MASS AND CHARGE AS VARIABLES AND CHARGE QUANTIZATION

By A. Herdegen

Institute of Physics, Jagellonian University, Cracow*

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In the classical relativistic mechanics of a particle whose internal structure is neglected the mass and the charge are treated as variables. In this way the manifestly covariant Hamiltonian formalism is derived. Upon quantization the charge quantization rule $e = ne_0$, $n = 0, \pm 1, \pm 2, \ldots$ is proved.

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1. Introduction

It is commonly assumed that if the internal structure of a particle is fixed and neglected then the mass and the charge of the particle are some fixed parameters (this is also true, up to renormalization, in quantum field theory). We shall, on the contrary, starting with the Lagrangian of a charged point particle placed in external gravitational and electromagnetic fields, assume that its mass and charge are additional variables. As the internal structure of the particle is neglected, the resulting system consists of a point particle of any kind subject to the action of these fields. In Section 2 we show what modifications of Lagrange formalism this train of thought entails. Section 3 contains the derivation of manifestly invariant Hamiltonian formalism and the discussion of the gauge invariance of change in parametrization. Section 4, which is central in this article, gives the proof of charge quantization which is based on the assumptions of the present approach.

2. The mass and the charge as variables

The action of a particle with charge e and mass m in external gravitational and electromagnetic fields described by the metric tensor $g_{\mu\nu}$ (with signature $(+1, -1, -1, -1)$) and the potential A_μ respectively is

$$S_{\mathrm{cl}} = \int \mathscr{L}_{\mathrm{cl}} d\tau = \int -mc\sqrt{g_{\mu\nu}(z)\dot{z}^\mu\dot{z}^\nu} - \frac{e}{c} A_\mu(z)\dot{z}^\mu d\tau, \tag{2.1}$$

* Address: Instytut Fizyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland.

where $z^\mu(\tau)$ is an arbitrarily parametrized trajectory of the particle. This action integral is invariant with respect to the gauge transformation of reparametrization and changes by only boundary terms under the gauge transformation of potential A_μ. It follows that the equations of motion are invariant with respect to these two gauge transformations. The Lagrangian $\mathscr{L}_{\mathrm{cl}}$ itself, however, consists of two terms multiplied by mc^2 and e respectively, which depend on the choice of gauges. Therefore, if we want to treat m and e as variables and write down the Lagrange-Euler equations while retaining the gauge invariance at the same time, we must add to these terms two additional generalized velocities, say $\dot{s}$ and $\dot{\lambda}$ respectively, to compensate the gauge changes. The resulting action is

$$S = \int \mathscr{L} d\tau = \int \left[mc^2 \left(\dot{s} - \frac{1}{c} \sqrt{g_{\mu\nu}(z)\dot{z}^\mu \dot{z}^\nu} \right) + e \left(\dot{\lambda} - \frac{1}{c} A_\mu(z) \dot{z}^\mu \right) \right] d\tau, \tag{2.2}$$

where the set of variables consists now of z^μ, m, e, s, λ. The action integral is gauge invariant if together with the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ the replacement $\lambda \to \lambda + \frac{1}{c} \Lambda(z)$ is performed. The equations of motion are

$$\dot{m} = 0, \tag{2.3}$$

$$\dot{e} = 0, \tag{2.4}$$

$$mc \left[\frac{d}{d\tau} \left(\frac{\dot{z}^\mu}{\sqrt{g_{\varrho\nu} \dot{z}^\varrho \dot{z}^\nu}} \right) + \frac{1}{\sqrt{g_{\varrho\nu} \dot{z}^\varrho \dot{z}^\nu}} \Gamma^\mu_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta \right] = \frac{e}{c} F^\mu_{\ \cdot\varrho} \dot{z}^\varrho, \tag{2.5}$$

$$\dot{s} = \frac{1}{c} \sqrt{g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu}, \tag{2.6}$$

$$\dot{\lambda} = \frac{1}{c} A_\mu \dot{z}^\mu, \tag{2.7}$$

where (2.3) and (2.4) have been used in (2.5). The equations (2.3) and (2.4) ensure what is demanded of mass and charge, namely that once measured to have values m and e respectively they remain constant along the whole trajectory. The equation for trajectories (2.5) is exactly the same as usual. From (2.6) it is clear that s controlls the parametrization and is increasing, but otherwise is not determined by equations of motion. Classically s is the proper time. Similarly λ controlls the electromagnetic potential along the trajectory of the particle. Classically both s and λ have no physical meaning or consequences.

