual solutions as well as pure non-self-dual vacuum solutions which in some cases may be considered as gravitational instanton solutions.

2. *Derivation of the curvature*

Let (M, g, σ) be a four-dimensional Riemannian manifold (signature $++++$) with a metric tensor g and a linear connection σ compatible with g. In choosing a local orthonormal basis σ^μ, we can put the metric of Euclidean space-time in the form

$$ds^2 = \eta_{\mu\nu}\sigma^\mu\sigma^\nu, \tag{1}$$

where $\eta_{\mu\nu} = (1, 1, 1, 1)$ is the Euclidean metric tensor. We take

$$\sigma^0 = \omega^0 = dt, \qquad \sigma^i = R_i\omega^i \text{ (no sum)}, \tag{2}$$

where ω^i are time-independent differential one-forms and where the R_i are functions of t only (here and henceforth Latin indices will assume the values 1, 2, 3, whereas Greek indices will assume the values 0, 1, 2, 3). The one-forms obey the relations

$$d\omega^i = -\tfrac{1}{2}\, C_{kl}{}^i\omega^k \wedge \omega^l, \qquad d\sigma^i = -\tfrac{1}{2}\,\gamma_{\alpha\beta}{}^i\sigma^\alpha \wedge \sigma^\beta, \tag{3}$$

where $C_{kl}{}^i$ are the structure constants, $\gamma_{\alpha\beta}{}^i$ the connection coefficients, and $\wedge$ denotes the exterior product. The structure constants for the Bianchi types I–IX are:

I: $\quad C_{ij}{}^k = 0, \forall\, i, j, k$

II: $\quad C_{23}{}^1 = 1,$

IV. $\quad C_{31}{}^1 = C_{23}{}^1 = C_{32}{}^2 = 1,$

V: $\quad C_{31}{}^1 = C_{32}{}^2 = 1$

$$\mathrm{VI}_h\colon C_{23}{}^1 = C_{13}{}^2 = 1,\ C_{31}{}^1 = C_{32}{}^2 = a,\ a := (-h)^{1/2}$$

$$\mathrm{VII}_h,\ \mathrm{VIII},\ \mathrm{IX}\colon C_{ij}{}^k = -\varepsilon_{ijk}n_k + (\delta_{ki}\delta_{i3}+\delta_{li}\delta_{k3})a,\ a := h^{1/2}, \tag{4}$$

where ε_{ijk} is the totally antisymmetric Levi-Civita pseudotensor and δ_{ik} is the Kronecker symbol. We have:

n_1	n_2	n_3	a
1	1	0	a, type-VII_h
1	1	-1	0, type-VIII
1	1	1	0, type-IX.

The invariant parameter h is required to subclassify types VI and VII. Bianchi type III is the same as VI_{-1}. The Bianchi classification gives two broad classes, class A for which $C_{il}{}^l = 0$, and class B ($C_{il}{}^l \neq 0$). In type VI_h, Bianchi's q is such that $h = -[(1+q)/(1-q)]^2$ while in VII_h we have $h = q^2/(4-q^2)$. Using the Ellis-MacCallum (1969) decomposition

$$C_{ij}{}^k = \varepsilon_{lij}n^{kl} + 2\delta^k{}_{[j}a_{i]} = -C_{ji}{}^k, \tag{5}$$

where

$$a_i = \tfrac{1}{2}C_{ij}{}^j, \qquad n^{ij} = \tfrac{1}{2}C_{kl}{}^{(i}\varepsilon^{j)kl}, \tag{6}$$

we obtain from the Jacobi identities

$$C_{[ij}{}^k C_{l]k}{}^m = 0. \tag{7}$$

$$n^{ij}a_j = 0. \tag{8}$$

The restriction imposed by $n^i{}_i = 0$ can apply only in Bianchi types I, V, VI_h and VIII and allows simplified field equations especially foı type VI_h.

The exterior derivatives of the orthonormal basis one-forms σ^μ are readily found by us of Eqs. (2) and substitution of the first of Eqs. (3). Comparison of these equations with the second of Eqs. (3) provides immediately the connection coefficients $\gamma_{\alpha\beta}{}^i$. These quantities enter into the formula

$$\sigma_{\mu\nu} = \tfrac{1}{2}(\gamma_{\mu\nu\alpha}+\gamma_{\mu\alpha\nu}-\gamma_{\nu\alpha\mu})\sigma^\alpha \tag{9}$$

to provide six affine connection one-forms $\sigma_{\mu\nu}$.

