Vol. B14 (1983) ACTA PHYSICA POLONICA No 11

ON CLASSICAL CHROMODYNAMICS OF EXTERNAL CHARGES AND FIELDS

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(*Received May 9, 1983*)

Classical mechanics of colored particles in external, classical Yang-Mills fields is discussed from quantum-mechanical point of view. Theory of classical Yang-Mills fields in the presence of external sources is presented.

PACS numbers: 11.10.Np

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Introduction

During the last several years quantum chromodynamics (QCD) has become the most promising candidate for the theory of strong interactions. In favour of it one could cite, e.g. the fact that it puts the phenomenologically correct old quark model in a consistent dynamical context, or experimental confirmation of the perturbative calculations of cross sections for deep inelastic scattering.

Nevertheless, QCD is not fully understood yet. There are fundamental problems within it for which no satisfactory solutions are known, e.g., the problem of explaining the spectrum of hadrons and related problem of quark confinement. There are good reasons to believe that these and other difficult problems can be made easier after gaining a detailed knowledge about properties of the corresponding unquantized theory, i.e. about classical chromodynamics (CCD). This is the main motivation for the widespread studying of CCD.

That unquantized theory is obtained by replacing the quantum fields of QCD by classical ones. In this way we obtain an extremely complicated set of intercoupled Dirac and Yang-Mills equations. Even in the much simpler Abelian case (intercoupled Maxwell and Dirac equations) it is difficult to extract information from such a set of equations. Therefore, it is natural to simplify the problem further, namely to consider color charged matter in external Nonabelian gauge fields, and vice versa, Nonabelian gauge fields generated by external color currents. Thus, in this way we would like to imitate the traditional scheme of electrodynamics.

The main stream of investigations of CCD has always been devoted to pure Yang--Mills sector, i.e. the quark fields have been put to zero, for references see [1][1]. Actually, we are aware only of a single paper on CCD with quarks published before 1978: the paper by Mandula [2] on Yang-Mills fields generated by a fixed point-like quark source. The fact that very little was known about classical Yang-Mills fields generated by a fixed distribution of quarks was realized independently by several authors [3–6]. Since then the interest in the subject has been constantly growing, see e.g. [7–28]. Nevertheless, the problem is still far from being explored.

Much more extensively was investigated the complementary problem of motion of colored particles placed in an external Yang-Mills fields, see e.g. [29–46].

We would like to cover these two subjects, i.e. classical Yang-Mills fields generated by external sources and classical motion of color charged matter in external Yang-Mills field with the term classical chromodynamics of external charges and fields.

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[1] The vast literature on monopoles and vortices deals with Higgs fields which are regarded as dynamical variables — they are not externally fixed. Coupling of the scalar Higgs fields to Yang-Mills fields is different from that of fermions because of spin of fermions and because of the fact that the color current of the Higgs fields contains the term $A^a_\mu|\varphi|^2$ mixing Yang-Mills field and the Higgs's. This term is absent in the color current of fermions. Moreover, it is well-known [1] that in most problems the Higgs's in the Prasad--Sommerfield limit can be reinterpreted as the zeroth component of Euclidean Yang-Mills field, i.e. the problem is then essentially the pure Yang-Mills problem. All these facts cause that the numerous results obtained for Yang-Mills theory in the presence of Higgs fields are not relevant for Yang-Mills fields in the presence of spin $\frac{1}{2}$ quarks.

This paper is a review of our work on classical chromodynamics of external charges and fields. The content of the paper is divided into two parts.

In the first part we consider color charged matter in the external SU(2) gauge field. Specifically, we investigate the particle-like classical limit of the Dirac equation with the external Yang-Mills field. We find that such a limit does not exist in general. When this limit exists, there is an interesting mixing between spin and color degrees of freedom of the particle.

In the second part we investigate classical Yang-Mills fields generated by external sources. We discuss the problem of gauge invariance of energy in the presence of an external current and we present certain topological characteristics of the external charge distribution. Next, we describe the exact Abelian Coulomb solution for a set of spatially separated color charges, we discuss a perturbative approach to solving Yang-Mills equations with weak external sources, and we construct within this perturbative approach a Nonabelian Coulomb solution for the external color charge characterized by Hopf index ± 1. We also consider certain modification of CCD in which the Yang-Mills potentials are coupled to gauge invariant external sources.

Part I: Color Charged Particle in an External Nonabelian Gauge Field

I.1. The equations of motion for classical particle with spin and color spin

Classical mechanics of colored particles [29–46] is the very interesting example of classical mechanics with internal degrees of freedom, even though one should not expect to observe colored particles in any experiment, according to the color confinement hypothesis.

Here we would like to propose a set of classical equations of motion for a colored and spinning particle interacting with an external SU(2) Nonabelian gauge field. We find that because of mixing between color and spin, it is necessary to introduce a new classical dynamical variable $[J^{ab}]$, $a, b = 1, 2, 3$. The constraint relations between $[J^{ab}]$, the classical spin $\vec{S}$, and the classical color spin $\vec{I}$ are also found. The full presentation of our results is published elsewhere [37].

The classical equations of motion for a spinless, colored particle were extracted by Wong [29] from the Heisenberg equations for the momentum and color spin operators derived from Dirac equation with external SU(2) gauge field. The replacement of the operators by c-number classical quantities in the Heisenberg equations led to the classical equations of motion. In the non-relativistic limit these equations are

$$m\ddot{\vec{x}} = gI^a\vec{E}^a + \frac{g}{c}\dot{\vec{x}}\times\vec{B}^aI^a, \tag{1}$$

$$\dot{I}^a = \frac{g}{\hbar}\varepsilon_{abc}\left(A_0^b - \vec{A}^b\frac{\dot{\vec{x}}}{c}\right)I^c, \tag{2}$$

where $\vec{I} = (I^a)$, $a = 1, 2, 3$ is the color spin vector of the particle, the dots denote differentiations with respect to time. The color electric and magnetic fields are defined as

$$E^{ai} = F^a_{0i}, \quad B^{ak} = -\tfrac{1}{2}\,\varepsilon_{krt}F^a_{rt}, \tag{3}$$

where $F^a_{\mu\nu}$ is the SU(2) field strength tensor, see formula (3) in Ch. 1, §1 of Part II.

Our derivation also starts from the Heisenberg equations, however we make two essential improvements. First, we use the Foldy-Wouthuysen representation for the Dirac equation with SU(2) gauge external field. This allows us to avoid the well-known problem [47] with interpretation of the $\vec{\alpha} = \gamma^0\vec{\gamma}$ matrices as the classical velocity (this problem is present in Ref. [29]). Second, we identify the classical quantities as the expectation values of operators, assuming the quantum state of a wave packet form. We avoid the ambiguous and formal procedure consisting of replacing quantum operators by c-number quantities used in Ref. [29]. This allows us to observe mixing between spin and color spin. Moreover, in our approach from the knowledge of classical quantities one can gain an information about quantum mechanical wave function of the particle, while in Ref. [29] the wave function is completely abandoned.

Our results were obtained with the approximate Hamiltonian H_2 in the Foldy-Wouthuysen representation. The obtained classical theory is a nonrelativistic one. Moreover, the equations (21), (23), (24) below for the internal degrees of freedom can not be rewritten in the relativistic form by merely introducing the proper time by $\gamma d/dt = d/d\tau$. The equations have truly nonrelativistic form.

The Dirac Hamiltonian in the Foldy-Wouthuysen representation calculated up to the order $(mc)^{-2}$ is [37]

$$\begin{aligned} H = {} & mc^2\beta + \frac{1}{2m}\,\beta\left(\vec{p} - \frac{g}{c}\,\hat{\vec{A}}\right)^2 + g\hat{A}_0 + \frac{g\hbar}{2mc}\,\varepsilon_{iks}\hat{S}^s F_{ik} \\ & - \frac{g\hbar}{4m^2c^2}\,\varepsilon_{iks}\hat{S}^s\left[\hat{E}^i\left(p^k - \frac{g}{c}\,\hat{A}^k\right) + \left(p^k - \frac{g}{c}\,\hat{A}^k\right)\hat{E}^i\right] - \frac{g\hbar^2}{8m^2c^2}\,D_i\hat{E}^i, \end{aligned} \tag{4}$$

where $D_k\hat{E}^i$ is the covariant derivative of the color electric field, $\hat{A}_\mu = A^a_\mu\hat{T}^a$, $\hat{T}^a = \dfrac{\sigma^a}{2}$ are the SU(2) generators in the fundamental representation. $\beta = \gamma^0$ is taken here to be diagonal

$$\beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix},$$

and $\hat{S}^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$ is the spin operator.

From the Hamiltonian (4) we obtain the following Heisenberg equations of motion. We introduce the mechanical momenta,

$$\pi^r = p^r - \frac{g}{c}\,\hat{A}^r, \tag{5}$$

then

$$\frac{dx^i}{dt} = \frac{1}{m}\beta\pi^i - \frac{g\hbar}{2m^2c^2}\varepsilon_{tis}\hat{S}^s\hat{E}^t, \tag{6}$$

$$\frac{d\pi^r}{dt} = -\frac{g}{2c}\left(\hat{F}_{ri}\frac{dx^i}{dt} + \frac{dx^i}{dt}\hat{F}_{ri}\right) + g\hat{F}_{0r} - \frac{g\hbar}{2mc}\varepsilon_{iks}\hat{S}^s Dr\hat{F}_{ik}$$

$$+\frac{g\hbar}{4mc^2}\beta\varepsilon_{iks}\hat{S}^s\left[D_r\hat{E}^i\frac{dx^k}{dt} + \frac{dx^k}{dt}D_r\hat{E}^i\right] + \frac{g\hbar^2}{8m^2c^2}D_r(D_i\hat{E}^i). \tag{7}$$

We consider also spin S^t and color spin T^a operators,

$$\frac{d\hat{S}^t}{dt} = -\frac{g}{mc}\hat{F}_{tk}\hat{S}^k + \frac{g}{4mc^2}\beta\hat{S}^p\left(\hat{E}^t\frac{dx^p}{dt} - \hat{E}^p\frac{dx^t}{dt} + \frac{dx^p}{dt}\hat{E}^t - \frac{dx^t}{dt}\hat{E}^p\right), \tag{8}$$

$$\frac{d\hat{T}^a}{dt} = \frac{g}{\hbar}\varepsilon_{abc}\left[A_0^b - \frac{1}{2c}\left(\frac{dx^i}{dt}A^{bi} + A^{bi}\frac{dx^i}{dt}\right)\right]\hat{T}^c + \frac{g}{2mc}\varepsilon_{abc}\varepsilon_{its}\hat{S}^s\hat{F}^b_{it}\hat{T}^c$$

$$+\frac{g}{4mc^2}\varepsilon_{ips}\varepsilon_{bac}\left(E^{bi}\frac{dx^p}{dt} + \frac{dx^p}{dt}E^{bi}\right)\beta\hat{S}^s\hat{T}^c + \frac{g\hbar}{8m^2c^2}\varepsilon_{bac}(D_i\hat{E}^i)^b\hat{T}^c. \tag{9}$$

Now let us turn to the expectation values. In this Section we assume the following "particle-like" form of the Dirac bispinor ψ in the Schrödinger picture in the positive energy sector [47]

$$\psi(\vec{x}, t) = \begin{pmatrix} u^{\alpha\eta}(\vec{x}, t)\varphi(\vec{x}-\vec{x}(t)) \\ 0 \end{pmatrix}, \tag{10}$$

where $\alpha = 1, 2$ refers to spin, $\eta = 1, 2$ refers to color. Here $\vec{x}(t)$ is the trajectory of the corresponding classical particle, $\varphi(\vec{x}-\vec{x}(t))$ is the c-number valued wave packet localized at $\vec{x}(t)$, with the average momentum

$$\langle\varphi|\vec{p}|\varphi\rangle = m\dot{\vec{x}}(t). \tag{11}$$

We neglect the quantum mechanical spreading out of the wave packet. We find it convenient to assume that for $\vec{x}$ close to $\vec{x}(t)$

$$u(\vec{x}, t) = \left(1 + \frac{ig}{\hbar c}\hat{A}^i(\vec{x}(t), t)\Delta x^i - \frac{ig}{2\hbar c}\partial_i\hat{A}^k(\vec{x}(t), t)\Delta x^i\Delta x^k\right.$$

$$\left.+\tfrac{1}{2}\left(\frac{ig}{\hbar c}\right)^2\hat{A}^k(\vec{x}(t), t)\hat{A}^i(\vec{x}(t), t)\Delta x^i\Delta x^k\right)u_0(t), \tag{12}$$

where $\Delta x^i = x^i - x^i(t)$. Furthermore, while calculating the expectation values we assume that due to the localizability

$$\langle\varphi|\Delta x^i|\varphi\rangle = \langle\varphi|\Delta x^i\Delta x^k\Delta x^s|\varphi\rangle = 0 \tag{13}$$

(in fact, the spherical symmetry of $\varphi(\vec{x}-\vec{x}(t))$ with $\vec{x}(t)$ as the center will do this), and that

$$\langle\varphi|\Delta x^i \Delta x^k|\varphi\rangle = \langle\varphi|\Delta x^i \Delta x^k \Delta x^s \Delta x^p|\varphi\rangle = 0. \tag{14}$$

Then we have the following formulae:

$$\langle\psi|\pi^r_{\mathrm{S}}|\psi\rangle = m\dot{x}^r(t), \qquad \langle\psi|\pi^r_{\mathrm{S}}\pi^t_{\mathrm{S}}|\psi\rangle = m^2\dot{x}^r\dot{x}^t, \tag{15}$$

$$I^a(t) \overset{\mathrm{df}}{=} \langle\psi|\hat{T}^a_{\mathrm{S}}|\psi\rangle = u_0^+(t)\hat{T}^a_{\mathrm{S}}u_0(t), \tag{16}$$

$$S^b(t) \overset{\mathrm{df}}{=} \langle\psi|\hat{S}^b_{\mathrm{S}}|\psi\rangle = u_0^+(t)\hat{S}^b_{\mathrm{S}}u_0(t), \tag{17}$$

$$\tfrac{1}{2}\langle\psi|\,[\hat{T}^a_{\mathrm{S}}, \pi^r_{\mathrm{S}}]_+\,|\psi\rangle = mI^a\dot{x}^r(t), \tag{18}$$

$$\tfrac{1}{2}\langle\psi|\hat{S}^b_{\mathrm{S}}[\hat{T}^a_{\mathrm{S}}, \pi^r_{\mathrm{S}}]_+\,|\psi\rangle = mu_0^+(t)\hat{S}^b_{\mathrm{S}}\hat{T}^a_{\mathrm{S}}u_0(t)\dot{x}^r, \tag{19}$$

where the subscript S denotes the Schrödinger picture operators, $|\psi\rangle$ is the Schrödinger wave function (10).

Let us now calculate

$$\frac{d}{dt}\langle\psi|\pi^r_{\mathrm{S}}|\psi\rangle = {}_{\mathrm{H}}\langle\psi|\,\frac{d\pi^r}{dt}\,|\psi\rangle_{\mathrm{H}}$$

$$= \langle\psi|\ \text{(the r.h.s. of (7) taken to the Schrödinger picture)}\ |\psi\rangle$$

$$= -\frac{g}{c}\,\dot{x}^i F^a_{ri}(\vec{x}(t), t)I^a(t) + gF^a_{0r}(\vec{x}(t), t)I^a,$$

where in the last step we have used (15) and the definition (16) of the classical color spin vector $\vec{I}$. Moreover, because we neglect the quantum mechanical spreading out of the wave packet, which gives the contribution of the order $\hbar$ to $d|\psi\rangle/dt$ we neglect all other terms of the same or higher order in $\hbar$ for consistency. Thus, we have obtained the following classical equation for the trajectory

$$m\ddot{x}^r(t) = -\frac{g}{c}\,\dot{x}^i F^a_{ri}I^a + gF^a_{0r}I^a. \tag{20}$$

Similarly, for the classical color spin vector we obtain

$$\frac{dI^a}{dt} = \frac{g}{\hbar}\,\varepsilon_{abc}\left(A^b_0 - \frac{\dot{x}^i}{c}A^{bi}\right)$$

$$+\frac{g}{2mc}\,\varepsilon_{itb}\varepsilon_{cad}\left[-F^c_{it} + \frac{1}{2c}\,(E^{ci}\dot{x}^t - E^{ct}\dot{x}^i)\right]J^{db}(t), \tag{21}$$

where we have to introduce the new classical quantity

$$J^{cs}(t) \overset{\mathrm{df}}{=} \langle\psi|\hat{T}^c_{\mathrm{S}}\hat{S}^s_{\mathrm{S}}|\psi\rangle = u_0^+(t)\hat{T}^c_{\mathrm{S}}\hat{S}^s_{\mathrm{S}}u_0(t). \tag{22}$$

For the classical spin vector

$$\frac{dS^t}{dt} = \frac{g}{mc}\left[-F^a_{tp} + \frac{1}{2c}(E^{at}\dot{x}^p - E^{ap}\dot{x}^t)\right] J^{ap}(t). \tag{23}$$

In order to have a closed set of equations we have to add the equation for $J^{ap}(t)$.