3. The Hamiltonian formalism and gauge invariance of reparametrization

Aiming at the canonical quantization we shall now go over to the Hamiltonian formalism. To this end let us introduce the momenta conjugated to z^μ

$$p_\mu \equiv -\frac{\partial \mathscr{L}}{\partial \dot{z}^\mu} = mc\frac{g_{\mu\kappa}\dot{z}^\kappa}{\sqrt{g_{\varrho\nu}\dot{z}^\varrho \dot{z}^\nu}} + \frac{e}{c}A_\mu \tag{3.1}$$

and let us denote

$$\pi_\mu \equiv p_\mu - \frac{e}{c}A_\mu. \tag{3.2}$$

The set of variables enlarged by p_μ satisfies one constraint equation

$$g^{\mu\nu}\pi_\mu\pi_\nu = m^2c^2. \tag{3.3}$$

Wishing to retain the manifest relativistic invariance we regard p_μ as independent variables and use (3.1) and (3.3) to eliminate m from Lagrangian $\mathscr{L}$, which yields

$$\mathscr{L} = e\dot{\lambda} - p_\mu \dot{z}^\mu + \operatorname{sgn} m \cdot \sqrt{g^{\mu\nu}\pi_\mu\pi_\nu}\, c\dot{s}.$$

Defining a new variable $\dot{u} = \dot{s} \cdot \operatorname{sgn} m$ and denoting $\mathscr{H}(z, p, e) = c\sqrt{g^{\mu\nu}\pi_\mu\pi_\nu}$ we obtain

$$\mathscr{L} = e\dot{\lambda} - p_\mu \dot{z}^\mu + \mathscr{H}(z, p, e)\dot{u}. \tag{3.4}$$

The set of independent variables is now z^μ, p_μ, e, λ, u. The Lagrange-Euler equation with respect to variable u is dependent on the remaining Lagrange-Euler equations for Lagrangian (3.4) which are equivalent to

$$\dot{e} = 0, \tag{3.5}$$

$$\dot{z}^\mu = \frac{\dot{u}c}{\sqrt{g^{\varrho\kappa}\pi_\varrho\pi_\kappa}} g^{\mu\nu}\pi_\nu, \tag{3.6}$$

$$\dot{\pi}_\mu = \frac{\dot{u}c}{\sqrt{g^{\varrho\nu}\pi_\varrho\pi_\nu}}\left[\Gamma^\kappa_{\mu\alpha}g^{\alpha\sigma}\pi_\sigma\pi_\kappa + \frac{e}{c}F_{\mu\cdot}{}^\kappa\pi_\kappa\right], \tag{3.7}$$

$$\dot{\lambda} = \frac{\dot{u}}{\sqrt{g^{\varrho\nu}\pi_\varrho\pi_\nu}} g^{\kappa\sigma}A_\kappa\pi_\sigma. \tag{3.8}$$

u is not determined by the equations of motion. If u is chosen as monotonic function of τ then the equations (3.5)–(3.8) are equivalent to the original set of equations of motion (2.3)–(2.7) for $m \neq 0$, appended by the definition of p_μ (3.1). The unrestrictive choice of positive m in the original equations is equivalent to the condition $\dot{u} > 0$ and classically u becomes then the proper time.

Let us now by (ξ, η) denote any of the pairs $(-\lambda, e)$, (z^μ, p_μ), $\mu = 0, 1, 2, 3$. The just discussed independent Lagrange-Euler equations of (3.4) have the form

$$\dot{\xi} = \frac{\partial \mathscr{H}}{\partial \eta}\dot{u}, \quad \dot{\eta} = -\frac{\partial \mathscr{H}}{\partial \xi}\dot{u}. \tag{3.9}$$

Any choice of $\dot{u} > 0$, even if it depends on trajectory, can be reduced by change of parametrization on each solution to one common value $\dot{u} = 1$; τ is then the proper time. By this choice we arrive at the standard Hamiltonian formalism

$$\{\xi, \eta\} = 1, \quad \dot{q} = \{q, \mathscr{H}\}, \quad (q = \xi, \eta), \tag{3.10}$$

which offers a starting point for quantization. On the other hand, should we quantize (3.9) before u has been eliminated, we would face the following difficulty of physical non-uniqueness: in quantum case u is an operator and its choice has physical content. This effect is demonstrated in the Appendix.

The equations (3.10) (for pure electromagnetic interaction) with λ however missing and e being a fixed constant, constituted the base for the relativistic quantum model of [1] (see also [2] and Refs. of [1]). In the following section we shall investigate the consequences of subjecting to quantization another pair of canonical variables $(-\lambda, e)$.