The curvature two-forms

$$\theta_{\mu\nu} = \sigma_\mu{}^\alpha \wedge \sigma_{\alpha\nu} + d\sigma_{\mu\nu} \tag{10}$$

can readily be computed by use of (9) and the compatibility equation

$$0 = d\eta_{\mu\nu} = \sigma_{\mu\nu}+\sigma_{\nu\mu}. \tag{11}$$

Out of this calculation, one reads the individual components $R^{\mu\nu}{}_{\alpha\beta}$ of the curvature tensor by using the second Cartan equation

$$\theta^{\mu\nu} = \tfrac{1}{2}R^{\mu\nu}{}_{\alpha\beta}\sigma^\alpha \wedge \sigma^\beta \tag{12}$$

as an identification scheme. Thus we can easily calculate the Ricci tensor $R_{\mu\nu} = -R^\alpha{}_{\mu\nu\alpha}$.

Since we are only interested in vacuum solutions of Einstein's field equations we have

$$R_{\mu\nu} = 0. \tag{13}$$

By requiring that the connection forms $\sigma_{\mu\nu}$ be (anti) self-dual, i.e.

$$\sigma_{\mu\nu} = \delta\tilde{\sigma}_{\mu\nu} = \frac{\delta}{2}\varepsilon_{\mu\nu\varrho\tau}\sigma_{\varrho\tau} \tag{14}$$

it follows that the Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$ is (anti) self-dual (or half-flat)

$$R^{\mu}{}_{\nu\alpha\beta} = \delta\tilde{R}^{\mu}{}_{\nu\alpha\beta} = \frac{\delta}{2}\varepsilon_{\alpha\beta\gamma\delta}R^{\mu}{}_{\nu}{}^{\gamma\delta}, \tag{15}$$

where $\delta = 1$ for self-dual and $\delta = -1$ for anti self-dual solutions. Taking the trace of this equation, we find that the Ricci tensor vanishes; hence the (anti)-self-dual space-times automatically satisfy the vacuum Einstein equations (13).

The condition (14) reduces the field equations to a system of first-order differential equations. Imposing only the self-dual curvature condition (15) we obtain second-order differential equations and thus a richer spectrum of possible solutions. However, any self--dual $R^{\mu}{}_{\nu\alpha\beta}$ can be considered to come from a self-dual connection if a "self-dual gauge" is chosen (Eguchi et al. 1980; Eguchi and Hanson 1979b). There are several reasons why the self-dual condition is overly restrictive and why one might want to construct space--times which are non-self-dual (neither self-dual nor anti-self-dual). Since all self-dual space-times satisfy the vacuum Einstein equations (with $\Lambda = 0$), they cannot contain any source fields (besides self-dual electromagnetic fields with vanishing energy-momentum tensor). This limits their use in quantum interaction physics. Recent constructions of non--self-dual "nonlinear gravitons" has been given by Yasskin and Isenberg (1982). However, since the problem of finding all self-dual solutions to Euclidean gravity has not arrived at the same degree of completeness as for the Yang-Mills theory (for recent progress see Hitchin (1979)), we are encouraged to construct further explicit (anti)-self-dual solutions.

3. *Self-dual Bianchi solutions*

Type-I

The (anti) self-duality condition (15) gives

$$(\ln R_i^2)' = 2\lambda_i R_j R_k, \tag{16}$$

where $(\ )' = d/d\eta$, $dt = R_1 R_2 R_3 d\eta$, i, j, k are in cyclic order and λ_i are constants obeying

$$\lambda_i \lambda_j = 0. \tag{17}$$

The case $\lambda_i = 0$ may be obtained directly from (14) without integration. The possible solutions are

$$R_i = a_i, \quad \lambda_i = 0, \tag{18a}$$

$$R_i = a_i, \quad R_j = b_j \exp(c_j \eta), \quad \exists_j, \lambda_j \neq 0, \quad i \neq j, \tag{18b}$$

where a_i, b_j, c_j are constants of integration.

The vacuum field equations $R_{\mu\nu} = 0$ are

$$(\ln R_i^2)'' = 0, \quad (\ln R_1)'(\ln R_2 R_3)' + (\ln R_2)'(\ln R_3)' = 0 \tag{19}$$

from which we obtain the Euclidean Kasner (1921) solution

$$R_i = c_i \exp(a_i \eta), \quad a_1 a_2 + a_1 a_3 + a_2 a_3 = 0, \tag{20}$$

$c_i = \text{const}$. By setting $a_1 + a_2 + a_3 = 1$ and $c_1 c_2 c_3 = 1$ it follows that

$$R_i = c_i t^{a_i}, \quad a_1^2 + a_2^2 + a_3^2 = 1. \tag{21}$$

Eq. (21) represents the Euclidean Kasner solution in its usual form.