In the case where spin and color decouple, $u_0^{\alpha\eta}(t) = \varphi^\alpha(t)\chi^\eta(t)$, we have $J^{ap}(t) = I^a(t)S^p(t)$. However, in the general case $J^{ap}(t)$ is an independent quantity, see next Section. The equation for $J^{ap}(t)$ follows from

$$\frac{d\hat{J}^{ap}}{dt} = \hat{T}^a \frac{d\hat{S}^b}{dt} + \frac{d\hat{T}^a}{dt}\hat{S}^b,$$

namely

$$\frac{dJ^{ab}}{dt} = \frac{g}{\hbar}\varepsilon_{acd}\left(A_0^c - \frac{\dot{x}^i}{c}A^{ci}\right)J^{db}$$

$$+ \frac{g}{8mc}\varepsilon_{ipb}\varepsilon_{cad}\left[-F^c_{ip} + \frac{1}{2c}(\dot{x}^p E^{ci} - \dot{x}^i E^{cp})\right] I^d$$

$$+ \frac{g}{4mc}\left[-F^a_{bp} + \frac{1}{2c}(E^{ab}\dot{x}^p - E^{ap}\dot{x}^b)\right] S^p. \tag{24}$$

The equations (20), (21), (23), and (24) are the classical equations of motion for the particle with spin and with color spin. The equation (20) coincides with the Wong's equation (1), others do not. Observe that from these equations it does not follow that $\vec{S}^2$ and $\vec{I}^2$ in general are constants of motion. They can under special circumstances become constants of motion, e.g. when $J^{ab} = I^a S^b$.

I.2. The constraint equations and the determination of the wave function $u_0(t)$

The equations (20), (21), (23), (24) have to be completed with constraint equations. The reason is that the fifteen numbers I^a, S^b, J^{ab} are expectation values in the single state $u_0(t)$. Therefore, these expectation values depend on 6 independent, real numbers forming $u_0(t)$ (because u_0 is normalized to 1 and because the overall phase factor of u_0 does not change the expectation values). Thus, the constraints are necessary if the classical mechanics based on the equations (20), (21), (23), (24) is to be related to the quantum mechanical Dirac particle. We find these constraint equations in this Section. We also shall show how to calculate $u_0(t)$ from known $I^a(t)$, $S^b(t)$, $J^{ab}(t)$, and we shall find the time evolution equation for $u_0(t)$, [37].

First, let us show that knowledge of the classical quantities $\vec{I}$, $\vec{S}$, J^{ab} determines the wave function $u_0(t)$ up to an arbitrary time dependent phase factor. This fact we shall regard as proof that the above set of classical dynamical variables describing the internal motion of the particle is complete, in the sense that any other classical, internal dynamical variable, i.e., the expectation value of an operator in the state u_0, is a function of $\vec{I}$, $\vec{S}$, $[J^{ab}]$.

To this end we shall regard the spinor $[u_0^{\alpha\eta}]$ as 2×2 matrix $\hat{u}_0$. Then, the normalization condition $u_0^{*\alpha\eta}u_0^{\alpha\eta} = 1$ takes the form

$$\mathrm{Tr}\,\hat{u}_0^+\hat{u}_0 = 1. \tag{25}$$

Furthermore,

$$I^a \equiv \langle u_0|\hat{T}^a|u_0\rangle = \tfrac{1}{2}\,\mathrm{Tr}\,(\hat{u}_0^*\sigma^a\hat{u}_0^{\mathrm{T}}) = \tfrac{1}{2}\,\mathrm{Tr}\,(\hat{u}_0\sigma^{a\mathrm{T}}u_0^+), \tag{26}$$

$$S^t = \tfrac{1}{2}\,\mathrm{Tr}\,(\hat{u}_0^+\sigma^t\hat{u}_0), \tag{27}$$

$$J^{ab} = \tfrac{1}{4}\,\mathrm{Tr}\,(\hat{u}_0^+\sigma^b\hat{u}_0\sigma^{a\mathrm{T}}), \tag{28}$$

where the star denotes the complex conjugation, and T denotes the transposition of the matrix. It is clear that we cannot determine the overall phase factor of u_0.

From (25)–(27) it follows that

$$\hat{u}_0\hat{u}_0^+ = \tfrac{1}{2}\,\sigma^0+\vec{S}\vec{\sigma}, \tag{29}$$

$$\hat{u}_0^+\hat{u}_0 = \tfrac{1}{2}\,\sigma^0+\vec{I}\vec{\sigma}^{\mathrm{T}}. \tag{30}$$

Equations (29), (30) imply that

$$|\det\hat{u}_0|^2 = \tfrac{1}{4}-\vec{I}^2 = \tfrac{1}{4}-\vec{S}^2. \tag{31}$$

Thus, we see that for SU(2)-colored particle

$$\vec{I}^2 = \vec{S}^2 \leqslant \tfrac{1}{4}. \tag{32}$$

It is easy to see that this fact is consistent with the equations (21), (23) only if

$$\varepsilon_{bac}I^aJ^{cs} = \varepsilon_{skr}S^kJ^{br}. \tag{33}$$

Utilising (26)–(30) it is easy to prove that the condition (33) is satisfied. Relations (32), (33) are examples of the constraint equations.

From (31) it follows that $\hat{u}_0$ is a singular matrix only when $\vec{S}^2 = \vec{I}^2 = 1/4$. It is easy to prove that $\det\hat{u}_0 = 0$ is equivalent to

$$u_0^{\alpha\eta} = \xi^\alpha\chi^\eta, \tag{34}$$

i.e., in this case the spin and color spin decouple. In this degenerate case knowledge of $\vec{I}$ and $\vec{S}$, together with the normalization conditions

$$\xi^+\xi = 1, \quad \chi^+\chi = 1,$$

determines ξ, χ up to the arbitrary time-dependent phase factor. For example, when $I^3 \neq -1/2$,

$$\chi = \exp\,[i\alpha(t)]\,(\tfrac{1}{2}+I^3)^{-1/2}\begin{pmatrix}\tfrac{1}{2}+I^3\\ I^1+iI^2\end{pmatrix}, \tag{35}$$

and if $I^3 = -1/2$

$$\chi = \exp\,[i\alpha(t)] \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Analogous formulae hold for ξ.

Let us remark here that the above relation between $\vec{I}(\vec{S})$ and the 2-spinor $\chi(\xi)$ can be refined by utilising the coherent states for the SU(2) group, [39].

In the degenerate case (34) we have $J^{ab} = I^a S^b$. The constraint (33) becomes trivialized to $0 = 0$. However, $[J^{ab}]$ does not cease to be an independent dynamical variable for the particle, as we shall argue below. This means that the relation $\det \hat{u}_0 = 0$ is not conserved in time.

Now, let us consider the general case, which also includes $\det \hat{u}_0 = 0$. In order to determine $\hat{u}_0$ we recall that any 2×2 matrix can be written in the form

$$\hat{u}_0 = HV, \tag{36}$$

where

$$H = H^+ = \sqrt{\hat{u}_0 \hat{u}_0^+} \tag{37}$$

is a positive definite, hermitean matrix, and V is a unitary matrix determined from (36). If $\hat{u}_0$ is not singular, the matrix V is determined uniquely, namely

$$V = H^{-1}\hat{u}_0. \tag{38}$$

From (29), (37) we obtain

$$H = \frac{1}{\sqrt{2\lambda}}(\lambda\sigma^0 + \vec{S}\vec{\sigma}), \tag{39}$$

where

$$\lambda = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 - 4\vec{S}^2}, \tag{40}$$

and σ^0 is the 2×2, unit matrix. Furthermore, because

$$\vec{\sigma}^{\mathrm{T}} = \tilde{\vec{\sigma}} = \begin{pmatrix} \sigma^1 \\ -\sigma^2 \\ \sigma^3 \end{pmatrix},$$

we obtain from (29), (30), (39) that

$$V^+ \vec{S}\vec{\sigma} V = \tilde{\vec{I}}\vec{\sigma}, \tag{41}$$

i.e., V represents a rotation which rotates $\vec{S}$ into $\tilde{\vec{I}}$, where

$$\tilde{\vec{I}} = \begin{pmatrix} I^1 \\ -I^2 \\ I^3 \end{pmatrix}.$$

For instance, when $\vec{S} \neq -\tilde{\vec{I}}$ we may take

$$V_0 = i\vec{\sigma}\,\frac{\vec{S}+\tilde{\vec{I}}}{|\vec{S}+\tilde{\vec{I}}|}\,. \tag{42}$$

Obviously, (41) does not determine V completely. Namely, we can take

$$V = V_0 C,$$

where C is any unitary matrix commuting with $\tilde{\vec{I}}\vec{\sigma}$. Any such C has the form

$$C = \exp\,[i\beta(t)]\exp\left[i\,\frac{\gamma(t)}{2}\,\frac{\tilde{\vec{I}}\vec{\sigma}}{|\tilde{\vec{I}}|}\right]. \tag{43}$$

Thus, we see that in the general case $\hat{u}_0$ is not determined by the knowledge of $\vec{I}(t)$ and $\vec{S}(t)$ — apart from the unessential phase factor $\exp\,[i\beta(t)]$ we do not know the function $\gamma(t)$.

Therefore $[J^{ab}]$ is the new dynamical variable for the classical particle, independent of $(I^a), (S^b)$. Observe that if at certain instant t_0

$$u_0^{\alpha\eta}(t_0) = \xi^{\alpha}(t_0)\chi^{\eta}(t_0), \tag{44}$$

then

$$J^{ab}(t_0) = I^a(t_0)S^b(t_0). \tag{45}$$

However, it is easy to check from (36), (38), (40) that the quantity $Q^{ab} = J^{ab} - I^a S^b$ is not a constant of motion. Therefore, in general (41), (42) do not hold for $t \neq t_0$, and therefore J^{ab} does not cease to be the independent dynamical variable.

From (39), (42), (43) we obtain

$$u_0 = \frac{1}{\sqrt{2\lambda}}\,\frac{1}{|\vec{S}+\tilde{\vec{I}}|}\exp\,[i\beta(t)]\left(\cos\frac{\gamma}{2} + i\,\frac{\lambda}{I}\sin\frac{\gamma}{2}\right)$$

$$[\tfrac{1}{2}\,|\vec{S}+\tilde{\vec{I}}|^2 + (\vec{S}\times\tilde{\vec{I}})\vec{\sigma} + \lambda(\vec{S}\vec{\sigma} + \tilde{\vec{I}}\vec{\sigma})]. \tag{46}$$

In the degenerate case $\lambda = I = S = 1/2$, and therefore in this case all dependence on $\gamma(t)$ takes the form of the undeterminable phase factor $\exp\,[i\gamma(t)/2]$. It can be shown that in this case (46) can be written in the form (34) with ξ, η given by (35).

In the general case however, the function $\gamma(t)$ does not appear in the form of a phase factor, so it has to be determined. This determination is possible if in addition to $\vec{I}, \vec{S}$ we also know the matrix $[J^{ab}]$. From (46) we obtain

$$J^{ab} = \tfrac{1}{4}\,\frac{I^a S^b}{\vec{I}^2} + \sin\gamma A^{ab} + \cos\gamma B^{ab}, \tag{47}$$

where

$$A^{ab} = \frac{\lambda - 2\vec{S}^2}{4\lambda|\vec{I}|}\left[\varepsilon_{ad\bar{b}} I^d - 2\varepsilon_{adc}\frac{I^d S^{\bar{c}}(S^b + I^{\bar{b}})}{|\vec{S}+\vec{\bar{I}}|^2}\right], \tag{48}$$

$$B^{ab} = -\frac{I^a S^b}{4\vec{I}^2} + \frac{\lambda - 2\vec{S}^2}{2\lambda|\vec{S}+\vec{\bar{I}}|^2}(S^{\bar{a}} + I^a)(S^b + I^{\bar{b}}) + \frac{1}{4\lambda}[2S^b I^a - (\lambda - 2\vec{S}^2)\delta^{ab}].$$

In these formulae the barred indices $\bar{a}$, $\bar{b}$, etc., denote the change of sign of the vector or the tensor when the value of the index equals to two, e.g.,

$$[\delta^{\bar{a}b}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The presence of the barred indices is due to the fact that $\sigma^{aT} = \sigma^{\bar{a}}$. From (47) we can determine γ if we know $\vec{I}$, $\vec{S}$, $[J^{ab}]$.

Thus, we have proved that $\vec{I}$, $\vec{S}$, $[J^{ab}]$ form the complete set of classical dynamical variables for the internal motion of the particle. Now, we shall find the constraints. For 15 real number valued quantities (I^a), (S^b), $[J^{ab}]$ we have to find 9 independent equations, in order to be left with 6 independent quantities.

In order to find the constraints, we consider the matrix $M^{\nu}{}_{\mu}$, defined by

$$M^{\nu}{}_{\mu} = \tfrac{1}{2}\operatorname{Tr}(\hat{u}_0^{+}\sigma^{\nu}\hat{u}_0\sigma^{\mu}). \tag{49}$$

Comparing (49) with definitions of I^a, S^b, J^{ab}, we see that

$$M^0{}_0 = \tfrac{1}{2},$$

$$M^b{}_0 = S^b, \quad M^0{}_a = I^{\bar{a}}, \quad M^b{}_a = 2J^{\bar{a}b}. \tag{50}$$

The matrix $M^{\nu}{}_{\mu}$ obeys the relation

$$M^{\nu}{}_{\mu} g_{\nu\varrho} M^{\varrho}{}_{\sigma} = |\det \hat{u}_0|^2 g_{\mu\sigma}, \tag{51}$$

where $(g_{\mu\nu}) = (1, -1, -1, -1)$ is the Minkowski space-time metrics. In the degenerate case this relation can be easily verified by a direct calculation. In the nondegenerate case, $\det \hat{u}_0 \neq 0$, this identity comes simply from the fact that $L^{\nu}{}_{\mu} = M^{\nu}{}_{\mu}/|\det \hat{u}_0|$ is a Lorentz transformation, because the matrix $\hat{u}_0/(\det u_0)^{1/2}$ is an element of the SL(2, C) group. Here we use the well-known relation between SL(2, C) and the proper, orthochronous Lorentz group, [48]. The Lorentz transformations obey the relation

$$L^{\nu}{}_{\mu} g_{\nu\varrho} L^{\varrho}{}_{\sigma} = g_{\mu\sigma}.$$

This relation leads to (51).

Let us recall that $|\det \hat{u}_0|$ is determined by $\vec{I}$ or $\vec{S}$ via (31).
From (50), (51) we obtain that

$$\tfrac{1}{4}-\vec{S}^2 = |\det \hat{u}_0|^2, \tag{52}$$

$$\tfrac{1}{4}I^a - J^{ab}S^b = 0, \tag{53}$$

$$4J^{ab}J^{db}-I^aI^d = |\det \hat{u}_0|^2\delta^{ad}. \tag{54}$$

Another set of relations is obtained from the fact that if $L^\nu{}_\mu$ is a Lorentz transformation, then $(L^\mathrm{T})^\nu{}_\mu$ is a Lorentz transformation too. The difference between L and L^T is equivalent to the interchange of $\vec{S}$ and $\vec{I}$, and to the replacement of $J^{\bar{a}b}$ by $J^{b\bar{a}}$. Thus, we obtain

$$\tfrac{1}{4}-\vec{I}^2 = |\det \hat{u}_0|^2, \tag{55}$$

$$\tfrac{1}{4}S^b - J^{ab}I^a = 0, \tag{56}$$

$$4J^{db}J^{dc}-S^bS^c = |\det \hat{u}_0|^2\delta^{bc}. \tag{57}$$

In the degenerate case these relations can be easily verified by a direct calculation.

The equations (52), (55) are equivalent to (31). The equations (53), (54) are the nine constraint equations. The equations (56), (57) are equivalent to (53), (54) because (53), (54) together form the sufficient condition for $L^\nu{}_\mu$ to be the Lorentz transformation. The previously found relations (32), (33) also follow from (52)-(54).

The next problem to be investigated is the question whether the classical equations of motion (21), (23), (24) respect the constraints, i.e., whether the above relations are conserved in time if $\vec{I}$, $\vec{S}$, $[J^{ab}]$ evolve in time according to the equations of motion.

It seems that the most illuminating way to find the answer to this question is to observe that those three classical equations of motion can be derived from a single equation for $\hat{u}_0(t)$. Then, the solutions to (21), (23), (24) can be regarded as the expectation values (26)–(28) in the state $\hat{u}_0(t)$ for all t — this would guarantee that together they form the matrix (49) for all t, i e., that the constraints are conserved.