The generalization to non-zero Λ obeying $R_{\mu\nu} = \Lambda g_{\mu\nu}$ can be obtained from the reduced field equations

$$(\ln R_1)^{\cdot}(\ln R_2 R_3)^{\cdot} + (\ln R_2)^{\cdot}(\ln R_3)^{\cdot} = -\Lambda, \tag{22a}$$

$$(\ln R_i)^{\cdot\cdot} + 3(\ln R)^{\cdot}(\ln R_i)^{\cdot} = -\Lambda, \tag{22b}$$

where $R^3 = R_1 R_2 R_3$ and $(\ )^{\cdot} := d/dt$. It follows that

$$(\ln R_i)^{\cdot} = (\ln R)^{\cdot} + A_i R^{-3} \tag{23a}$$

and

$$(\ln R)^{\cdot 2} = -\frac{\Lambda}{3} + \tfrac{1}{2} a^2 R^{-6}, \tag{23b}$$

where $\sum_{i=1}^{3} A_i = 0$ and $3a^2 = \sum_{i=1}^{3} A_i^2$. The field equations can now be integrated to give the most general Euclidean Bianchi type-I solutions with $\Lambda \neq 0$:

$$R_i = (\sin \omega t)^{1/3} \left(\tan \frac{\omega}{2} t\right)^{A_i/b\omega}, \quad \Lambda > 0, \tag{24a}$$

$$R_i = (\sinh \omega t)^{1/3} \left(\tanh \frac{\omega}{2} t\right)^{A_i/b\omega}, \quad \Lambda < 0, \tag{24b}$$

where $\omega^2 = 3|\Lambda|$ and $b^2 = 3a^2/2|\Lambda|$.

The corresponding non-Euclidean version of these solutions has been first given by Saunders (1969) (see also Kramer et al. 1980). The locally rotationally symmetric Euclidean solution with $\Lambda \neq 0$ and $R_2 = R_3$ has been found by us recently (Lorenz 1983b).

Type-II

The (anti) self-dual equations to be considered are

$$(\ln R_i^2)' = n_i R_1^2 - 2\lambda_i n_i R_j R_k, \tag{25a}$$

where $n_1 = -n_2 = -n_3 = 1$ and

$$\lambda_1 = -\lambda_2\lambda_3, \quad \lambda_1\lambda_2 = \lambda_1\lambda_3 = 0. \tag{25b}$$

It follows that $\lambda_1 = 0$. Introducing the new variable r by $dr = (R_1^4/r^3)d\eta$ the solution with $\lambda_i = 0$ is given by

$$R_1^2 = \delta\frac{r^4}{2}\left[1-\left(\frac{a}{r}\right)^4\right], \quad R_2^2 = R_3^2 = bR_1^{-2}, \tag{25c}$$

where a, b are constants. In addition we find as special solutions the following Eguchi--Hanson types

$$\text{I:} \qquad R_1 = rg = br^{-1}, \quad R_2 = R_3 = r, \quad fg = 2\delta, \tag{26a}$$

$$\text{II:} \qquad R_1 = r, \quad R_2 = R_3 = rg = br^{-1}, \quad f = -2\delta g^2, \tag{26b}$$

where b = const and $dt = fdr$.

In case of $\lambda_2 \neq 0$, $\lambda_3 = 0$ we obtain the solution

$$R_1^2 = [-\delta(\eta-\eta_0)]^{-1}, \quad R_2^2 = -\delta(\eta-\eta_0)\exp[2\lambda_2 b(\eta-\eta_1)],$$
$$R_3^2 = -b\delta(\eta-\eta_0), \tag{27}$$

where b, η_i are constants. This is the first general triaxial Bianchi type-II solution. The LRS-Euclidean-Taub-Bianchi type-II solution with $R_2 = R_3$ can be obtained from $R_{\mu\nu} = 0$

$$R_1^2 = q[\sinh q(\eta-\eta_0)]^{-1},$$
$$R_2^2 = \frac{1}{q}\sinh q(\eta-\eta_0)\exp q(\eta-\eta_1), \quad q = \text{const.} \tag{28}$$

(Taub 1951; Lorenz 1980a).

Types-III, VI_h

The Bianchi type-III model is a special subcase of Bianchi type-VI_h with $h = -1$. There are two distinct cases that arise: cases $n^i{}_i = 0$ and $n^i{}_i \neq 0$. We first consider the case $n^i{}_i = 0$. As noted in Chapter 2 this restriction allows simplified field equations. The self--dual field equations to be solved are

$$(\ln R_i^2)' = 2\lambda_i R_j R_k. \tag{29}$$

In addition we have the constraint equations

$$(A+1)\,[\ln(R_1/R_2)]' = (A-1)\,[\ln(R_1/R_3)]' = 0,$$
$$\lambda_1\lambda_2 = -(A+1)^2R_2/R_1, \quad \lambda_1\lambda_3 = -(A-1)^2R_2/R_1,$$
$$\lambda_2\lambda_3 = -(A^2-1)R_2R_3/R_1^2, \quad A^2 = -h. \tag{30}$$

The possible solutions are

$$R_1 = [-2\lambda_1 b(\eta-\eta_0)]^{-1} = R_2, \quad R_3 = b, \quad \lambda_1^2 = \lambda_2^2 = -4, \quad \lambda_3 = 0, \quad A = 1, \tag{31a}$$

$$R_1^2 = R_2^2 = R_3^2 = [-2\lambda_1(\eta-\eta_1)]^{-1}, \quad \lambda_1^2 = \lambda_2^2 = \lambda_3^2 = -1, \quad A = 0. \tag{31b}$$

Thus only complex self-dual Bianchi types-III and VI_0 solutions are allowed. However, it has been pointed out by Boutaleb-Joutei et al. (1981) that complex solutions should be studied systematically.