Such an equation for $\hat{u}_0$ can be derived in the following manner. From the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = H_2\psi$$

we obtain for the wave packet (10) that

$$i\hbar\frac{\partial\tilde{\varphi}}{\partial t}u_0 + i\hbar\tilde{\varphi}\frac{\partial}{\partial t}u_0 = H_2\tilde{\varphi}u_0, \tag{58}$$

$$\psi(\vec{x},t) = u(\vec{x},t)\varphi(\vec{x}-\vec{x}(t)) \equiv \tilde{\varphi}(\vec{x},t)u_0(t).$$

Here we regard u_0 as the 2-spinor, not as a 2×2 matrix.

Now we assume that

$$u_0^+ \frac{\partial}{\partial t} u_0 = 0. \tag{59}$$

In fact, this assumption does not restrict the generality of our arguments, because if a certain $\bar{u}_0$ does not obey (59), then

$$u_0(t) = \exp\left[- \int_0^t \left(\bar{u}_0^+ \frac{\partial}{\partial t'} \bar{u}_0 \right) dt' \right] \bar{u}_0(t)$$

obeys it. Because $\bar{u}_0^+ \bar{u}_0 = 1$, the above exponential is a time-dependent phase factor (the integrand is imaginary). Therefore, (59) is merely a restriction on the overall phase of u_0, which has no effect on $\vec{I}$, $\vec{S}$, $[J^{ab}]$, as is clear from (16), (17), (22).

From (58), (59) we obtain that

$$i\hbar \frac{\partial \varphi}{\partial t} = (u^+ H_2 u)\varphi - \frac{g}{c} A_k^a \dot{x}^k I^a \varphi, \tag{60}$$

and therefore

$$i\hbar \frac{\partial}{\partial t} u_0 = \bar{H}_2 u_0 - (u_0^+ \bar{H}_2 u_0) u_0, \tag{61}$$

where

$$\bar{H}_2 = \int d^3\vec{x} \varphi^* \left(1 - \frac{ig}{\hbar c} \hat{A}^i \Delta x^i\right) H_2 \left(1 + \frac{ig}{\hbar c} \hat{A}^k \Delta x^k\right) \varphi - \frac{g}{c} \hat{A}^i \dot{x}^i + \frac{g}{c} A^{ai} I^a \dot{x}^i \tag{62}$$

is the "effective" Hamiltonian for spin and color degrees of freedom. It is easy to check that $\bar{H}_2$ has the form (we neglect the terms of order $\hbar^2$ because we neglect the spreading out of the wave packet which gives the contribution of order $\hbar$ to $\partial\varphi/\partial t$):

$$\bar{H}_2 = \tfrac{1}{2} m \dot{\vec{x}}^2 - \frac{g}{c} \hat{A}^i \dot{x}^i + g\hat{A}_0 + \frac{g}{c} A_i^a I^a \dot{x}^i$$

$$+ \frac{g\hbar}{2mc} \varepsilon_{iks} \hat{S}^s \hat{F}_{ik} - \frac{g\hbar}{2mc^2} \varepsilon_{iks} \dot{x}^k \hat{S}^s \hat{E}^i, \tag{63}$$

where we have used (15). Of course, $\bar{H}_2$ is a hermitian matrix.

Thus, the time evolution of u_0 is governed by the nonlinear equation (61). In fact, the nonlinear term in (61) is superficial. The simple change of phase of u_0, performed by passing to

$$w(t) = \exp\left[\frac{i}{\hbar} \int_0^t (u_0^+ \bar{H}_2 u_0) dt' \right] u_0(t) \tag{64}$$

removes this term. Namely, $w(t)$ obeys the linear equation

$$i\hbar \frac{\partial w}{\partial t} = \bar{H}_2 w. \tag{65}$$

The nonlinear term in (61) is necessary in order to ensure (59). Of course, $w(t)$ does not obey (59) in general.

From (64), (65) we see that

$$\frac{d}{dt}(u_0^+ \hat{P} u_0) = \frac{i}{\hbar} u_0^+ [\bar{H}_2, \hat{P}] u_0, \tag{66}$$

where $\hat{P}$ denotes $\hat{T}^a$, $\hat{S}^b$ or $\hat{J}^{ab}$. It is easy to check that this equation leads to the equations (21), (23), (24) for I^a, S^b, J^{ab}.

Now we can prove that the constraints are conserved in time. The proof is based on the plausible assumption that the equations (21), (23), (24) for the fixed trajectory $\vec{x}(t)$ have a unique solution determined by the initial data $\vec{I}(t_0)$, $\vec{S}(t_0)$, $[J^{ab}(t_0)]$. If the initial data are specified in such a way that the constraints are satisfied, then there exists $u_0(t_0)$ such that (26–28) are true for $t = t_0$. Next, we solve (61) for $u_0(t)$ with the $u_0(t_0)$ as the initial data. Applying (26–28) again with the calculated $u_0(t)$ we obtain the solution $\vec{I}(t)$, $\vec{S}(t)$, $[J^{ab}(t)]$ of the equations (21), (23), (24) with the chosen initial values. As for this solution (26–28) are true for all t, the constraints are conserved in time.

Finally, let us state once more the most interesting result of this Section: the internal degrees of freedom for the classical particle with spin and SU(2) color spin are described by a 4×4 matrix $[M^\nu{}_\mu]$, which is closely related to an element of the SO(3,1) group, due to the constraint equations.

Finally let us note that we do in fact not take the classical limit for spin and color degrees of freedom. We just consider the expectation values of quantum spin and color operators for spin $\frac{1}{2}$ and color $\frac{1}{2}$ particle.

I.3. Limitation of the concept of the classical colored particle

The derivation of the classical equations presented above relies on the "particle-like" Ansatz (10) and the assumptions (13) and (14). If they are true, the classical equations form a classical approximation to the Dirac equation with the external SU(2) gauge field. If they are not true, the classical equations can still be considered as a selfconsistent basis for a classical mechanics of a spinning, colored particle. However, in this case the classical mechanics ceases to be a classical approximation to the quantum mechanics based on the Dirac equation.

In fact we would like to present a simple example which suggests that the relevance of the classical mechanics for the classical limit of quantum mechanics of a particle in the external Yang-Mills field seems to be restricted, [38]. The point is that we find examples in which the wave equation, here for simplicity we consider the Schrödinger equation, does not allow for a satisfactory notion of a classical trajectory of a single point-like particle even in the limit $\hbar \to 0$. However, in the particular circumstances where such a trajectory can be defined, Eqs. (1) and (2) can be derived, e.g. by the semiclassical method as we will show in this Section. In the other case, the Eqs. (1) and (2) can still be considered as a self-consistent basis for the very interesting example of classical mechanics with internal degrees of freedom, however without correspondence to quantum mechanics of a point-like particle in the external Yang-Mills field.

We consider the Schrödinger equation for a spinless particle with the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - \frac{g}{c}\vec{A}^a\hat{T}^a\right)^2 + gA_0^a\hat{T}^a \tag{67}$$

where $\hat{T}^a = \frac{\sigma^a}{2}$ are the generators of the SU(2) gauge group. We shall consider the simple case of a gauge potential of the "Abelian" type

$$A_\mu^a(\vec{x}, t) = h^a A_\mu(\vec{x}, t), \tag{68}$$

where $\vec{h} = (h^a)$ is a constant vector in the color space, and $\vec{h}^2 = 1$.

Our argument for the lack of a satisfactory notion of the classical trajectory is based on the investigation of time evolution of a wave function which at the moment $t = t_0$ has the form of the localized at $\vec{x} = \vec{x}_0$ wave packet

$$\psi^\eta(\vec{x}, t_0) = u_0^\eta(t)\varphi(\vec{x}-\vec{x}_0), \tag{69}$$

where the index η describes color degrees of freedom. We assume also that the wave packet is localized in the momentum space, the average momentum being $m\vec{v}_0$.

Let $\vec{e}_\pm$ be the normalized eigenvectors of the Hermitean matrix $h^a\hat{T}^a$,

$$(h^a\hat{T}^a)\vec{e}_\pm = \pm\tfrac{1}{2}\vec{e}_\pm. \tag{70}$$

The Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\vec{\psi} = H\vec{\psi}$$

can be projected one $\vec{e}_\pm$, yielding

$$i\hbar\frac{\partial}{\partial t}\varphi_\pm = H_\pm\varphi_\pm, \tag{71}$$

where

$$\varphi_\pm(\vec{x}, t) = \vec{e}_\pm\vec{\psi}(\vec{x}, t), \tag{72}$$

$$\vec{\psi}(\vec{x}, t) = \vec{e}_+\varphi_+(\vec{x}, t)+\vec{e}_-\varphi_-(\vec{x}, t), \tag{73}$$

and

$$H_\pm = \frac{1}{2m}\left(\vec{p}\mp\frac{g}{2c}\vec{A}\right)^2 \pm \frac{g}{2}A_0. \tag{74}$$

From Eqs. (71)–(74) we see that any time dependent wave function can be represented as the sum of two wave functions, each evolving in time independently of the other. The equations (71) can be regarded as the Schrödinger equations for two scalar (i.e., colorless) particles with the opposite electric charges $\pm g/2$, placed in the external electromagnetic field A_μ. Applying these observations to the initial wave packet (69) we obtain that

$$\varphi_\pm(\vec{x}, t_0) = c_\pm \varphi(\vec{x}-\vec{x}_0),$$

where

$$\vec{u}_0(t_0) = c_+\vec{e}_+ + c_-\vec{e}_-$$

evolve as the independent wave packets. Hence, if the usual conditions for the classical limit of quantum mechanics are satisfied, we can approximately write that

$$\varphi_\pm(\vec{x}, t) = c_\pm \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} dt'\left[\pm\frac{g}{2}\left(-A_0+\frac{\dot{\vec{x}}_\pm\vec{A}}{c}\right) + \frac{m}{2}\dot{\vec{x}}_\pm^2\right]\right\}\varphi(\vec{x}-\vec{x}_\pm(t)), \tag{75}$$

where $\vec{x}_+(t)$ ($\vec{x}_-(t)$) is the classical trajectory of the particle with the electric charge $+g/2$ $(-g/2)$, placed in the gauge field A_μ with the initial data $\vec{x}(t_0) = \vec{x}_0$, $\dot{\vec{x}}(t_0) = \vec{v}_0$. The external field A_μ is taken at the point $\vec{x} = \vec{x}_\pm(t)$, respectively.

The form (75) of the wave function can easily be justified within the framework of the Feynman path integrals [49]. In particular, the phase factor in Eq. (75) is just $\exp(i/\hbar S[\vec{x}_\pm])$ where $S[x_\pm]$ is the classical action for the classical trajectory $\vec{x}_\pm(t)$ — it is the common overall factor in any semiclassical approximation.

In general the trajectories $\vec{x}_+(t), \vec{x}_-(t)$ are of course different, although the initial positions and velocities are identical. This means that the time dependent wave function $\vec{\psi}(\vec{x}, t)$ does not in general have the form of a single wave packet. The initial wave packet (69) has dissociated in two separate wave packets. Observe that the difference between $\vec{x}_+(t)$ and $\vec{x}_-(t)$ does not vanish even in the limit $\hbar \to 0$ because $\vec{x}_\pm(t)$ obey different classical Newton equations which do not contain $\hbar$. In this sense, the dissociation of the wave packets in the Nonabelian gauge field is a macroscopic phenomenon.

In the Abelian case one could also have a dissociation of the initial wave packet, e.g., in experiments of the Stern-Gerlach type. However, in this case the separation is due to the coupling of spin to the external field. Because spin couplings are proportional to $\hbar$, the separation between wave packets vanishes in the limit $\hbar \to 0$ — therefore in the Abelian case the satisfactory notion of classical trajectory for a single classical particle in the external field can be introduced in that limit.

When

$$\vec{x}_+(t) \approx \vec{x}_-(t) \equiv \vec{x}(t), \tag{76}$$

what implies also that $\dot{\vec{x}}_+(t) \approx \dot{\vec{x}}_-(t)$, we can write that

$$\vec{\psi}(\vec{x}, t) = \exp\left(\frac{im}{2\hbar}\int_{t_0}^{t} dt'\dot{\vec{x}}^2\right)\varphi(\vec{x}-\vec{x}(t))u_0(t), \tag{77}$$

where

$$u_0(t) = c_+ \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} dt'\left(-\frac{g}{2}A_0+\frac{g}{2c}\dot{\vec{x}}\vec{A}\right)\right]\vec{e}_+$$

$$+c_- \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} dt'\left(\frac{g}{2}A_0-\frac{g}{2c}\dot{\vec{x}}\vec{A}\right)\right]\vec{e}_-. \tag{78}$$

In this case it is easy to show that

$$I^a \overset{\text{df}}{=} \langle\psi|\hat{T}^a|\psi\rangle \tag{79}$$

obeys the Wong's equation (2) using the formula

$$I^a(t) = \tfrac{1}{2}(|c_+|^2-|c_-|^2)h^a+2\,\mathrm{Re}\left\{m^a c_+^* c_- \exp\left[\frac{i}{\hbar}g\int_{t_0}^{t}\left(A_0-\frac{\dot{\vec{x}}}{c}\vec{A}\right)dt'\right]\right\}, \tag{80}$$

where $\vec{m} \overset{\text{df}}{=} \vec{e}_+^{\,+}\vec{T}\vec{e}_-$ obeys the relation $\vec{h}\times\vec{m} = i\vec{m}$. Eq. (80) follows from Eqs. (77), (78), and (79). The Wong's equation (1) can also be obtained in this case. Namely, $\vec{x}_\pm(t)$ obey the equations

$$m\ddot{\vec{x}}_\pm(t) = \pm\frac{g}{2}\left(\vec{E}+\frac{1}{c}\dot{\vec{x}}_\pm\times\vec{B}\right). \tag{81}$$

Then it is easy to see that the center of mass,

$$\vec{x}(t) = |c_+|^2\vec{x}_+(t)+|c_-|^2\vec{x}_-(t)$$

obeys Eq. (1), because $\vec{h}\vec{I} = \frac{1}{2}(|c_+|^2-|c_-|^2)$ as it follows from Eq. (80).

The condition (76) is obeyed for sufficiently small t because of the equality of initial positions and velocities. However, for larger t it is not satisfied in general because of the influence of the external field. Then there is no satisfactory notion of a classical trajectory of a pointlike particle. Also, Wong's equation (2) then gives wrong prediction for $\vec{I}(t)$. For the potential (68), namely, it predicts in general a rotation around $\vec{h}$ for all t, while Eqs. (79), (73) and (75) give

$$\vec{I}(t) = \tfrac{1}{2}(|c_+|^2-|c_-|^2)\vec{h} = \text{const} \tag{82}$$

when the two wave packets $\varphi_\pm$ become spatially separated.

Now we would like to present another argument for the lack of the point-particle-like classical limit. As is well-known, in the Abelian case the classical limit can be obtained by the substitution

$$\psi = \exp\left(\frac{i}{\hbar} S\right)$$

into the Schrödinger equation and letting $\hbar \to 0$. The real number valued function S turn out to satisfy the classical Hamilton-Jacob equation. In the Nonabelian case we can sub stitute

$$\vec{\psi} = \exp\left(\frac{i}{\hbar} S\right)\vec{f}.$$

In the leading order in $\hbar$ we obtain

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\vec{f}^{+}\left(\frac{\partial S}{\partial \vec{x}} - \frac{g}{c}\vec{A}^{a}\hat{T}^{a}\right)\vec{f} + gA_0^a\vec{f}^{+}\hat{T}^{a}\vec{f}, \tag{83}$$

and

$$\left(\frac{\partial S}{\partial \vec{x}} - \frac{g}{c}\vec{A}^{a}\hat{T}^{a}\right)^2\vec{f} = \left[\vec{f}^{+}\left(\frac{\partial S}{\partial \vec{x}} - \frac{g}{c}\vec{A}^{a}\hat{T}^{a}\right)^2\vec{f}\right]\vec{f}. \tag{84}$$

The last equation implies that $\vec{f}$ is an eigenvector of $h^a\hat{T}^a$ because A_i^a has the form (68). This means that $\vec{f} = \vec{e}_{\pm}$, as calculated in Eq. (70). Thus, from (83) we obtain two Hamilton-Jacobi equations, one with electric charge $+g/2$, the other one with $-g/2$. Thus we again find the two independent classical motions.

Let us also consider the gauge potential of a more Nonabelian type that Eq. (68) namely

$$A_0^a = 0, \quad A_i^a = A\delta_i^a, \tag{85}$$

A — constant. Such a potential gives $E^{ai} = 0$, $B^{ai} = \frac{Ag}{\hbar c}\delta^{ai}$. It is easy to check that the spectrum of the Hamiltonian (67) has two branches

$$E_{\pm}(\vec{p}) = \frac{1}{2m}\vec{p}^2 \mp \frac{Ag}{2mc}|\vec{p}| + \frac{3g^2A^2}{8mc^2}. \tag{86}$$

Therefore, the initial wave packet (69) will have two components, each moving with different group velocity

$$\vec{v}_{\text{group}} = \left.\frac{\partial E(\vec{p})}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_0} = \frac{1}{m}\vec{p}_0 \mp \frac{1}{2}\frac{gA}{mc}\frac{\vec{p}_0}{|\vec{p}_0|}, \tag{87}$$

where p_0 is the center of the wave packet (6) in the momentum space[2].