The Euclidean vacuum equations to be solved are

$$\begin{aligned}
(\ln R_1^2)'' &= -4(A^2+1)R_2^2R_3^2,\\
(\ln R_2^2)'' &= -4A(A+1)R_2^2R_3^2,\\
(\ln R_3^2)'' &= -4A(A-1)R_2^2R_3^2,\\
(\ln R_1)'(\ln R_2R_3)'+(\ln R_2)'(\ln R_3)' &= -(3A^2+1)R_2^2R_3^2,\\
2A(\ln R_1)'-(A+1)(\ln R_2)'+(1-A)(\ln R_3)' &= 0.
\end{aligned} \tag{32}$$

Introducing the new time variable τ by $d\tau = R_2R_3 d\eta$ we obtain the general Euclidean vacuum solutions

$$\begin{aligned}
R_1^2 &= (\sin 2A\tau)^{(A^2+1)/A^2}(\tan A\tau)^{m/A^2},\\
R_2^2 &= (\sin 2A\tau)^{(A+1)/A}(\tan A\tau)^{m/A},\\
R_3^2 &= (\sin 2A\tau)^{(A-1)/A}(\tan A\tau)^{-m/A}, \quad A \neq 0,
\end{aligned} \tag{33}$$

where $m^2-3A^2-1 = 0$, $m = \text{const}$ and

$$\begin{aligned}
R_1^2 &= \exp[-(1/a^2)\exp 2a(\eta-\eta_1)-2a(\eta-\eta_2)],\\
R_2^2 &= \exp a(\eta-\eta_1) = R_3^2, \quad A = 0, \quad a = \text{const}.
\end{aligned} \tag{34}$$

The corresponding non-Euclidean solutions have been first given by Ellis and MacCallum (1969). Solution (34) can be reexpressed in its Ellis-MacCallum form

$$R_1^2 = 2\tau, \quad R_2^2 = R_3^2 = \tau^{-1/2}\exp(-\tau^2). \tag{35}$$

We now consider the case $n^i{}_i \neq 0$. The self-duality condition (15) leads to the equation

$$(\ln R_i^2)' = \delta[n_jR_j^2+n_kR_k^2-n_iR_i^2]-2\lambda_iR_jR_k, \tag{36}$$

where $n_1 = -n_2 = 1$, $n_3 = 0$. In addition we have the constraint equations

$$\begin{aligned}
A[\ln(R_1/R_3)]' &= A[\ln(R_2/R_3)] = 0,\\
4[(\ln R_1)'(\ln R_2R_3)'+(\ln R_2)'(\ln R_3)'] &= -[(R_1^2+R_2^2)^2+12AR_1^2R_2^2].
\end{aligned} \tag{37}$$

It follows that $A = 0$, i.e. $h = 0$ and $\lambda_1 = 0$. In case of $\lambda_2 = \lambda_3 = 0$, which may be obtained directly from (14) without integration, we obtain the solution

$$R_1^2 = 2r^2F^{1/2}[\delta(1-(a_1/r)^4)]^{-1},$$
$$R_2^2 = 2r^2F^{1/2}[-\delta(1-(a_2/r)^4)]^{-1},$$
$$R_3^2 = a_3^{-1}r^6F^{1/2}, \tag{38}$$

where $dr = r^{-3}(R_1R_2R_3)^2 d\eta$ and $F^{1/2} = r^{-6}(R_1R_2R_3)^2$, $a_i = \text{const.}$

There aie no Eguchi-Hanson Bianchi type-VI_0 solutions. In case of $\lambda_3 \neq 0$, $\lambda_2 = 0$ we obtain

$$R_1^2 = a\tan[-a\delta(\eta-\eta_0)], \quad R_2^2 = a^2R_1^{-2},$$
$$R_3^2 = \sin[-2a\delta(\eta-\eta_0)]\exp[-\lambda_3(\eta-\eta_1)], \tag{39}$$

where a, η_i are constants.

We finally should like to point out that the general vacuum solution of Bianchi type-VI_h $(n^i{}_i \neq 0)$ in unknown.

Type-IV

The non-Euclidean Bianchi type-IV model has been considered by Harvey and Tsoubelis (1977), Harvey et al. (1979), Tsoubelis (1976), and Siklos (1978, 1980). The model has a numbei of remarkable features. It is first found that it cannot be persistently diagonal in either the vacuum or perfect fluid case; the simplest non-diagonal case possess a solution only if the cosmological constant is zero and no matter is present. Howevei, it can be made compatible with an electromagnetic null field.