[2] Let us note in passing that in this case the Wong's equations (1) and (2) have interesting integral of motion, namely $m\dot{\vec{x}} + \hbar\vec{I}$, which resembles the well-known example of the Nonabelian monopole field [34], [35] where $\vec{L} + \hbar\vec{I}$ is an integral of motion, $\vec{L}$ — angular momentum of the particle.

The example (85) can be easily generalized to arbitrary constant potentials, A^a_μ = const.

In both the considered cases, Eqs. (68) and (85), the splitting of the initial wave packet is due to the presence of the color operators $\hat{T}^a$. Therefore we think that this phenomenon is typical for most of Nonabelian gauge potentials, although we cannot exclude a possibility of the existing of particular gauge potentials in which the initial wave packet will not disperse on the macroscopic scale. On the whole we are led to the general conclusion that the relevancy of the classical mechanics of colored particles for the description of the classical limit of quantum mechanics of colored particles is restricted. Nevertheless, such a classical mechanics remains to be the very interesting example of selfconsistent classical mechanics of a particle with internal degrees of freedom for any external Nonabelian gauge field.

Our considerations can easily be generalized to SU(n) fields. Then, the equation (70) will have n different eigenvalues, and the initial wave packet will in general dissociate into n separate wave packets.

Let us end this Section with the following remark connected with the real, physical theory, i.e. QCD. The above presented considerations suggest that the color charged matter, i.e. the quark matter, placed in an external classical Nonabelian gauge field in general will tend to disperse all over the space. For example, one can consider the potential of the form $A^a_\mu(\vec{x}) = A_\mu(\vec{x})h^a(\vec{x})$, where $h^a(\vec{x})$ is constant in each of regions Ω_i covering the whole space, however the direction of $\vec{h}$ changes from region to region. Then the initial wave packet will dissociate into many separate wave packets, the number of them depends on the number of the crossed regions Ω_i. According to e.g. Ref. [50] the QCD vacuum is filled with randomly fluctuating Nonabelian gauge fields. In such a vacuum, in order to have a localized clot of color charged matter propagating in a definite direction it is necessary to have a strong force binding the matter together in order to prevent it from dispersing over the space. The commonly conjectured confinement force could be such a force. Without this force all matter would disperse all over the space!

1.4. Final remarks

The formal aspects of classical mechanics of colored spinless particle were investigated in a number of papers — see [30, 32, 33, 46] and references therein. In particular, the Lagrangian and Hamiltonian formulations of the classical mechanics were constructed, and usefulness of Grassmann variables was advocated. The classical mechanics of particles with spin also was considered in numerous papers, see [51, 52] and references therein. Particles bearing both spin and color spin were considered earlier in [30, 33], and recently in [44], however within an entirely different framework. Our work indicates the possibility of nontrivial mixing between spin and color, resulting in the new classical observable $[J^{ab}]$.

Much less investigated was the problem of correspondence between the classical mechanics and quantum mechanics of colored particles. The treatment in the pioneering paper [29] was not satisfactory. We find that such a relation is very limited — it is difficult to extract the notion of classical trajectory of the colored particle from quantum mechanics. This fact corresponds rather well with the commonly believed confinement of colored particles.

Finally, let us mention that even though classical particle with color spin is not a physical object, the classical mechanics of colored particles can be a useful theoretical model. For example, it helps to formulate hypotheses about properties of color interactions, see e.g. [31, 35, 36]. Also it can be applied directly in quantum calculations within the framework of the proper time method [53], see e.g. [54–56].

Part II: Nonabelian Gauge Fields Generated by External Sources

II.1. Rudiments of the theory of Nonabelian gauge fields in the presence of external sources

II.1.1. Energy of the system of Nonabelian gauge fields and external sources

In the following we consider SU(2) gauge fields. We frequently use matrix notation

$$\hat{A}_\mu = A_\mu^a T^a, \tag{1}$$

where $T^a = \sigma^a/2$ are the generators of SU(2) gauge group,

$$[T^a, T^b] = i\varepsilon_{abc} T^c. \tag{2}$$

The field strength tensor is

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + ig[\hat{A}_\mu, \hat{A}_\nu].\ ^3 \tag{3}$$

In the presence of an external source Yang-Mills equations have the form

$$D_\mu \hat{F}^{\mu\nu} = \hat{j}^\nu, \tag{4}$$

where $\hat{j}^\nu$ is the external source, and

$$D_\mu \hat{F}^{\mu\nu} = \partial_\mu \hat{F}^{\mu\nu} + ig[\hat{A}_\mu, \hat{F}^{\mu\nu}] \tag{5}$$

is the covariant derivative of the field strength tensor. From (4) it follows that

$$D_\mu \hat{j}^\mu = 0, \tag{6}$$

where

$$D_\mu \hat{j}^\alpha = \partial_\mu \hat{j}^\alpha + ig[\hat{A}_\mu, \hat{j}^\alpha].$$

This condition is a consistency condition. If $\hat{A}_\mu$ does not obey (6), then it can not solve (4). Observe that because (6) involves $\hat{A}_\mu$, it can not be regarded as a simple condition on

[3] In this part we use the units $c = \hbar = 1$.

$\hat{j}_\mu$. This is in contiast to the Abelian case, in which we have $\partial_\mu j^\mu = 0$ what can be obeyed just by an appropriate choice of j_μ. In practice (6) is the fifth (matrix) equation for $\hat{A}_\mu$.

The Nonabelian gauge transformations have the form

$$\hat{A}'_\mu = \omega \hat{A}_\mu \omega^{-1} + \frac{i}{g} \partial_\mu \omega \omega^{-1}, \tag{7}$$

$$\hat{F}'_{\mu\nu} = \omega \hat{F}_{\mu\nu} \omega^{-1}, \tag{8}$$

$$\hat{j}'_\mu = \omega \hat{j}_\mu \omega^{-1}, \tag{9}$$

where $\omega = \omega(\vec{x}, t) \in SU(2)$. The Yang-Mills equations (4) are covariant under these transformations.

The equation following from (4) for $\nu = 0$ is the Nonabelian Gauss law

$$D_i \hat{F}^{i0} = \hat{j}^0. \tag{10}$$

It is easy to check that it is sufficient to impose (10) at certain instant $t = t_0$ — due to the other Yang-Mills equations it will be obeyed for all t. This follows from the equation

$$D_0(D_i \hat{F}^{i0} - \hat{j}^0) = 0.$$

Thus, (10) is in fact a constraint on the initial data for time-dependent Yang-Mills equations.

Now we shall consider the energy of the system of external sources and gauge fields. The energy-momentum tensor for Yang-Mills fields is (for a pedagogical derivation see, e.g. [57])

$$T^{\mu\nu} = -F^{a\mu\beta} F^{a\nu}{}_\beta + \tfrac{1}{4} g^{\mu\nu} F^{a\alpha\beta} F^a{}_{\alpha\beta}. \tag{11}$$

It is easy to check that from (4) it follows that

$$\partial_\mu T^{\mu\nu} = -j^a_\beta F^{a\nu\beta}. \tag{12}$$

Thus, T^{00} is not conserved in general. However, in the particular case when $j^a_i = 0$ we have

$$\partial_\mu T^{\mu 0} = 0, \tag{13}$$

and therefore

$$E = \int d^3\vec{x}\, T^{00} = \tfrac{1}{2} \int (E^{ai} E^{ai} + B^{ai} B^{ai}) d^3\vec{x}, \tag{14}$$

where

$$E^{ai} \equiv F^a_{0i}, \quad B^{ai} = -\tfrac{1}{2} \varepsilon_{iks} F^{aks} \tag{15}$$

are the color electric and magnetic fields respectively, can be adopted as the conserved energy of the system of Yang-Mills fields and external color charge $\hat{j}_0$.

The physical meaning of the formula (14) becomes a little bit more transparent when one decomposes the electric field into longitudinal and transverse components, [7]. The longitudinal component is not an independent variable because it can be calculated from the Gauss law (10).

In the Abelian case, we have

$$\vec{E} = \vec{E}_T + \vec{E}_L,$$

where

$$\operatorname{div} \vec{E}_T = 0, \quad \operatorname{rot} \vec{E}_L = 0, \quad \vec{E}_L = -\operatorname{grad} \chi,$$

and from the Abelian Gauss law

$$\Delta \chi = -\varrho,$$

where $\varrho \equiv j_0$. Then, the energy can be written as

$$E = \tfrac{1}{2} \int d^3\vec{x}(\vec{E}^2 + \vec{B}^2) = \tfrac{1}{2} \int d^3\vec{x}(\vec{E}_T^2 + \vec{B}^2) + \frac{1}{8\pi} \int \frac{\varrho(\vec{x})\varrho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x} d^3\vec{x}'. \tag{16}$$

The first term on the r.h.s. of (16) gives the energy stored in the degrees of freedom which are intrinsic to the gauge field, while the second term is the energy due to the presence of the external charge. It is also clear that $\vec{E}_L$ is not a dynamical variable because it is fixed completely by the Gauss law constraint.

In the Nonabelian case such a simple interpretation of (14) is not possible. Namely, from the Nonabelian Gauss law (10) we obtain

$$\tfrac{1}{2} \int (\vec{E}_L^a)^2 d^3\vec{x} = -\tfrac{1}{2} \int (\hat{\delta}^{-1} \varrho^{\text{tot}}) \Delta (\hat{\delta}^{-1} \varrho^{\text{tot}}) d^3\vec{x}, \tag{17}$$

where

$$\varrho^{\text{tot}}(\vec{x}, t) = \hat{j}_0(\vec{x}) + ig[\hat{A}_i(\vec{x}, t), \hat{\vec{E}}_T(\vec{x}, t)], \tag{18}$$

and $\hat{\delta}$ is the operator with matrix elements

$$\hat{\delta}^{ab} = \delta_{ab} \Delta - \varepsilon_{abc} A_i^c \partial_i \tag{19}$$

where δ_{ab} is the Kronecker delta, Δ is the laplacian. The operator $\hat{\delta}$ becomes symmetric when A_i^a obeys the Coulomb gauge condition $\partial_i A_i^a = 0$. In (18) we have noted that the gauge fields can in principle depend on time, although the external charge $\hat{j}_0$ is static. From (17) we obtain

$$E = \tfrac{1}{2} \int d^3\vec{x}[(\vec{E}_T^a)^2 + (\vec{B}^a)^2] - \tfrac{1}{2} \int d^3\vec{x}(\hat{\delta}^{-1} \varrho^{\text{tot}}) \Delta (\hat{\delta}^{-1} \varrho^{\text{tot}}). \tag{20}$$

The formula (20) differs from the Abelian formula (16) in two important respects. First, both terms on the r.h.s. of (20) contain $\hat{A}_i$ fields. Thus, the separation on pure gauge field part and pure external charge part does not happen in this case. Secondly, the $\hat{\delta}$ operator in some cases is not invertible. This problem is closely related to the existence of so called infinitesimal Gribov copies in the Coulomb gauge. The meaning of this fact is that $\vec{E}_L^a$ still contains certain degrees of freedom which should be regarded as dynamical ones because $\vec{E}_L^a$ is not determined uniquely by the external charge and $\vec{A}^a$, $\vec{E}_T^a$ via the Gauss law constraint. To our best knowledge, up to now no one was able to succeed in separating those degrees of freedom from $\vec{E}^a_L$.

Now, let us consider the general case, in which j^a_i can be different from zero. It is not difficult to check that the quantity

$$E_0 = \frac{1}{2}\int d^3\vec{x}[(E^{ai})^2+(B^{ai})^2]-\int d^3\vec{x}j^{ai}A^{ai} \tag{21}$$

is constant in time if A^a_μ obeys (4). However, it is easy to check that the term

$$\int d^3\vec{x}j^{ai}A^{ai}$$

is not gauge invariant [21], [58]. Therefore, (21) is not a satisfactory expression for the energy. We propose two possible ways to solve this difficulty.

The first possibility [27] is to add to (21) another gauge noninvariant contribution describing the internal energy of the external color currents, chosen in such a way that the total expression becomes gauge invariant. For example, we can do this by reinterpreting $\hat{j}^i$ as a quark current

$$j^{a\mu}(\vec{x}) = g\bar{\psi}(\vec{x})\gamma^\mu T^a\psi(\vec{x}), \tag{22}$$

where the bispinor $\psi(\vec{x})$ transforms under fundamental representation of SU(2). We assume that $\psi(\vec{x})$ does not depend on time in order to have time-independent $j^{a\mu}$. Then, we can add to the r.h.s. of (21) the term

$$E_{\text{int}} = -i\int \bar{\psi}(\vec{x})\gamma^i\partial_i\psi(\vec{x})d^3\vec{x}+m\int \bar{\psi}(\vec{x})\psi(\vec{x})d^3\vec{x}. \tag{23}$$

It is easy to see that

$$E = E_0+E_{\text{int}} \tag{24}$$

is gauge invariant and constant in time. Observe also that when $\hat{j}_i = 0$, then E_{int} becomes gauge invariant. Therefore, E_{int} can be included into E also when $\hat{j}_i = 0$. This rather natural solution to the problem of gauge invariance of the energy has the following important consequence: it is not enough to specify $\hat{j}_\mu(\vec{x})$ in order to calculate the energy of the system. One should know rather the external field $\psi(\vec{x})$ instead of the external source $\hat{j}_\mu(\vec{x})$ in order to have the full specification of the system.

The second possible way to avoid the difficulty is to change the form of the coupling of the gauge field to the external source $\hat{j}_\mu$, i.e. instead of

$$\int j^{a\mu}A^a{}_\mu d^3\vec{x}$$

to write a different, gauge invariant expression. We will describe our proposal in more detail in Section II.3. Here we would like only to mention that this general line of improving the theory of external colored sources was pursued also by Nambu and Venturi, [58], although in different manner. In our opinion the first proposed solution, based on (22)–(24) is more natural from the point of view of the standard CCD, in which selfinteracting Yang-Mills fields are coupled to quark fields.

II.1.2. Gauge invariant characteristics of the external color charges

From (9) it follows that the color 4-current density $\hat{j}_\nu$ is not gauge invariant, in contrary to the Abelian case in which the electric current is gauge invariant.

It is natural to ask what are gauge invariant features of the external color charge. Of course, gauge invariant are the local quantities

$$h_{\mu\nu}(\vec{x}, t) = \frac{1}{2} \operatorname{Tr} [\hat{j}_\mu(\vec{x}, t)\hat{j}_\nu(\vec{x}, t)]. \tag{25}$$

The diagonal elements $h_{\nu\nu}$ are the moduli of the components of the current density $\hat{j}_\nu$.

We would like to point out that there also exist certain global characteristics of the external color current, [28] which are invariant under so called small gauge transformations. For definiteness, let us consider the time-independent color charge density $\hat{j}_0(\vec{x}) = j_0^a(\vec{x})\frac{\sigma^a}{2}$. We also assume that

$$j_0^a(\vec{x}) = e^a(\vec{x})f(\vec{x}), \tag{26}$$

where $|\vec{e}| = 1$, and $f(\vec{x}) \neq 0$, possibly except for the single point $\vec{x} = 0$. This assumption about $f(\vec{x})$ is a technical one. It can be avoided at price of complicating of the analysis. We assume also that $\hat{j}_0(\vec{x})$ is continuous. The color structure of $\hat{j}_0$ is entirely described by $\vec{e}(\vec{x})$.

The gauge invariant characteristics of $\vec{e}(\vec{x})$ is given by the winding number $\pi_2[\vec{e}]$, called also the Kronecker index — an element of the second homotopy group π_2. This topological number is present here because $\vec{e}(\vec{x})$ defines a continuous mapping of the sphere $S^2 : |\vec{x}| = R$, $R > 0$, into the sphere $S^2 : |\vec{e}| = 1$, and such mappings are classified accordingly to π_2, [59, 60]. This number is invariant under continuous deformations of the mapping $\vec{e}(\vec{x}) : S^2 \to S^2$. From this fact it follows that $\pi_2[\vec{e}]$ does not depend on the radius $R(\neq 0)$. It also follows that $\pi_2[\vec{e}]$ is gauge invariant. Namely, mappings $S^2 \ni \vec{x} \to \omega(\vec{x}) \in SU(2) = S^3$ are topologically trivial [59, 60]. Therefore one can continuously deform $\omega(\vec{x})$, $|\vec{x}| = R$, into the unit matrix. This deformation of $\omega(\vec{x})$ gives also a continuous deformation of $\hat{e}'(\vec{x})$, defined as

$$\hat{e}'(\vec{x}) = \omega(\vec{x})\hat{e}(\vec{x})\omega^{-1}(\vec{x})$$

into $\hat{e}(\vec{x})$. Thus, $\pi_2[\vec{e}]$ is gauge invariant, because $\pi_2[\hat{e}'] = \pi_2[\hat{e}]$.

Of course, when $\pi_2[\vec{e}] \neq 0$, $\vec{e}(\vec{x})$ has to be singular at least at one point. With the assumptions following (26) this point is $\vec{x} = 0$. In this case we have to take $f(0) = 0$ in order to keep $\hat{j}_0(\vec{x})$ continuous.

Example of $\vec{e}(\vec{x})$ such that $\pi_2[\vec{e}] = k$ is given by

$$\hat{e}(\vec{x}) = \begin{pmatrix} \sin\vartheta \ \cos k\varphi \\ \sin\vartheta \ \sin k\varphi \\ \cos\vartheta \end{pmatrix}, \tag{27}$$

ϑ, φ — spherical angles, k — integer. The case $k = 1$ was investigated in [7, 8]. For $|k| > 1$ no exact solution is known. Likely, such solutions can not be spherically symmetrical. However, it is possible to obtain approximate solutions by perturbative expansion in powers of $\hat{j}_0$, see Section II.2.2.

Now, let us consider in more detail the class of $\vec{e}(\vec{x})$ characterized by $\pi_2[\vec{e}] = 0$. Utilising regular gauge transformations it is possible to transform this class of external color charges into the Abelian gauge frame, in which

$$\hat{e}(\vec{x}) = \frac{\sigma^3}{2}. \tag{28}$$

This follows from the facts that: 1) such $\vec{e}(\vec{x})$ can be continuously deformed to the constant map, $\vec{e}(\vec{x}) = \text{const}$, for each $R\ (= |\vec{x}|)$, 2) continuous deformation of $\vec{e}(\vec{x})$ is equivalent to local rotations of $\vec{e}(\vec{x})$, i.e., it is equivalent to the gauge transformations. Moreover, $\vec{e}(\vec{x})$ does not have to have the topological singularity at $\vec{x} = 0$ induced by behaviour of $\vec{e}(\vec{x})$ at $\vec{x} = 0$. Therefore, we assume that $\vec{e}(\vec{x})$ is regular everywhere. Now we will show that this class of $\vec{e}(\vec{x})$ admits for a topological subclassification.

Namely, consider $\vec{e}(\vec{x})$ such that

$$\hat{e}(\vec{x}) \to \frac{\sigma^3}{2} \tag{29}$$

when $|\vec{x}| \to \infty$. Then, $\vec{e}(\vec{x})$ becomes constant at spatial infinity and therefore we can pass from R^3 to its compactification S^3, still keeping $\vec{e}(\vec{x})$ continuous. Thus, such $\vec{e}(\vec{x})$ can be considered as a continuous mapping from S^3 into S^2. Such mappings are known to be classified by the Hopf index $h[\vec{e}]$, [59, 60].

Of course, even if $h[\vec{e}] \neq 0$, one can find $\omega(\vec{x})$ such that

$$\omega^{-1}\hat{e}(\vec{x})\omega = \frac{\sigma^3}{2}. \tag{30}$$

However, it is possible to prove that such $\omega(\vec{x})$ themselves are characterized by nonzero winding number $\pi_2[\omega]$:

$$\pi_3[\omega] = h[\vec{e}]. \tag{31}$$

Here, $\pi_3[\omega]$ is an element of the third homotopy group, which classifies mappings from S^3 (compactified R^3) to another S^3 (formed by the manifold of SU(2) group). The equality (31) follows from another equality (38), proved below.

Gauge transformations $\omega(\vec{x})$ such that $\pi_3[\omega] \neq 0$ are called the large gauge transformations. It is well-known that they have to be regarded on a different footing than small gauge transformations, i.e., those with $\pi_3[\omega] = 0$, [61]. The existence of the large gauge transformations implies the θ-vacuum and instanton tunneling in the quantized Yang--Mills theory. Therefore, it would be natural to expect that the external charges with nonzero Hopf index would play an essential role in a classical gauge theory which somehow takes into account the presence of the θ-vacuum.

It is well-known that one can construct the conserved current $J^{\mu}_{\mathrm{H}}[\vec{e}]$ corresponding to the Hopf index regarded as a conserved (topological) charge, [62]. This current is constructed as follows. Introduce the antisymmetric tensor

$$f_{\mu\nu} = (\partial_{\mu}\vec{e} \times \partial_{\nu}\vec{e})\vec{e} \tag{32}$$

and introduce the potentials

$$f_{\mu\nu} = \partial_{\mu}a_{\nu} - \partial_{\nu}a_{\mu}. \tag{33}$$

Then, the current is given by the following formula

$$J^{\mu}_{\mathrm{H}}[\vec{e}] = \frac{1}{32\pi^2}\,\varepsilon^{\mu\nu\varrho\sigma} f_{\varrho\sigma} a_{\nu}. \tag{34}$$

The Hopf index is given by

$$h[\vec{e}] = \int d^3\vec{x} J^{0}_{\mathrm{H}}[\vec{e}]. \tag{35}$$

Similarly, one can construct the conserved current corresponding to the winding number $\pi_3[\omega]$:

$$J^{\mu}_{\mathrm{W}}[\omega] = \frac{1}{24\pi^2}\,\varepsilon^{\mu\nu\varrho\sigma}\,\mathrm{Tr}\,[\partial_{\nu}\omega\omega^{-1}\partial_{\varrho}\omega\omega^{-1}\partial_{\sigma}\omega\omega^{-1}]. \tag{36}$$

The winding number is given by

$$\pi_3[\omega] = \int d^3\vec{x} J^{0}_{\mathrm{W}}[\omega]. \tag{37}$$

From (34), (36) it is clear that spatial topological currents are present when the external color charge is time-dependent. In the time-independent case they vanish, and the four-currents reduce to the Hopf index density and the winding number density.

Now we shall prove that

$$J^{\mu}_{\mathrm{H}}[\vec{e}] = J^{\mu}_{\mathrm{W}}[\omega], \tag{38}$$

where

$$\vec{e}\vec{\sigma} = \omega\sigma^3\omega^{-1}. \tag{39}$$

From (38) taken for $\mu = 0$, together with (35), (37), follows the equality (31).

The proof of (38) is the following. Substituting (39) into (32) it is easy to find the potentials a_{ν} defined by (33),

$$a_{\nu} = -i\,\mathrm{Tr}\,[\sigma^3\omega^{-1}\partial_{\nu}\omega].$$

As the next step, we prove that

$$J^{\mu}_{\mathrm{H}}[\vec{e}] = \frac{1}{6\pi^2}\,\varepsilon^{\mu\nu\varrho\sigma}\,\mathrm{Tr}\,[\hat{a}^{+}_{\nu}\partial_{\varrho}\hat{a}_{\sigma}], \tag{40}$$

where

$$\hat{a}_\nu = \frac{\sigma^3}{2}\omega^{-1}\partial_\nu\omega.$$

The proof of (40) is nothing more than tedious algebra based on the following parametrization of ω:

$$\mathrm{SU}(2) \ni \omega = \varphi^0\sigma^0 + i\vec{\varphi}\vec{\sigma}, \quad (\varphi^0)^2 + \vec{\varphi}^2 = 1.$$

It is easy to see that the r.h.s. of (40) is equal to $J^\mu_{\mathrm{W}}[\omega]$.

Let us generalize the above considerations to the time-dependent external color charges $\hat{j}_0(\vec{x}, t)$. From the condition (6) it follows that

$$\partial_0\hat{j}_0 - ig[\hat{A}_0, \hat{j}_0] = 0, \tag{41}$$

i.e.,

$$\hat{j}_0(\vec{x}, t) = u(t, 0|\vec{x})\hat{j}_0(\vec{x}, 0)u^{-1}(t, 0|\vec{x}), \tag{42}$$

where

$$u(t, 0|\vec{x}) = \mathrm{T}\exp\left[ig\int_0^t \hat{A}_0(\vec{x}, t')dt'\right] \tag{43}$$

is an element of SU(2), T denotes time ordering. From (42) it follows that the time evolution of $\hat{j}_0$ is described by time-dependent gauge transformations. As we know, continuous gauge transformations can not change the topological number $\pi_2[\vec{e}]$. Therefore, the basic classification of $\hat{j}_0$, based on the second homotopy group π_2, holds also for time-dependent external color charges $\hat{j}_0$, provided that the time evolution is continuous.

What concerns the Hopf index subclassification in the sector $\pi_2[\vec{e}] = 0$, it also holds for time-dependent charges under some assumptions. Because time evolution (42) is assumed to be continuous, it can be regarded as a continuous deformation of the vector field $\vec{e}(\vec{x}, t = 0)$, and therefore it can not change the value of the Hopf index $h[\vec{e}(\vec{x}, t)]$ which takes discrete (integer) values only. However, it is important that (42) does not change the boundary conditions imposed on $\vec{e}(\vec{x})$ at spatial infinity. This is equivalent to the assumption that $\lim_{|\vec{x}|\to\infty} \vec{e}(\vec{x}, t)$ is time independent $\left(\text{and equal to } \frac{\sigma^3}{2}\right)$. Only if this assumption is satisfied, the Hopf index subclassification can be extended to time-dependent color charges.

The presented topological classification of external charges is kinematical one in the sense that it is entirely based on the fact that the color current has values in 3-dimensional space. It has not been clarified whether the topological characteristics of the external charge somehow manifest themselves in the form of solutions of Yang-Mills equations with the external charge. In this respect, an interesting example is discussed in Section II.2.3 — an external color charge with Hopf index ± 1 is shown to support perturbative Nonabelian Coulomb solution.

It is also not clear how to generalize the above classification to SU(n), $n > 2$, gauge groups.

II.2. Examples of Nonabelian gauge fields generated by external sources

II.2.1. The Abelian Coulomb solution for the system of spatially separated external charges

Classical Yang-Mills equations with nonvanishing external sources provide us with a relatively simple example of color interactions. Solutions of these equations possess a number of rather peculiar properties. This can be observed even in the simplest case of the Abelian Coulomb solution presented below, [6].

The equations have the form (4) of the Chapter II.1

$$D_\mu \hat{F}^{\mu\nu} = \hat{j}^\nu, \tag{1}$$

where the external color current $\hat{j}^\nu$ obeys the condition (6) of the Chapter II.1

$$D_\mu \hat{j}^\mu = 0. \tag{2}$$

We use the matrix notation

$$\hat{j}^\nu = j^{a\nu}\frac{\sigma^a}{2}, \qquad \hat{F}^{\mu\nu} = F^{a\mu\nu}\frac{\sigma^a}{2},$$

$a = 1, 2, 3$, σ^a — Pauli matrices. The covariant derivative has the form

$$D_\mu(\,\cdot\,)^\varrho = \partial_\mu(\,\cdot\,)^\varrho + ig[\hat{A}_\mu, (\,\cdot\,)^\varrho].$$

In the following we shall consider only the case without spatial currents

$$\hat{j}^\nu(\vec{x}, t) = \delta^{\nu 0}\hat{j}_0(\vec{x}, t) \tag{3}$$

with $\hat{j}_0(\vec{x}, t)$ continuous.

Let us now consider the case of time independent $\hat{j}_0(\vec{x})$, nonvanishing only in several disconnected, bounded regions $\Omega_1, \Omega_2, \ldots, \Omega_N$ of space. Then, we can write

$$\hat{j}_0(\vec{x}) = g\sum_{\alpha=1}^{N} \hat{K}_\alpha(\vec{x}), \tag{4}$$

where

$$\hat{K}_\alpha(\vec{x}) = \begin{cases} \neq 0 & \text{for} \quad \vec{x} \in \Omega_\alpha, \\ = 0 & \text{for} \quad \vec{x} \notin \Omega_\alpha. \end{cases}$$

Utilising gauge transformations

$$\hat{A}'_\mu = \omega\hat{A}_\mu\omega^{-1} + \frac{i}{g}\,\partial_\mu\omega\omega^{-1}, \tag{5}$$

$$\hat{j}'_\mu = \omega\hat{j}_\mu\omega^{-1}, \tag{6}$$

where $\omega \in SU(2)$ is appropriately chosen, it is sometimes possible to rotate in color space the charge density $\hat{j}_0(x)$ in such a way that it will become parallel or antiparallel to the third directions in color space, i.e.,

$$\hat{j}'_0(\vec{x}) = \frac{\sigma^3}{2} j'^3_0(\vec{x}). \tag{7}$$

This is the so called Abelian gauge frame.

The sign of $j'^3_0(\vec{x})$ has to be constant in each region Ω_α, i.e., $j'^a_0(\vec{x})$ has to be parallel or antiparallel to the third axis in color space in the whole Ω_α. Otherwise, the gauge transformation ω determined from (6), (7) would not be continuous. Uncontinuous gauge transformations are not allowed because of differentiations present in (5). (In fact, it is easy to give examples of singular gauge transformations which change physical characteristics of the system, like its energy.) However, the relative sign of $j'^3_0(\vec{x})$ between different Ω_α's is not determined. This follows from the fact that in between the regions Ω_α the gauge transformation ω can be chosen freely, because $\hat{j}_0 = 0$ there. Therefore, one can always find a smooth gauge transformation $\omega(\vec{x})$ rotating $j'^a_0(\vec{x})$ from the antiparallel orientation to the parallel one, or vice versa, in a chosen region Ω_β, while not rotating $j'^a_0(\vec{x})$ in the other regions Ω_α, $\alpha \neq \beta$. For example, one can take $\omega(\vec{x}) = \omega_\beta(\vec{x})$, where

$$\omega_\beta(\vec{x}) = \begin{cases} \exp(i\sigma^2\pi) & \text{for} \quad \vec{x} \in \Omega_\beta, \\ 1 \quad \text{for} \quad \vec{x} \in \Omega_\alpha, & \alpha \neq \beta, \end{cases} \tag{8}$$

and for $\vec{x} \notin \Omega_\alpha$, $\alpha = 1, \ldots, N$, $\omega_\beta(\vec{x})$ interpolates smoothly between the values (8). Thus, we can write

$$\hat{j}'_0(\vec{x}) = \frac{\sigma^3}{2} g \sum_{\alpha=1}^{N} q_\alpha K'_\alpha(\vec{x}) \tag{9}$$

where $q_\alpha = \pm 1$, and $K'_\alpha(\vec{x}) > 0$. The value of q_α is not determined — it can be changed by the gauge transformations of the type (8).

With the external color charge (9) we assume the following Ansatz for $\hat{A}'_\mu$:

$$\hat{A}'_i = 0, \qquad \hat{A}'_0 = \frac{\sigma^3}{2} A'^3_0(\vec{x}). \tag{10}$$

Then (2) is satisfied automatically and (1) leads to the ordinary Poisson equation

$$\Delta A'^3_0(\vec{x}) = g \sum_{\alpha=1}^{N} q_\alpha K'_\alpha(\vec{x})$$

with the solution (vanishing at infinity)

$$A'^3_0(\vec{x}) = -\frac{g}{4\pi} \sum_\alpha q_\alpha \int \frac{K'_\alpha(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'.$$

Performing the gauge transformation inverse to (5), (6) we calculate $\hat{A}_\mu$, thus obtaining a solution to (1) and (2) with the external color charge (3). This solution is called the Coulomb solution.

The energy for this solution is

$$E = \frac{g^2}{32\pi^2} \sum_{\alpha,\beta=1}^{N} q_\alpha q_\beta \int \frac{K'_\alpha(\vec{x}) K'_\beta(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} \, d^3\vec{x} d^3\vec{x}\,'. \tag{11}$$

It depends on the choice of values for q_α. Thus, the fact that $\hat{j}_0(\vec{x})$ is not gauge invariant has lead to appearance of several physically different solutions. Therefore it would be interesting to identify certain gauge invariant features of the external color charge distribution $\hat{j}_0(\vec{x}, t)$. We have attempted to achieve this by ascribing to $\hat{j}_0(\vec{x}, t)$ topological numbers, Section II.1.2. The above method of solving Yang-Mills equations with external charges was proposed in [3] for continuous charges and in [6] for a set of pointlike charges.

In literature there are given also many other types of solutions besides the Coulomb solutions, like the "screening" solutions and "magnetic dipole" solution of Sikivie and Weiss [3] or time-dependent generalization of the Coulomb solutions, the "Nonabelian Coulomb" solution and its time-dependent generalization, and bifurcating solutions of Jackiw and collaborators, [7, 8]. There is no explanation why do all these solutions exist and what are they for. Only rather partial results on this are available. For example, below we shall argue that the Nonabelian Coulomb solution is intimately related to the charge with nonzero Hopf index.