The simplest choice of the differential one-forms σ^i, which yields a nontrivial model is

$$\sigma^0 = \omega^0 = dt, \quad \sigma^1 = R_1\omega^1, \quad \sigma^2 = R_2\omega^2, \quad \sigma^3 = BR_3\omega^2 + R_3\omega^3, \tag{40}$$

where $B = B(t)$. Even with this rather special choice of the metric the self-dual field equations to be considered turn out to be very complicated. Assuming $R_2 = R_3$ as in the Harvey-Tsoubelis approach we obtain the condition

$$\dot{B}(3+4\dot{B}/R_1) = 0 \tag{41}$$

from which it follows that $\dot{B} = 0$ or $\dot{B} = -(3/4)R_1$. However, it can be shown that both cases are in contradiction with the remaining field equations.

The Euclidean vacuum field equations to be solved are

$$(\ln R_1R_2R_3)^{\cdot\cdot} + (\ln R_1)^{\cdot 2} + (\ln R_2)^{\cdot 2} + (\ln R_3)^{\cdot 2} + \tfrac{1}{2}\left(\frac{R_3}{R_2}\dot{B}\right)^2 = 0, \tag{42a}$$

$$(\ln R_1)^{\cdot\cdot} + (\ln R_1)^{\cdot}(\ln R_1R_2R_3)^{\cdot} + \frac{2}{R_1^2} + \tfrac{1}{2}\left(\frac{R_3}{R_1R_2}\right)^2 = 0, \tag{42b}$$

$$(\ln R_2)^{\cdot\cdot} + (\ln R_2)^{\cdot}(\ln R_1R_2R_3)^{\cdot} + \frac{2}{R_1^2} + \tfrac{1}{2}\left(\frac{R_3}{R_1R_2}\right)^2 + \tfrac{1}{2}\left(\frac{R_3}{R_2}\dot{B}\right)^2 = 0, \tag{42c}$$

$$(\ln R_3)^{\cdot\cdot}+(\ln R_3)^{\cdot}(\ln R_1R_2R_3)^{\cdot}+\frac{2}{R_1^2}-\tfrac{1}{2}\left(\frac{R_3}{R_1R_2}\right)^2-\tfrac{1}{2}\left(\frac{R_3}{R_2}\dot{B}\right)^2=0, \tag{42d}$$

$$\frac{1}{R_1}\left[(\ln (R_1^2/R_2R_3))^{\cdot}-\tfrac{1}{2}\left(\frac{R_3}{R_2}\right)^2\dot{B}\right]=0, \tag{42e}$$

$$\tfrac{1}{2}\frac{R_3}{R_2}\left[\ddot{B}+\dot{B}(\ln R_3^3/R_2)^{\cdot}+\frac{2}{R_1^2}+\dot{B}(\ln R_1)^{\cdot}\right]=0. \tag{42f}$$

In the locally rotationally symmetric case $R_2 = R_3$ we obtain from (42c) and (42d)

$$\dot{B}^2 = -1/R_1^2. \tag{43}$$

Substitution of this into Eq. (42f) yields

$$(\ln R_2)^{\cdot} = -1/\dot{B}R_1^2. \tag{44}$$

We can now solve (42e) to give

$$R_1 = \pm\tfrac{5}{4}\, it+a, \qquad a = \text{const}. \tag{45}$$

It follows that

$$R_2 = b(\pm\tfrac{5}{4}\, it+a)^{4/5}, \qquad B = \ln (c(\pm\tfrac{5}{4}\, it+a))^{4/5}. \tag{46}$$

Thus we have obtained a complex Euclidean vacuum Bianchi type-IV solution.

Type-V

For the Euclidean Bianchi type-V model we obtain the (anti) self-dual field equations

$$(\ln R_i^2)' = 2\lambda_i R_j R_k,$$

$$(\ln R_1)' = (\ln R_2)' = (\ln R_3)',$$

$$(\ln R_i)'(\ln R_j)' = -1/R_3^2, \tag{47}$$

from which it follows that $R_1 = R_2 = R_3$ and $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = -1$. Thus equations (47) cannot be obtained from the (anti) self-dual condition (14). The complex solution is given by

$$R_1^2 = [-2\lambda_1(\eta-\eta_1)]^{-1}. \tag{48}$$

The Euclidean vacuum field equations to be considered are

$$(\ln R_i^2)'' = -4(R_1R_2)^2,$$

$$(\ln R_3^2/R_1R_2)' = 0,$$

$$(\ln R_1)'(\ln R_2R_3)'+(\ln R_2)'(\ln R_3)' = -3. \tag{49}$$

Introducing the new time variable τ by $d\tau = R_1R_2 d\eta$ we obtain the real solution

$$R_1^2 = (\sin 2\tau)(\tan \tau)^m, \qquad R_2^2 = (\sin 2\tau)(\tan \tau)^{-m}, \qquad R_3^2 = \sin 2\tau, \tag{50}$$

where $m = 3^{1/2}$. The corresponding non-Euclidean solution has been first given by Joseph (1969).