II.2.2. The perturbative approach to Yang-Mills equations with weak external sources

Among many very interesting developments in the subject of classical Yang-Mills equations with external sources there is also an idea of a perturbative expansion of the classical solution in powers of the external charge, [7]. This idea is very attractive, because it is the proposal of systematic and calculable procedure for solving Yang-Mills equations even with complicated external charges. In the paper [7] only the lowest order approximation is presented. Next, it is there applied in order to show existence of a new interesting type of solutions, namely Nonabelian Coulomb solutions (see the definition in the next Section). The authors give two examples of external color charge densities for which the perturbative Nonabelian Coulomb solution exist. In this Section we investigate this perturbative method in more detail, [27].

It should be clearly stated that the perturbative method, although very useful, is not universal. There are solutions which can not be reached in the perturbative way. For example, type II solutions discovered in [7] do not exist for sufficiently small q.

We consider Yang-Mills equations with fixed, static color charge density

$$\hat{j}_\mu = \delta_{\mu 0} q \hat{\varrho}(\vec{x}) \equiv \delta_{\mu 0} \hat{j}_0(\vec{x}), \tag{12}$$

where q is a small parameter and $\hat{\varrho}$ is a smooth function vanishing at infinity. Here we again use the matrix notation. e.g. $\hat{\varrho} = \varrho^a \frac{\sigma^a}{2}$. The gauge group is SU(2). We assume that solutions of these equations have the following form

$$\hat{A}_0 = q\hat{A}_0^{(1)} + q^3\hat{A}_0^{(3)} + \ldots, \tag{13}$$

$$\hat{A}_i = q^2\hat{A}_i^{(2)} + q^4\hat{A}_i^{(4)} + \ldots, \tag{14}$$

consistent with an analysis of powers of q in the equations. We find that Yang-Mills equations with external charge (12) admit the perturbative solution (13), (14) only if there is also present a spatial current

$$\hat{j}_i = q^2\hat{j}_i^{(2)} + q^4\hat{j}_i^{(4)} + \ldots . \tag{15}$$

This unpleasant fact was not observed in [7]. The current $\hat{j}_i$ is not arbitrary. It is induced by the assumed color charge (12) and it is necessary for consistency of the perturbative method. The phenomenon reminds very much examples from the quantum field theory, e.g. the inducing of $\lambda\varphi^4$ term is scalar electrodynamics, [63]. There the quartic term is necessary for applicability of perturbation theory, namely for its renormalizability.

If $\hat{\varrho}(\vec{x})$ obeys the condition

$$\Delta(\lambda^{-1}(\vec{x})\hat{\varrho}(\vec{x})) = -\hat{\varrho}(\vec{x}), \tag{16}$$

where $\lambda(\vec{x})$ is an arbitrary, non-vanishing regular function of $\vec{x}$, then the second order contribution to the current vanishes, $\hat{j}_i^{(2)} = 0$ although the color magnetic field in general need not vanish in this order. Therefore, this class of external charges is particularly interesting. It can support perturbative solutions of the magnetic type. In this case the possible presence of the magnetic field is due to particular features of the charge $\hat{\varrho}$, because the induced current of order q^4 at least cannot produce magnetic field of order q^2. The condition (16) appears in the lowest order in q and therefore it was discovered in [7], although in a different way and with different interpretation. We give new examples of charges obeying this condition. The analysis of the perturbative procedure is presented in Section 2 of our paper.

We expect that for small q and smooth and localized $\hat{\varrho}(\vec{x})$ the perturbative series is convergent to an exact solution of Yang-Mills equations, because there is rather little room for a pathological behaviour under such strong regularity conditions. However, one assumption has to be added. The reason is that the perturbative procedure involves taking inverse of the operator of linearized Yang-Mills equations around the zeroth order solution. For the Ansatz (13), (14) the zeroth order solution is $\hat{A}_\mu = 0$. As it is well known, in gauge theories such operator is not invertible, unless one restricts the space of functions by adding a gauge condition. In our paper we assume that Coulomb gauge condition

$$\partial_i\hat{A}_i = 0.$$

For static, regular, and vanishing at infinity fields this condition is sufficient for invertibility of the linear operators. Of course, the solution calculated in the Coulomb gauge can afterwards be transformed to another gauge, if desired.

The Yang-Mills equations in the static case can be written in the form ($g = 1$)

$$\partial_k \hat{F}^{k0} + i[\hat{A}_k, \hat{F}^{k0}] = q\hat{\varrho}, \tag{17}$$

$$\partial_k \hat{F}^{ki} + i[\hat{A}_0, \hat{F}^{0i}] + i[\hat{A}_k, \hat{F}^{ki}] = -\hat{j}_i, \tag{18}$$

where

$$\hat{F}^{k0} = -\partial_k \hat{A}_0 - i[\hat{A}_k, \hat{A}_0], \tag{19}$$

$$\hat{F}^{ki} = \partial_k \hat{A}_i - \partial_i \hat{A}_k + i[\hat{A}_k, \hat{A}_i]. \tag{20}$$

These equations have to be completed with the well-known consistency condition,

$$-\partial_i \hat{j}_i - i[\hat{A}_k, \hat{j}_k] + i[\hat{A}_0, q\hat{\varrho}] = 0. \tag{21}$$

In this Section (21) is regarded as an equation for $\hat{j}_k$. Inserting (13), (14) and (15) we obtain order by order in q the following equations

$$\partial_k \hat{j}_k^{(2)} = i[\hat{A}_0^{(1)}, \hat{\varrho}], \tag{22}$$

$$\partial_k \hat{j}_k^{(4)} = -i[\hat{A}_k^{(2)}, \hat{j}_k^{(2)}] + i[\hat{A}_0^{(3)}, \hat{\varrho}], \tag{23}$$

etc. We do not display higher order equations to save space. Their derivation is equally straightforward.

From these equations $\hat{j}_k^{(2n)}$ is determined up to a curl. In order to remove this non-uniqueness we can assume e.g., that the covariant curl of $\hat{j}_k$ vanishes,

$$\varepsilon_{iks}(\partial_k \hat{j}_s + i[\hat{A}_k, \hat{j}_s]) = 0. \tag{24}$$

This condition determines the curl of $\hat{j}_k^{(2n)}$,

$$\varepsilon_{iks}\partial_k \hat{j}_s^{(2)} = 0, \tag{25}$$

$$\varepsilon_{iks}\partial_k \hat{j}_s^{(4)} = -i\varepsilon_{iks}(\hat{A}_k^{(2)}\hat{j}_s^{(2)} - \hat{j}_s^{(2)}\hat{A}_k^{(2)}), \tag{26}$$

etc., what together with the divergence given by (22) and (23) fixes $\hat{j}_k^{(2n)}$ uniquely provided that we also assume that $\hat{j}_k^{(2n)}$ vanishes at $|\vec{x}| \to \infty$.

From (22) and (23) it follows that in general $\hat{j}_k^{(2n)} \neq 0$, because the commutators on the r.h.s. of (22) and (23) do not vanish. Thus, for the consistency of the method it is necessary to accompany the given external charge $\hat{\varrho}(\vec{x})$ by the external current $\hat{j}_k$ of order q^2 or higher. This phenomenon reflects the Nonabelian nature of the theory because it is caused by nonvanishing commutators.

From (22) we see that if

$$[\hat{A}_0^{(1)}, \hat{\varrho}] = 0, \tag{27}$$

then we have $\hat{j}_i^{(2)} = 0$. However, $\hat{j}_i^{(4)} \neq 0$ in this case, in general. Vanishing $\hat{j}_i^{(4)}$ requires to satisfy another nontrivial condition, namely $[\hat{A}_0^{(3)}, \hat{\varrho}] = 0$.

Now let us turn to the equations for gauge potentials. From Eqs. (13), (14) and (17–20) we obtain the following equations for $\hat{A}_0^{(2n-1)}$,

$$\Delta \hat{A}_0^{(1)} = -\hat{\varrho}, \tag{28}$$

$$\Delta \hat{A}_0^{(3)} = -2i[\hat{A}_k^{(2)}, \partial_k \hat{A}_0^{(1)}], \tag{29}$$

$$\Delta \hat{A}_0^{(5)} = -2i[\hat{A}_k^{(2)}, \partial_k \hat{A}_0^{(3)}] - 2i[\hat{A}_k^{(4)}, \partial_k \hat{A}_0^{(1)}] + [\hat{A}_k^{(2)}, [\hat{A}_k^{(2)}, \hat{A}_0^{(1)}]], \tag{30}$$

etc., when we have used the assumed Coulomb gauge condition for $\hat{A}_k$. The equations for the vector potential are the following

$$\Delta \hat{A}_i^{(2)} = -\hat{j}_i^{(2)} - i[\hat{A}_0^{(1)}, \partial_i \hat{A}_0^{(1)}], \tag{31}$$

$$\Delta \hat{A}_i^{(4)} = -\hat{j}_i^{(4)} + [\hat{A}_0^{(1)}, [\hat{A}_i^{(2)}, \hat{A}_0^{(1)}]] - i[\hat{A}_0^{(3)}, \partial_i \hat{A}_0^{(1)}]$$
$$- i[\hat{A}_0^{(1)}, \partial_i \hat{A}_0^{(3)}] + i[\hat{A}_k^{(2)}, \partial_i \hat{A}_k^{(2)}] - 2i[\hat{A}_k^{(2)}, \partial_k \hat{A}_i^{(2)}], \tag{32}$$

etc.

In the first step of the procedure we determine $\hat{A}_0^{(1)}$ and then $\hat{j}_i^{(2)}$ and $\hat{A}_i^{(2)}$. In the next step we calculate $\hat{A}_0^{(3)}$ and $\hat{j}_k^{(4)}$, $\hat{A}_k^{(4)}$. Equations (28)–(32) require to specify the boundary conditions at infinity. We choose them to be the same as in the electromagnetics, i.e. for the localized external source all potentials should vanish at infinity.

Let us now come back to the condition (27). For SU(2) gauge group it implies that

$$\hat{\varrho}(\vec{x}) = \lambda(\vec{x}) \hat{A}_0^{(1)}(\vec{x}), \tag{33}$$

where $\lambda(\vec{x})$ is an arbitrary function. The equation (28) then takes the form of the condition (16) for $\hat{\varrho}(\vec{x})$. Equivalently, one can consider

$$\Delta \hat{A}_0^{(1)}(\vec{x}) = -\lambda(\vec{x}) \hat{A}_0^{(1)}(\vec{x}). \tag{34}$$

The condition (34) is of course obeyed by any Abelian configuration, $\hat{A}_0^{(1)} \sim \frac{\sigma^3}{2}$. The external charge with nonzero Hopf index ± 1 is another example in which the condition (16) is obeyed. It will be considered in the next Section. Yet another example is provided by external charges whose orientation in color space is characterized by arbitrary, nonzero Kronecker index K. That is, we take

$$\vec{\varrho}(\vec{x}) = \varrho(\vec{x}) \vec{e}(\vec{x}),$$

where

$$\vec{e}(\vec{x}) = \begin{pmatrix} \sin \vartheta \quad \cos K\varphi \\ \sin \vartheta \quad \sin K\varphi \\ \cos \vartheta \end{pmatrix},$$

and $\varrho(\vec{x})$ is a function vanishing for $\vec{x} = 0$ in order to ensure continuity of $\vec{\varrho}(\vec{x})$ at $\vec{x} = 0$. Because of (33) we have

$$\vec{A}_0^{(1)}(\vec{x}) = \psi(\vec{x})\vec{e}(\vec{x}), \quad \psi(\vec{x}) = \lambda^{-1}(\vec{x})\varrho(\vec{x}).$$

Substituting this to (34) we obtain three equations, from which it follows in spherical coordinates that

$$\psi(r, \vartheta, \varphi) = f(r)(\sin\vartheta)^{\frac{K^2-1}{2}}, \tag{35}$$

where $f(r)$ is any regular function of $r = |\vec{x}|$ vanishing at $r = 0$ in order to ensure continuity of $\vec{A}_0^{(1)}$ at the origin. From the same equations it follows that

$$\lambda(\vec{x}) = -\frac{1}{r^2}\frac{1}{f(r)}\frac{d}{dr}\left(r^2\frac{d}{dr}f\right) - \frac{(K^2-1)^2}{4r^2\sin^2\vartheta} + \frac{K^4-1}{4} + K^2 + 1.$$

Therefore,

$$\varrho(\vec{x}) = \frac{1}{r^2}\left[-\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - f(r)\frac{(K^2-1)^2}{4\sin^2\vartheta} + f(r)\left(\frac{K^4-1}{4} + K^2 + 1\right)\right](\sin\vartheta)^{\frac{K^2-1}{2}}. \tag{36}$$

From (36) it follows that $f(r)$ should behave like $r^{2+\delta}$, $\delta > 0$, for $r \to 0$, in order $\varrho(0) = 0$. We also see that only for $K^2 = 1$ the external charge $\vec{\varrho}(\vec{x})$ becomes spherically symmetrical (this case was considered in [7]), and that for $K = 2$ it has singularity for $\vartheta = 0, \pi$ of the integrable type.

The number of examples of charges obeying (16) (which is equivalent to (34)) can also be sometimes increased by utilising the well-known behaviour of the Laplace operator under inversions. It is easy to check that if $\vec{A}_0^{(1)}(\vec{x})$ obeys (34), then

$$\vec{A}_0'(\vec{x}) = \frac{1}{r}\vec{A}_0^{(1)}\left(\frac{R^2}{r}, \vartheta, \varphi\right), \tag{37}$$

where $\vec{x} = (r, \vartheta, \varphi)$ and R is a fixed radius, also obeys (34) with $\lambda(\vec{x})$ replaced by

$$\lambda'(\vec{x}) = \frac{R^4}{r^4}\lambda\left(\frac{R^2}{r}, \vartheta, \varphi\right).$$

The above examples are so general that from (37) we do not obtain a new example. However, for the example discussed in the next Section, the transformation (37) leads to a new form of the external color charge.

II.2.3. The perturbative Nonabelian Coulomb solution

The perturbative method of solving of classical Yang-Mills equations is exceptionally well-suited for investigations of the Nonabelian Coulomb solutions, because they are defined through their behaviour in the limit $q \to 0$. The definition, given in [7], requires first to transform $\hat{\varrho}$ to the so called Abelian gauge frame in which

$$\hat{\varrho}'(\vec{x}) = \varrho(\vec{x})\frac{\sigma^3}{2}. \tag{38}$$

Then, the Nonabelian Coulomb (NC) solution is the static, finite energy solution which in the limit $q \to 0$ does not vanish and becomes the pure gauge

$$\hat{A}_0 = 0, \quad \hat{A}_i = i\partial_i \omega\omega^{-1}. \tag{39}$$

It should be added that $\omega(\vec{x})$ is a gauge transformation of such a type that $\hat{A}_i$ in (39) cannot be gauge transformed to zero by gauge rotations around the 3rd axis, which leave the charge (38) invariant. The purpose of this complex definition is twofold. First, to distinguish NC solutions from more trivial Abelian Coulomb (AC) solutions, which exist for any charge of the form (38) and have the property that $\hat{A}_0$ and $\hat{A}_i$ vanish when $q \to 0$ in the Abelian gauge frame (38) (up to static gauge transfomations leaving (38) invariant). Secondly, the pure gauge limit (39) for $q \to 0$ ensures that the nonzero strengths for $q \neq 0$ are entirely due to the external charge, that is that the solution is not a superposition of a solution of sourceless Yang-Mills equations with e.g. AC solution.

In this Section we apply the perturbative method to the external charge (12) with $\hat{\varrho}(\vec{x})$ characterized by Hopf index ± 1, [27]. We adjust $\hat{\varrho}(\vec{x})$ in order to obey the condition (16), and we calculate the solution in the lowest non-vanishing order. The solution is found to be of the NC type, in accordance with our prediction [28] that the external charge with nonzero Hopf index supports NC type solution.

The external charges characterized by nonzero topological currents (34) of the Chapter II.1 (and zero Kronecker index) lead in a very natural way to the Nonabelian Coulomb solution. To see this, we observe that in the Abelian gauge frame we still have some residual gauge freedom, namely the gauge transformations of the form

$$\omega_0(\vec{x}, t) = \exp\left[\frac{i\sigma^3}{2}\chi(\vec{x}, t)\right], \tag{40}$$

because they do not change the form (38) of the external charge. We shall have the Nonabelian Coulomb solution only if the limit (39) can not be compensated by gauge transformations (40). If the term (39) can be compensated, then

$$\partial_\mu \omega\omega^{-1} = \partial_\mu \omega_0 \omega_0^{-1}. \tag{41}$$

This however implies that $J^\mu_W[\omega] = 0$ because now all the terms on the r.h.s. of (36) of the Chapter II.1 commute, due to (41) and to the fact that ω_0 contains only commuting matrices σ^0 and σ^3. Thus, if $J^\mu_W[\omega] = J^\mu_H[\vec{e}] \neq 0$, we can have the Nonabelian Coulomb solution. It is sufficient to find any solution of Yang-Mills equations for $\hat{j}_0(\vec{x})$ characterized by $J^\mu_H[\vec{e}] \neq 0$ and such that it tends to zero when $\hat{j}_0 \to 0$ ($\vec{e}$ is related to $\hat{j}_0$ by (26) of the Chapter II.1). Then, after passing to the Abelian gauge frame by gauge transformations and after performing the limit $\hat{j}_0' \to 0$, we are left with the gauge term (39) which can not be compensated by gauge transformations (40).

The task of explicit obtaining the Nonabelian Coulomb solution is very difficult, because the charge $\hat{j}_0$ such that $J_H[\vec{e}] \neq 0$ is merely cylindrically symmetrical, implying cylindrically symmetrical solution of Yang-Mills equations at best. Precisely speaking, no explicit example of the exact Nonabelian Coulomb solution is known up to now.

Our perturbative solution has the limit (39) with $\omega(\vec{x})$ characterized by the winding number ± 1. Therefore, our solution could be related to a gauge field generated by weak external charge of the form (38) immersed in a topologically nontrivial (i.e., nonzero winding number) sector of Yang-Mills θ-vacuum, [61]. This is one of the reasons that we find the external charge with nonzero Hopf index interesting.

Our example of the NC solution is nicer than the two examples presented in [7], because our external charge can be gauge rotated to the Abelian gauge frame by a regular and simple gauge transformation. In the first example presented in [7], the gauge transformation is singular. The second example in [7] contains three subcases. 1) The external charge already being in the Abelian gauge frame — then, the solution is not of the NC type. 2) The external charge can be homotopically deformed to a standard charge with nonzero Kronecker index — then the gauge transformation is singular. 3) The gauge transformation is regular, however it is very complicated — this makes it difficult to compare NC solution with AC solution for this charge.

The Hopf index enumerates continuous mappings from S^3 into S^2. Its definition can be found, e.g. in [59, 60]. The continuous color charge distribution $\vec{\varrho}(\vec{x})$ can define such a mapping. Namely, we assume that $\vec{\varrho}(\vec{x}) \neq 0$. Then $\vec{e}(\vec{x}) = \vec{\varrho}(\vec{x})/|\vec{\varrho}(\vec{x})|$ has values in S^2. In order to compactify R^3 to S^3, we include the point at infinity. For continuity of the mapping it is then necessary to assume that $\lim_{|\vec{x}|\to\infty} \vec{e}(\vec{x}) = \text{const.}$, i.e. that it does not depend on angles. In this way $\vec{e}(\vec{x})$ gives a continuous mapping from S^3 into S^2.

It has been proved, in Section II.1.2, that the color charge distribution with orientation in color space characterized by the Hopf index H can be obtained from the charge in the Abelian gauge frame (38) by a topologically nontrivial gauge transformation, given by $\omega(\vec{x})$ with the winding number equal to H. The form of such a gauge transformation can be taken as

$$\omega_0(\vec{x}) = \cos\frac{\chi(r)}{2} + i\frac{\vec{x}}{r}\vec{\sigma}\sin\frac{\chi(r)}{2}, \tag{42}$$

where

$$\chi(0) = 0, \quad \chi(\infty) = 2\pi H. \tag{43}$$

Thus we have

$$\hat{\varrho}(\vec{x}) = \varrho(\vec{x})\omega_0(\vec{x})\frac{\sigma^3}{2}\omega_0^{-1}(\vec{x}), \tag{44}$$

where $\varrho(\vec{x}) \neq 0$. This gives

$$\hat{\varrho}(\vec{x}) = \tfrac{1}{2}\varrho(\vec{x})\left(\cos\chi(r)\vec{a}\vec{\sigma} + \sin\chi(r)\frac{\vec{a}\times\vec{x}}{r}\vec{\sigma} + 2\sin^2\frac{\chi(r)}{2}\frac{\vec{\sigma}\vec{x}}{r}\frac{\vec{a}\vec{x}}{r}\right), \tag{45}$$

where we have introduced the vector $\vec{a} = (0, 0, 1)$ in order to simplify the formula.

The functions $\varrho(\vec{x})$, $\chi(r)$ we shall determine from the condition (16) for the magnetic type solutions. We assume from the beginning that $\varrho(\vec{x})$ and $\lambda(\vec{x})$ are spherically symmetrical. With this assumption the condition (16) can be obeyed only if $H^2 = 1$. Denoting $\varrho(r)/\lambda(r)$ by $\psi(r)$ as before, we obtain from (16) the following equations (dots denote differentiations with respect to r)

$$\ddot{\chi}\psi + \frac{2}{r}\dot{\chi}\psi + 2\dot{\chi}\dot{\psi} - \frac{2}{r^2}\psi\sin\chi = 0, \tag{46}$$

$$\ddot{\psi} - \psi\dot{\chi}^2 + \frac{2}{r}\dot{\psi} + \frac{2}{r^2}\psi\cos\chi - \frac{2}{r^2}\psi + \lambda\psi = 0, \tag{47}$$

$$\ddot{\psi} + \frac{2}{r}\dot{\psi} - \frac{4}{r^2}\psi + \frac{4}{r^2}\psi\cos\chi + \lambda\psi = 0. \tag{48}$$

Subtracting the last two equations we obtain

$$\dot{\chi}^2 + \frac{2}{r^2}(\cos\chi - 1) = 0, \tag{49}$$

which can be easily integrated. Taking into account the boundary conditions (43) we obtain

$$\chi(r) = 4\,\mathrm{arctg}\,(\mu r H), \tag{50}$$

where $H = \pm 1$ and μ is an arbitrary constant scale, $\mu > 0$.

Now, the equation (46) becomes the first order equation for $\psi(r)$. Its integration yields

$$\psi(r) = d(1+\mu^2 r^2)^{-1/2}, \tag{51}$$

where d is a constant. Next, we calculate $\varrho = \lambda\psi$ from, e.g. Eq. (48). The result is

$$\varrho(r) = 35\mu^2 d(1+\mu^2 r^2)^{-5/2}. \tag{52}$$

Observe that this function does not vanish for any finite r, as required at the beginning of this Section.

Thus, we have found the external color charge with $H^2 = 1$ obeying the magnetic condition (16). Because of (33) we also know $\hat{A}_0^{(1)}$,

$$\hat{A}_0^{(1)}(\vec{x}) = \tfrac{1}{2}\psi(r)\Big[\cos\chi(r)\vec{a}\vec{\sigma} + \sin\chi(r)\frac{\vec{a}\times\vec{x}}{r}\vec{\sigma} + 2\sin^2\frac{\chi(r)}{2}\frac{\vec{\sigma}\vec{x}}{r}\frac{\vec{a}\vec{x}}{r}\Big], \tag{53}$$

and we also have $\hat{j}_i^{(2)} = 0$.

Now we would like to calculate $\hat{A}_i^{(2)}$. From (31) we have

$$\Delta\hat{A}_i^{(2)} = -i[\hat{A}_0^{(1)}, \partial_i\hat{A}_0^{(1)}], \tag{54}$$

where $\hat{A}_0^{(1)}$ is given by (53). The calculation of $\hat{A}_i^{(2)}$ from (54) is rather straightforward, although it is very tedious and leads to rather uninstructive formulae. Therefore we calculate here only the leading term for large r.

Because the r.h.s. of (54) behaves like r^{-4} for large r, we can write

$$\hat{A}_i^{(2)}(\vec{x}) = \frac{i}{4\pi}\int d^3\vec{x}'\,\frac{1}{|\vec{x}-\vec{x}'|}\,[\hat{A}_0^{(1)}(\vec{x}'),\partial_i\hat{A}_0^{(1)}(\vec{x}')]. \tag{55}$$

Next we use the standard multipole expansion [64]. Because the commutator on the r.h.s. of (55) has vanishing divergence, the magnetic pole term is absent. The magnetic dipole term is different from zero. For large r

$$\hat{A}_i^{(2)}(\vec{x}) = \varepsilon_{iks}\frac{\hat{m}_k x^s}{r^3}, \tag{56}$$

where the magnetic dipole moment is

$$(\hat{m}_1, \hat{m}_2, \hat{m}_3) = -\frac{\pi}{12}\frac{d^2}{\mu^3}(\sigma^1, \sigma^2, \tfrac{35}{8}\sigma^3). \tag{57}$$

From (57) it follows that the vector potential (56) contains mixing between color and space, and that is not spherically symmetrical.

The spatial current $\hat{j}_i^{(2)}$ vanishes. Nonvanishing current can appear in the order q^4 or higher. In order to check this it is necessary to check whether $[\hat{A}_0^{(3)}, \hat{A}_0^{(1)}] = 0$. This is a very tedious calculation. It requires first to calculate $\hat{A}_0^{(3)}$ from (29). We have not done it because in any case the nonvanishing current is at least of order q^4, and therefore can create magnetic field of order at least q^4, as it follows from Eqs. (31) and (32). Thus, our perturbative solution is a new example of the known phenomenon, [3], of creating magnetic fields by static color charge densities. Because the magnetic field is here of order q^2, it can not be attributed to possible nonvanishing $\hat{j}_i^{(4)}$.

The obtained perturbative solution is of the NC type. As it is seen from (44), the external color charge with Hopf index H can be gauge rotated to the Abelian gauge frame by the gauge transformation $\omega_0^{-1}(\vec{x})$, where $\omega_0(\vec{x})$ is given by Eq. (42). In the Abelian gauge frame we obtain

$$\lim_{q\to 0}\hat{A}_i'(\vec{x}) = -i\omega_0^{-1}\partial_i\omega_0. \tag{58}$$

This pure gauge potential can be regarded as belonging to the sector of the θ-vacuum with the winding number $\mp H$.

Let us now compare our solution with the AC solution for the color charge distribution (45), (50) and (52). This exact solution is obtained by gauge rotating $\hat{\varrho}(\vec{x})$ to the Abelian gauge frame, and next substituting into Yang-Mills equations the Ansatz

$$\hat{A}_0' = A_0\frac{\sigma^3}{2}, \quad \hat{A}_i' = 0. \tag{59}$$

The resulting linear Poisson equation for A_0,

$$\Delta A_0 = -q\varrho(r)$$

yields

$$A_0(r) = \tfrac{35}{3}\, qd(1+\mu^2 r^2)^{-1/2}. \tag{60}$$

Next we gauge rotate the color charge back to the initial form (45) using $\omega_0(\vec{x})$. The corresponding gauge transformation of the gauge potentials gives

$$\hat{A}_0^{C}(\vec{x}) = A_0(r)\omega_0(\vec{x})\frac{\sigma^3}{2}\omega_0^{-1}(\vec{x}), \tag{61}$$

$$\hat{A}_i^{C}(\vec{x}) = i\partial_i\omega_0(\vec{x})\omega_0^{-1}(\vec{x}), \tag{62}$$

where $\omega_0 \dfrac{\sigma^3}{2} \omega_0^{-1}$ can be read off from Eqs. (44) and (45).

This AC solution is exact. It has zero color magnetic field and the potentials vanish when $q \to 0$ in the Abelian gauge frame. On the other hand, the vector potential (56) of the NC solution does not have the Abelian form in the Abelian gauge frame, $\hat{A}_i'^{NC} \neq A_i \dfrac{\sigma^3}{2}$. The AC solution differs from the NC one also by magnitude of $\hat{A}_0$,

$$\hat{A}_0^{C}(\vec{x}) = \tfrac{35}{3}\,\hat{A}_0^{NC}. \tag{63}$$

From (63) it follows that the energy of the AC solution is $(\frac{35}{3})^2$ times greater than the energy of the NC solution. This relation is true only for small q, because we have neglected all contributions to the energy of the NC solution of order higher than q^2. The observed fact that the NC solution has lower energy than the AC solution is in full accordance with the general argument given in [7] for all NC type solutions. However, the big magnitude of difference is somewhat surprising.

We have used the perturbative approach to solve the classical Yang-Mills equations with the external charge with nonzero Hopf index. This method allowed us to perform some calculations. The more ambitious task to find the NC solution exactly seems to be very difficult, because one should not expect that the gauge potentials will possess more than merely cylindrical symmetry, if any. This leads to untractable set of nonlinear equations. The problem is even more difficult because of the possibility that $\hat{j}_i^{(4)} \neq 0$. Therefore, the solution with $\hat{j}_i = 0$ need not exist — if it exists, then it has to be singular in the limit $q \to 0$, or it has to cease to exist for small q (such solutions were found in Yang-Mills theory, [7]).

II.3. Fields generated by gauge invariant external sources

In this Chapter we would like to describe the anounced in Section II.1.1 modification of the coupling of an external source to the Nonabelian gauge field, [65]. The motivation is twofold. The first one is based on the difficulties with gauge invariance of energy for the standard coupling $A_\mu^a j_\mu^a$. The other one is the following.

In spite of numerous efforts it has not been possible to obtain confinement of quarks within classical chromodynamics. On the other hand, it is commonly believed that quantum chromodynamics does confine quarks. Therefore, CCD can not be regarded as a long distance limit of QCD. One is tempted to guess the form of a classical theory describing the effective long-distance structure of QCD (up to small quantum fluctuations). Below we present an effort in this direction. Namely, we assume that a physical source of Nonabelian gauge field has to be described by a gauge invariant mathematical object. This would correspond to the expectation that only color singlets are the physical states in QCD. Therefore, we shall consider the gauge invariant external sources. Such sources can not be coupled directly to A^a_μ because the total action would not be gauge invariant. We assume that they are coupled to gauge invariant, nonlocal objects (NGIO). We shall consider the following examples of NGIO:

$$W(y, x|C) = \bar{\chi}(y)\,\mathrm{P}\exp\,[ig \oint_{C,x}^{y} \hat{A}_\mu dz^\mu]\psi(x), \tag{1}$$

where x, y are points in Minkowski space-time, C is a path connecting y and x, P denotes path ordering of exponentials along C, χ, ψ are fermion fields, and $\hat{A}_\mu = A^a_\mu T^a$, where T^a are generators of SU(2) (or its representation). As the fermionless NGIO we take

$$W^0(x, x|C) = \mathrm{Tr}\,\mathrm{P}\exp\,[ig \oint_C \hat{A}_\mu dz^\mu], \tag{2}$$

where the trace is with respect to colour indices and C is a closed contour which starts and terminates at the same point x.

The nonlocal, gauge invariant objects, constructed from Nonabelian gauge potentials, were considered in a number of papers, e.g. [66]. Presently, there exists a hope that such objects provide a string picture of hadrons within the framework of Nonabelian gauge theories. They are expected to be directly related to the long distance structure of the Nonabelian gauge theory. The elementary fermion and Nonabelian gauge fields are not expected to reflect the long distance structure of the theory because of confinement of quarks and gluons.

In order to calculate S-matrix in terms of NGIO it is necessary to consider Green functions for such objects [67]. As an intuitive starting point for this calculation one could take the Feynman path formula for the generating functional for Green functions. Apart from the gauge fixing and F–P ghost terms which are not important on the classical level, the total action in such a formula would be

$$S = S_{\mathrm{YM}} + S_{\mathrm{F}} + \int JW, \tag{3}$$

where

$$S_{\mathrm{YM}} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}, \qquad a = 1, 2, 3,$$

is the Yang-Mills action, S_{F} is the Dirac action for fermions, W denotes NGIO and $\int JW$ is specified below. Our considerations can be regarded as an investigation of the static classical approximation to the above sketched problem. Our expectation is that this can be an easy way to get important information about properties of Green functions for

NGIO. We restrict ourselves to the most interesting gluonic sector of the theory by neglecting S_F.

Here we shall investigate the question what are the classical stationary points of the action $S_{YM}+\int JW$. As W we take the NGIO (1) and (2). The fermion fields present in (1) are regarded as a priori fixed external fields. In other words, we try to find classical gauge potentials generated by the external source J. This external source is, of course, gauge invariant, as it is coupled to the gauge invariant NGIO.

We observe that such sources imply classical Yang-Mills equations with an external current of color along the path C. In the case of NGIO given by (1) we show that the Yang-Mills equations are inconsistent, unless the fermion fields satisfy certain condition. When fermions do satisfy the condition, the external current in Yang-Mills equations vanishes and the external source decouples from Yang-Mills equations. This gives zero gauge field for such a source.

The fermionless case (2) is more complicated. The gauge potentials generated by the current of color flowing along the path C can be easily found. Because the line C has zero thickness, the potentials are singular on C. This is an unpleasant difficulty for the classical approach, because the external current of color contains explicitly $\hat{A}_\mu(z)$ taken for $z \in C$ and this is infinite. Of course, this difficulty could be resolved by a quantum smearing of the curve C. One should use some smooth $J(x, y|C)$ and to average (2) with it. We say "quantum smearing" because results of papers [66b] strongly suggest that $J(x, y|C)$ can be interpreted as a wave functional for a string.