Type-VII$_h$

It is convenient to investigate the Bianchi type-VII$_h$ model in a canonical frame with rotating axis in which the metric is diagonalized (Doroshkevich et al. 1973)

$$\sigma^0 = \omega^0 = dt, \quad \sigma^1 = R_1(\cos\phi\omega^1 - \sin\phi\omega^2),$$
$$\sigma^2 = R_2(\sin\phi\omega^1 + \cos\phi\omega^2), \quad \sigma^3 = R_3\omega^3, \tag{51}$$

where $\phi = \phi(t)$ is the angle of rotation in the (ω^1, ω^2) tetrad plane. The self-dual equations to be solved are

$$\dot{H}_i + H_i^2 + \varepsilon_{3ik}\left[3\left(\frac{R_k}{R_i}\right)^2 - \left(\frac{R_i}{R_k}\right)^2 - 2\right]\dot{\phi}^2$$
$$= \frac{\delta}{2}(2\delta_{i2} - \delta_{ii})\left\{\varepsilon_{ikl}\left[\frac{n_i R_i}{R_k R_l}(H_k + H_l - 2H_i) + \left(\frac{n_k R_k}{R_i R_l} - \frac{n_l R_l}{R_i R_k}\right)(H_k - H_l)\right]\right.$$
$$\left. + a\varepsilon_{3ik}\left(\frac{R_k}{R_i} - \frac{R_i}{R_k}\right)\right\}, \tag{52a}$$

$$H_i H_k + \tfrac{1}{2}\varepsilon_{ikl}\left(\frac{n_k n_l}{R_i^2} + \frac{n_i n_l}{R_k^2} - \frac{n_i n_k}{R_i^2}\right) - \tfrac{1}{4}\varepsilon_{3ik}\left(\frac{R_k}{R_i} - \frac{R_i}{R_k}\right)^2\dot{\phi}^2$$
$$+ \tfrac{1}{4}\varepsilon_{ikl}\left[\left(\frac{n_i R_i}{R_k R_l}\right)^2 + \left(\frac{n_k R_k}{R_i R_l}\right)^2 - 3\left(\frac{n_l R_l}{R_i R_k}\right)^2\right] - \left(\frac{a}{R_3}\right)^2$$
$$= \frac{\delta}{2}(2\delta_{i2} - \delta_{ii})\left\{\varepsilon_{ikl}\left[\left(\frac{n_i R_i}{R_k R_l} - \frac{n_k R_k}{R_i R_l}\right)(H_i - H_k) + \frac{n_l R_l}{R_i R_k}(H_i + H_k - 2H_l)\right]\right.$$
$$\left. + \frac{a}{R_3}\delta_{k3}\left(\frac{R_2}{R_1} - \frac{R_1}{R_2}\right)\dot{\phi}\right\}, \tag{52b}$$

$$\tfrac{1}{2}\varepsilon_{3ik}\left\{\left[\frac{R_k}{R_i}(3H_k - H_i) + \frac{R_i}{R_k}(H_k - 3H_i)\right]\dot{\phi} + \left(\frac{R_k}{R_i} - \frac{R_i}{R_k}\right)\ddot{\phi}\right\}$$
$$= \frac{\delta}{R_3}\varepsilon_{3ik}\left\{a(H_i - H_3) + \tfrac{1}{4}\left[3\left(\frac{R_k}{R_i}\right)^2 - \left(\frac{R_i}{R_k}\right)^2 - 2\right]\dot{\phi}\right\}, \tag{52c}$$

$$\frac{a}{R_3^2}\left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right) + \frac{H_3}{2}\left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right)\dot{\phi}$$
$$= \frac{\delta}{R_3}\left\{a(H_1 - H_3) + \tfrac{1}{4}\left[3\left(\frac{R_2}{R_1}\right)^2 - \left(\frac{R_1}{R_2}\right)^2 - 2\right]\dot{\phi}\right\}, \tag{52d}$$

$$a(H_1 + H_2 - 2H_3) + 2\dot{\phi}\sinh^2\frac{\mu}{2} = 0, \tag{52e}$$

where $H_i := \dot{R}_i/R_i$, $(\,)^{\cdot} := d/dt$, $n_1 = n_2 = 1$, $n_3 = 0$ and i, j, k are in cyclic order. δ_{ij} denotes the Kronecker symbol and ε_{ijk} is the Levi-Civita pseudotensor with $\varepsilon_{123} = 1$. The quantity μ is defined according to

$$\mu = 2 \ln (R_1/R_2). \tag{53}$$

It seems to be unlikely that the field equations (52) can be integrated in general. The case $\mu = 0$ reduces to the Bianchi type-V model discussed in the previous section. Assuming $a = 0$, it follows from (52e) that $\dot{\phi} = 0$, and by means of a rotation one can always select $\phi = 0$. Introducing the new time variable η by $dt = R_1 R_2 R_3 d\eta$ Eqs. (52a) can now be integrated to give

$$(\ln R_i^2)' = -\delta[n_j R_j^2 + n_k R_k^2 - n_i R_i^2] - 2\lambda_i R_j R_k \tag{54}$$

and

$$\delta\lambda_1 = -\lambda_2\lambda_3, \quad \delta\lambda_2 = -\lambda_1\lambda_3, \quad \lambda_1\lambda_2 = 0, \tag{55}$$

which follows from (52b). In case of $\lambda_i = 0$ we obtain the solution

$$R_i^2 = r^2(2F)^{1/2}\left[-\delta\left(1-\left(\frac{a_i}{r}\right)^4\right)\right]^{-1}, \quad i = 1,2,$$
$$R_3^2 = r^6 F^{1/2} a_3, \tag{56}$$

where $dr = r^{-3}(R_1R_2R_3)^2 d\eta$, $F^{1/2} = r^{-6}(R_1R_2R_3)^2$ and $a_i = \text{const}$.

It follows from (55) that we can have only $\lambda_3 \neq 0$. The corresponding solution may be given in the form

$$R_1^2 = a \coth[-a\delta(\eta-\eta_0)], \quad R_2^2 = a^2 R_1^{-2},$$
$$R_3^2 = \sinh[-2a\delta(\eta-\eta_0)] \exp[-\lambda_3(\eta-\eta_1)], \tag{57}$$

where a, η_i are constants of integration. The solutions (56) and (57) are the first triaxial Bianchi type-VII$_0$ solutions.

After long manipulations we obtain the Euclidean Bianchi type-VII$_h$ vacuum equations

$$(R_1R_2)'' + 4a^2 R_1 R_2 = 0,$$
$$\left(\phi' R_1 R_2 \sinh^2 \frac{\mu}{2}\right)' = -2aR_1R_2 \sinh^2 \frac{\mu}{2},$$
$$(\mu' R_1 R_2)' - 4R_1R_2(1+\phi'^2)\sinh\mu = 0,$$
$$a(\ln R_3^2/R_1R_2)' = 2\phi' \sinh^2 \frac{\mu}{2},$$

$$(\ln R_3)'(\ln R_1R_2)' + (\ln R_1)'(\ln R_2)' = -3a^2 - (1-\phi'^2)\sinh^2\frac{\mu}{2}, \tag{58}$$

where $a^2 = h$, $(\,)' = d/d\xi$ and $d\xi = R_1R_2 d\eta$. No general solutions of these complicated field equations are known. However one may attempt the following "Ansatz" to obtain a class of special solutions of Eqs (58). One seeks solutions obeying $\phi' = k = \text{const}$. This "Ansatz" is called the Lukash ansatz since it has been first used by Lukash (1974; 1976) with success in the non-Euclidean vacuum type-VII$_h$ case (see also Jantzen 1980).

For $k = i$ we obtain the complex solution

$$\mu = \ln\left[(-i)^{1/2}\cot(a\xi)\right],$$

$$R_1R_2 = A\sin(2a\xi), \quad R_3^2 = B(\sin(2a\xi))^{1-1/2a^2}\left[\exp(i\xi/a)\right], \tag{59}$$

where A, $B = \text{const}$ and $a^2 = 4/11$.

Types-VIII, IX

The self-dual condition (15) leads to the equations

$$(\ln R_i^2)' = -\delta[n_jR_j^2 + n_kR_k^2 - n_iR_i^2] - 2\lambda_iR_jR_k, \tag{60a}$$

$$\delta\lambda_i n_i = -\lambda_j\lambda_k, \tag{60b}$$

where $n_1 = n_2 = -n_3 = 1$ for type-VIII and $n_1 = n_2 = n_3 = 1$ for type-IX. It seems to be impossible to integrate Eqs (60a) in case of $\lambda_i \neq 0$ except when $R_1 = R_2$ which leads to the self-dual Taub-NUT metrics already obtained by Hawking (1977) (see also Lorenz (1983a) for type-VIII). Case $\lambda_i = 0$ may be solved completely to give

$$R_i^2 = 2r^2F^{1/2}\left[-\delta n_i - \left(\frac{a_i}{r}\right)^4\right]^{-1}, \tag{61}$$

where $dr = r^{-3}(R_1R_2R_3)^2 d\eta$, $F^{1/2} = r^{-6}(R_1R_2R_3)^2$ and $a_i = \text{const}$. The Bianchi type-IX solution ($n_i = -\delta = 1$) has been first given by Belinskii et al. (1978) (see also Gibbons and Pope 1979). The solutions (61) are the first triaxial Bianchi types-VIII and IX solutions.