However, there still exists a possibility of a classical description. Namely, one could think of such a wave functional $J(x, y|C)$ that it can be described classically by some very complicated curve C, so complex that $\hat{A}_\mu$ will be finite on C. In fact, in the quantum theory the curve C (being the shape of the string) strongly fluctuates, and therefore there is no reason why the best classical description should be given by geometrically simplest lines. We consider a curve C that fills in a torus, the twodimensional manifold. Such a curve could be considered intuitively as a limiting case of a curve winding around some given circle, when the number of windings increases to infinity. Continuous curves filling more than onedimensional manifolds are known in mathematics, e.g., Sierpiński curve [68]. The corresponding solution of Yang-Mills equations is given by some color magnetic field restricted to the inside of the torus and zero electric field. The classical selfenergy of the source is, unexpectedly, quantized through a selfconsistency condition. For a thin torus the energy spectrum is linear. Such a toroidal magnetic flux tube we would like to interpret as a classical picture of a glueball.

Two remarks are in order:

1. We consider the simplest, so called Abelian, solutions of Yang-Mills equations. It is already known that for a given external source Yang-Mills equations admit also other types of solutions. A similar phenomenon should be expected also in the case of the gauge invariant external sources.

2. Note that the action S is not in general real because of the term $S_{ext} = \int JW$. It is interesting that in spite of this, one could choose an overall constant in S_{ext} in such a way that the stationary points are given by real gauge potentials A^a_μ.

A. Yang-Mills equations with the gauge invariant sources involving fermions

We search for a stationary point of the action

$$S = -\tfrac{1}{4}\int d^4x F^a_{\mu\nu}F^{a\mu\nu} + S_{\text{ext}}, \tag{4}$$

where

$$S_{\text{ext}} = \int d^4x d^4y \int [dc_\mu] J(y, x|C) W(y, x|C), \tag{5}$$

$W(y, x|C)$ being given by (1). Here $[dc_\mu]$ denotes a functional measure in a space of paths C, connecting the points x, y in Minkowski space-time. Observe that the terms $\overline{\psi}\gamma A\psi$, present in the neglected S_F, would act as an additional gauge noninvariant external source for $\hat{A}_\mu$. Such sources were already investigated. In this Chapter we are not concerned with them.

In the following we assume that the external source J is strictly localized in three-space and that it is static, i.e.,

$$J(y, x|C) = q\delta(\vec{x}-\vec{x}_a)\delta(\vec{y}-\vec{x}_b)\delta(x_0-y_0)\,[\delta(C_0-x_0)]\,[\delta(\vec{C}-\vec{C}^{(0)})], \tag{6}$$

where $[\delta(\vec{C}-\vec{C}^{(0)})]$, $[\delta(C_0-x_0)]$ are the functional delta functions, $\vec{x}_a$, $\vec{x}_b$ are fixed points in three-space and $C^{(0)}$ is a curve connecting x_a, x_b. The two deltas, $\delta(x_0-y_0)$ and $[\delta(C_0-x_0)]$ make the configuration to be equal time configuration. The fact that x_0 is unspecified implies that the configuration is the static one. We assume that $\bar{\chi}$, ψ are constant in time. q is a constant characterizing the strength of the external source.

The action (4) implies the usual equations for the gauge potentials

$$D_\mu \hat{F}^{\mu\nu} = \hat{j}^\nu_{\text{ext}}, \tag{7}$$

where

$$j^{a\nu}_{\text{ext}}(x) = -\frac{\delta S_{\text{ext}}}{\delta A^a_\nu} \tag{8}$$

and $D_\mu = \partial_\mu + ig[\hat{A}_\mu, \cdot]$.

In order to calculate explicitly $j^{a\nu}_{\text{ext}}$ we parametrize the line $C^{(0)}$: $\lambda \in [0, 1]$, $C^{(0)}_\mu = C^{(0)}_\mu(\lambda) \equiv z_\mu(\lambda)$, $C^{(0)}_\mu(0) = x_{a\mu}$, $C^{(0)}_\mu(1) = x_{b\mu}$. Of course, $x_{a0} = x_{b0} = z_0(\lambda)$. Then

$$\mathrm{P}\exp\Big[ig\int_{x_a}^{x_b} A_\mu dz^\mu\Big] = \mathrm{P}\exp\Big[ig\int_0^1 d\lambda v^\mu A_\mu\Big],$$

where $v^\mu = \dfrac{dz^\mu}{d\lambda}$, and the variational derivative in (8) yields after a straightforward calculation

$$j^{a0}_{\text{ext}} = 0,$$

$$j^{ai}_{\text{ext}} = gq\int_0^1 d\lambda v^i(\lambda) I^a(\vec{z}(\lambda))\delta(\vec{x}-\vec{z}(\lambda)), \tag{9}$$

where

$$I^a(\vec{z}(\lambda)) = i\bar{\chi}(\vec{x}_b)V_C(\vec{x}_b, \vec{z}(\lambda))T^a V_C(\vec{z}(\lambda), \vec{x}_a)\psi(\vec{x}_a), \tag{10}$$

and

$$V_C(\vec{x}, \vec{y}) = \mathrm{P}\exp[ig \int_{C,\vec{y}}^{\vec{x}} \hat{A}_i dz^i].$$

Thus, $j^{ai}_{ext} = 0$, except for the line $\vec{C}^{(0)}$, along which there is a flow of color charge.

Yang-Mills equations (7) imply the constraint (6) of the Chapter II.1, i.e.,

$$\partial_\mu j^{a\mu}_{ext} - g\varepsilon_{abc}A^b_\mu j^{c\mu}_{ext} = 0 \tag{11}$$

for any current on the r.h.s. of them. For gauge noninvariant external sources this constraint is a nontrivial condition to be satisfied. For the gauge invariant sources, the current (9) satisfies the constraint identically on the whole $\vec{C}^{(0)}$, excluding the end points $\vec{x}_a$, $\vec{x}_b$. At these points the constraint is not satisfied unless the fermion wave functions $\bar{\chi}$, ψ obey certain condition. Namely, from (9) we get

$$\partial_\mu j^{a\mu}_{evt} = (\partial_i I^a) j^i + I^a \partial_i j^i, \tag{12}$$

where $j^i(\vec{x}) = gq \int_0^1 d\lambda v^i(\lambda)\delta(\vec{x}-\vec{z}(\lambda))$ is the usual current obeying the static continuity equation $\partial_i j^i = 0$ for $\vec{x} \neq \vec{x}_a, \vec{x}_b$. Thus, the last term on the r.h.s. of (12) vanishes for $\vec{x} \neq \vec{x}_a, \vec{x}_b$. It is easy to verify that the first term on the r.h.s. of (12) cancels with the term $\varepsilon_{abc}A^b_\mu j^{\mu c}_{ext}$ present on the l.h.s. of the constraint (11). For $\vec{x} = \vec{x}_a$, $\vec{x} = \vec{x}_b$ we have $\partial_i j^i \sim \delta(\vec{x}-\vec{x}_{a,b})$ and therefore the constraint (11) implies

$$I^a(\vec{x}_a) = I^a(\vec{x}_b) = 0. \tag{13}$$

However, from (10) it follows that

$$I^a(\vec{z}) = D^{ab}(V_c) I^b(\vec{x}_b),$$

where $D^{ab}(V_C)$ is the matrix of the adjoint representation of SU(2), corresponding to the group element $V_C(\vec{z}(\lambda), \vec{x}_b)$. Therefore the condition (13) implies $I^a(\vec{x}) = 0$ for all $\vec{x} \in \vec{C}^{(0)}$, i.e. the external source decouples from Yang-Mills equations. This means that the external source does not generate any gauge field.

To summarize, either the external source decouples from classical Yang-Mills equations or it is not consistent with them. We would like to interpret this result as an indication that NGIO for which $I^a(\vec{x}) = 0$ are, in some sense, favored by the Nonabelian gauge theory.

B. The gauge invariant sources without fermions

Now we shall consider the action (4), (5) with W replaced by $W^0(x, x|C)$ given by (2) and $J = J(x|C) = q\delta(\vec{x}-\vec{x}_a)\,[\delta(\vec{C}-\vec{C}^{(0)})]\,[\delta(C_0-x_0)]$. In the corresponding Yang-Mills equations (7) the external courrent $j^{a\nu}_{ext}$ has the form (9), where now

$$I^a(\vec{z}) = i\,\mathrm{Tr}\,[\tilde{V}_C(\vec{x}_a, \vec{x}) T^a V_C(\vec{x}, \vec{x}_a)]. \tag{14}$$

$\tilde{V}_C$ is calculated along the other arc of $\vec{C}^{(0)}$ than that used in V_C. One can verify that the external current satisfies the constraint (11) on the whole $\vec{C}^{(0)}$.

At first sight Yang-Mills equations look as very complicated integro-differential nonlinear equations. Still one could find a solution foı them. Namely, we observe that because gauge transformations just rotate the color spin vector (I^a), one can perform such a gauge transformation that the resulting (I^a) will point in the 3-rd direction for all $\vec{z} \in \vec{C}^{(0)}$, i.e. $I^a = Ie^a_z$, where $e^a_z = \delta^{a3}$, [6]. Then, the Ansatz $A^a_\mu(x) = \delta^{a3}A_\mu(x)$ reduces Yang-Mills equations to ordinary Maxwell equations for $A_\mu(x)$ with the external static current

$$j^i_{\text{ext}}(\vec{x}) = Igq \int_0^1 d\lambda v^i \delta(\vec{x} - \vec{C}^{(0)}(\lambda))$$

along the line $\vec{C}^{(0)}$. The corresponding solutions are known from a text-book electrodynamics. As the next step, we insert the solutions for A_μ on the r.h.s. of (14). Because the l.h.s. is given $(I^a = \delta^{a3}I)$, this yields a consistency condition from which one could try to determine some constants present in A_μ.

Unfortunately, because of zero thickness of the line $\vec{C}^{(0)}$, $\hat{A}_\mu$ has a logarithmic singularity on $\vec{C}^{(0)}$ and we meet the difficulty mentioned earlier. As it was explained, we assume that $\vec{C}^{(0)}$ is at least a twodimensional structure. The simplest possibility is to assume that $\vec{C}^{(0)}$ is a torus, i.e. the current j^v_{ext} forms a torus-like coil. This assumption is not as peculiar as it may look at first sight. Firstly, we recall that in Yang's formulation of Nonabelian gauge theories [66a], one uses the exponentials exp $[ig\hat{A}_\mu dx^\mu]$ independently at each space--time point x. There is no reason why one should arrange these infinitesimal exponentials just along the simplest lines and to neglect more refined possibilities. Secondly, some support comes also from energy considerations. Namely, the classical selfenergy of an infinitely thin, static, linear current diverges logarithmically. On the other hand, classical selfenergy of the current forming the torus is finite, equal to the energy of magnetic field inside the torus.

Thus, we assume that $\vec{C}^{(0)}$ forms a torus-like coil. The solution of the Maxwell equations is given by some magnetic field inside the coil. It remains to check the consistency condition. Because we have $A^{ai} = \delta^{a3}A^i$, $T^a = \frac{1}{2}\sigma^a$, σ^a — Pauli matrices, then

$$V_C(\vec{x}, \vec{x}_a) = \text{Tr} \exp\Big[-ig \int_{C^{(0)},\vec{x}_a}^{\vec{x}} A^i dx^i T^3\Big] = \exp[-igT^3\Phi d(\vec{x}, \vec{x}_a)\kappa],$$

and

$$\tilde{V}_C(\vec{x}_a, \vec{x}) = \exp[-igT^3\Phi\tilde{d}(\vec{x}_a, \vec{x})\kappa].$$

Here Φ is the flux of the magnetic field through the torus, $d(\vec{x}, \vec{x}_a)$ is the length of the torus between the points $\vec{x}_a$, $\vec{x}$ (that is the distance along the big circle of the torus between the points obtained by perpendicular projections of the points $\vec{x}_a$, $\vec{x}$ on the big circle), $\tilde{d}(\vec{x}_a, \vec{x})$ also is the distance between the projections of $\vec{x}_a$ and $\vec{x}$ but taken along the other arc of the big circle of the torus, κ is an unknown coefficient describing the density of windings of the current around the torus. The path ordering was dropped out because now the exponentials commute. From (14) we obtain that

$$I^1 = I^2 = 0, \quad I^3 = I = \sin\tfrac{1}{2}\, g\Phi l\kappa,$$

where $l = d(\vec{x}, \vec{x}_a)+\tilde{d}(\vec{x}_a, \vec{x})$ is the perimeter of the torus along its big circle. Because for the thin, toroidal coil $\Phi \cong qIg\kappa S_\perp$, where $S_\perp$ is the area of the perpendicular cross-section of the torus, we get the consistency condition

$$I = \sin\left[\tfrac{1}{2}(g\kappa)^2 I_q V\right], \tag{15}$$

where V is the volume of the torus. Of course, our considerations require $\kappa \to \infty$. In order to obtain finite results we parallelly take $g \to 0$, in such a way that $g\kappa \equiv \lambda = \text{const}$.

The condition (15) can be read over in at least two ways. Straightforwardly, it can be regarded as an equation for I, in which V, λ, q are fixed parameters describing the given torus. The energy of the torus is $E = \frac{1}{4} H^2 V = \frac{1}{4} q^2\lambda^2 I^2 V$. For sufficiently large $\lambda^2 qV$ there are many values of I obeying (15).

However, the condition (15) can be interpreted also as a quantization condition for $\lambda^2 V$. Namely, when I is a priori fixed, (15) implies

$$\lambda^2 V = \frac{2}{Iq}(\pm \arcsin I + 2k_\pm \pi), \qquad k_+ = \text{integer},\ k_- = \text{integer} + \tfrac{1}{2}. \tag{16}$$

Observe that (15) implies also that q is a positive number, $q > 0$. Thus we have to assume $k_+ \geqslant 0$, $k_- \geqslant \frac{1}{2}$ in order to ensure $\lambda^2 V > 0$. Then, the energy is

$$E_\pm = \tfrac{1}{2} qI(\pm \arcsin I + 2k_\pm \pi), \tag{17}$$

where $\arcsin I < \dfrac{\pi}{2}$. Observe that the spectrum of energy is linear and that it depends only on the strength of the external source q and on the constant I. The constant I in this case is not determined, except for the condition $|I| < 1$. The cases $I = 0, 1$ we exclude as the trivial ones. It is natural to take $I = \frac{1}{2}$ because we have used the fundamental representation for T^a.

Our results indicate that classical gauge fields can be created in a gauge invariant manner on the classical level only if the accompanying fermions satisfy the condition (13). Then, the external current vanishes and the external source decouples from Yang-Mills equations. In particular, this means that the classical selfenergy of such a source is given entirely by the classical selfenergy of the set of accompanying fermions. This selfenergy can be calculated from Yang-Mills equations with the external current $\bar{\psi}T^a\psi$ (this current, coupled directly to $\hat{A}_\mu$, is a gauge noninvariant external source of gauge fields and therefore it was neglected in our considerations). Unfortunately, we cannot relate the condition (13) to the common requirement that the fermions should form a color singlet state.

In the case without fermions, the classical approach seems to require rather complicated objects instead of a simple, closed contour C. We have considered C to be a torus. The question arises whether the resulting magnetic flux tube is stable. Presently we have no answer to this question. It seems that there is no reason for a topological stability. On the other hand, the spectrum (17) is bounded from below and, intriguingly, does not depend on the size of the torus — this suggests an energetistic stability.

The above results can be easily extended to SU(n) groups. Of course, the number of types of NGIO then increases.

This description of the gauge invariant external sources ends our investigations of classical chromodynamics of external charges.

II.4. Final remarks

The question what is the classical Yang-Mills field generated by a given distribution of color charges is far from being satisfactorily answered. Just the opposite, numerous investigations [2–28] reveal unexpectedly complex situation. It was discovered that a fixed color charge distribution allows for infinitely many fields obeying Yang-Mills equations (4) of the Chapter II.1, all these solutions having finite energy and vanishing at infinity. There is no clear cut principle which could tell us which of the solutions is to be adopted as "the physical one". Much more work is required in order to reach a satisfactory understanding of this situation.

We are convinced that further efforts in this direction will lead to important insights into Yang-Mills dynamics on classical as well as quantum levels. Already at the moment one can make interesting observations. For example, solutions with lowest energy as a rule contain nonzero color magnetic fields. Thus, the presence of color magnetic fields results in lowering the energy of Yang-Mills system. This remarkably well corresponds with the belief that the vacuum state in confining QCD is filled in with magnetic fields [50], although the precise relation of the two things is not known at the moment.

A possible way of incorporating into quantized Yang-Mills theory the knowledge gained from classical Yang-Mills equations with external sources is through a modification of gluon propagator. An effort in this direction is presented in [69] — unfortunately, it requires to invent a way of dealing with infrared divergences.

The author would like to thank Professor J. Rayski for encouragement and support. Helpful discussions with Professors K. Zalewski, A. Staruszkiewicz and H. B. Nielsen are also acknowledged.

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