The Eguchi-Hanson types I and II solutions (Eguchi and Hanson 1978; Eguchi and Hanson 1979a, b; Eguchi et al. 1980) for Bianchi types-VIII and IX can be obtained from the equations

I:
$$R_1 = R_2 = rg, \; R_3 = r,$$
$$2g^2 = -\delta f(2g^2 - n_3), \quad 2g(rg' + g) = -\delta n_3 f, \tag{62a}$$

II:
$$R_1 = R_2 = r, \quad R_3 = rg,$$
$$2 = -\delta n_3 fg, \quad 2(rg' + g) = -\delta f(2 - n_3g^2), \tag{62b}$$

where $dt = f dr$ and $(\;)' := d/dr$. The corresponding solutions are

I:
$$g^2 = \frac{n_3}{2}\left[1 \pm \left(1 - n_3\left(\frac{a}{r}\right)^4\right)^{1/2}\right], \tag{63a}$$

II:
$$g^2 = n_3 + \left(\frac{a}{r}\right)^4, \tag{63b}$$

where $a = \text{const}$. The original Eguchi-Hanson solutions are those with $n_3 = -\delta = 1$.

The vacuum Einstein equations $R_{\mu\nu} = 0$ reduce to the form:

$$(\ln R_i^2)'' = n_i^2 R_i^4 - (n_j R_j^2 - n_k R_k^2)^2 \tag{64}$$

and

$$4[(\ln R_1)'(\ln (R_2 R_3))' + (\ln R_2)'(\ln R_3)'] = 2[n_1 n_2 R_1^2 R_2^2 + n_1 n_3 R_1^2 R_3^2 + n_2 n_3 R_2^2 R_3^2] - [n_1^2 R_1^4 + n_2^2 R_2^4 + n_3^2 R_3^4], \tag{65}$$

where i, j, k are in cyclic order. (65) is a first integral of (64). Equations (64) and (65) may be integrated completely if we impose the conditions that two of the R_i are equal. For type-VIII we can equate only R_1 with R_2 obtaining a symmetry about the third axis. This can be seen from the fact that the field equations for type-VIII do not turn into each other under any permutation of the indices i, j, k. For type-IX the intrinsic geometry of three-space does not privilege any direction of space. The solutions are given by

$$R_1^2 = R_2^2 = \tfrac{1}{4} q \sinh [q(\eta - \eta_2)] \operatorname{sech}^2 [\tfrac{1}{2} q(\eta - \eta_1)],$$

$$R_3^2 = q \operatorname{csch} [q(\eta - \eta_2)], \tag{66a}$$

$$R_1^2 = R_2^2 = \tfrac{1}{4} q \sinh [q(\eta - \eta_2)] \operatorname{csch}^2 [\tfrac{1}{2} q(\eta - \eta_1)],$$

$$R_3^2 = q \operatorname{csch} [q(\eta - \eta_2)], \tag{66b}$$

where $q = \text{const}$. The solution (66a) is new and represents an Euclidean Bianchi-Taub solution of type-VIII (see also Lorenz (1980)). The solution (66b) has been first given by Gibbons and Pope (1979). The more familiar forms

$$u(r)(r^2 - l^2) = k(r^2 + l^2) - 2mr, \quad k = n_3 \tag{67}$$

can be obtained by transforming (66a, b) to a canonical r system

$$r^2 = R_1^2 + l^2, \quad u(r) = R_3^2 \tag{68}$$

where $R_1^2 R_3 d\eta = u^{-1/2} dr$, $m = \text{const}$ and l denotes the NUT-parameter (see also Lorenz (1983b)).

4. *Conclusion*

We have given a complete discussion of all (anti)-self-dual solutions for diagonal Euclidean Bianchi types I–IX models. Besides being new exact solutions to Einstein's field equations, the solutions are double self-dual solutions of O(4) Yang-Mills equations (Oleson 1977; Wilczek 1977). The global properties, including the topological invariants χ (Euler characteristic) and τ (signature), as well as the regularity considerations of the obtained solutions will be discussed in a forthcoming paper (Lorenz 1983c).

The restriction of considering only diagonal (with the one exception of Bianchi type-IV) metrics is legitimated from the fact that no general exact non-diagonal solution of Einstein's field equations has been obtained until now. In addition, it was shown by us in this paper that no non-diagonal self-dual Bianchi type-IV solution can exist. We thus conclude that to best of our knowledge it seems to be very unlikely to construct non-diagonal self-dual solutions of Bianchi types I–IX. We finally should like to remark that a systematic investigation of the combined Bianchi-Einstein-Maxwell equations, made by us in the past few years and published in many papers, has shown that rather all integrable cases have now been found.

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