4. *The electromagnetic gauge invariance and charge quantization*

While we got rid of the gauge freedom of reparametrization the electromagnetic gauge freedom persists and, as we have remarked, the change in potential

$$A'_\mu = A_\mu + \partial_\mu \Lambda \tag{4.1}$$

is accompanied by

$$\lambda' = \lambda + \frac{1}{c}\Lambda(z) \tag{4.2}$$

and obviously

$$z^{\mu'} = z^\mu. \tag{4.3}$$

In order to remain within the Hamiltonian formalism we demand that (4.2), (4.3) form part of a canonical transformation. This yields the transformation of momenta

$$e' = e, \tag{4.4}$$

$$p'_\mu = p_\mu + e\frac{1}{c}\partial_\mu \Lambda(z). \tag{4.5}$$

The gauge transformation (4.1)–(4.5) leaves the Hamiltonian and the classical physical state unchanged. Going over to the quantum case we shall demand the gauge independence of physical states to be retained.

The quantization of canonical variables gives

$$[\hat{\lambda}, \hat{e}] = i\hbar, \tag{4.6}$$

$$[\hat{p}_\mu, \hat{z}^\nu] = i\hbar\delta^\nu_\mu. \tag{4.7}$$

The Hilbert space of the canonical relations (4.7) has no bearing on the present discussion and will simply multiply the Hilbert space in which (4.6) is represented. As translation in λ by arbitrary constant, a special case of (4.2), is an allowable transformation leaving all other quantities unchanged, it is clear that the variable λ is continuous in nature. It seems therefore reasonable to represent (4.6) in the Hilbert space $H = L^2(\boldsymbol{R}, d\mu(\lambda))$ in which $\hat{\lambda}$ is the operatoı of multiplication by λ. We make no a priori assumptions about the measure μ itself.

The assumptions about the charge operator $\hat{e}$ and the set of physical states are given in the following two points:

1) $\hat{e}$ is a selfadjoint operator in the Hilbert space $H = L^2(\boldsymbol{R}, d\mu(\lambda))$. The action of the unitary group $U(\varepsilon) = \exp\left(\frac{i}{\hbar}\varepsilon\hat{e}\right)$ can be represented as translation in λ. This means that if $\psi_\varepsilon = U(\varepsilon)\psi$ then functions which represent the vectors ψ and ψ_ε (and which will be denoted by the same symbols) can be chosen so as to fulfil

$$\psi_\varepsilon(\lambda) = \psi(\varepsilon+\lambda).$$

2) Physical states are represented by all these rays in H which are invariant with respect to the group $U(\varepsilon)$. These rays lie in the range $\mathscr{D}(\hat{e})$ of operator $\hat{e}$ and span the whole Hilbert space H.

The first point translates the *heuristic* relation (4.6) into mathematically sound statement[1]. The second, beside the usual assumptions about physical states, introduces their invariance with respect to the special gauge transformations: translations in λ.

Proposition. The whole spectrum σ_e of operator $\hat{e}$ consists of isolated eigenvalues, whose separations are bounded from below. The physical states are charge eigenstates.

Proof. Let $\psi \in H$ represent a physical state. Then according to 2) $\psi_\varepsilon \equiv U(\varepsilon)\psi = \exp$ (i phase (ε))ψ. Taking the strong limit $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\psi_\varepsilon - \psi)$, which exists due to $\psi \in \mathscr{D}(\hat{e})$ we obtain $\hat{e}\psi = e\psi$, $e \in \sigma'_e$, where σ'_e is the set of all eigenvalues of operator $\hat{e}$. Conversely, if ψ is an eigenstate of $\hat{e}$, then $\psi_\varepsilon \equiv U(\varepsilon)\psi = \exp\left(\frac{i}{\hbar}\varepsilon e\right)\psi$ and ψ is a physical state. If we choose representants of ψ and ψ_ε as in 1) then $\psi(\lambda) = \psi_\lambda(0) = \psi(0)\exp\left(\frac{i}{\hbar}e\lambda\right)$. The measure μ must therefore be finite and can be normalized. The set (of all physical states) $\left\{\exp\left(\frac{i}{\hbar}e\lambda\right)\right\}_{e\in\sigma'_e}$ forms an orthonormal basis in H. It follows that $\sigma_e = \overline{\sigma'_e}$ i.e. σ_e consists of the set of eigenvalues and its limit points. The latteı, however, are absent as the separations between different points of σ'_e are bounded from below. Indeed, if $e, e' \in \sigma'_e$, $e'-e \neq 0$

[1] We do not insist on (4.6) itself to be fulfilled in the strict mathematical sense.

then due to orthogonality $e'-e$ is one of the zero points of the function $f(\alpha) = \int \exp\left(\frac{i}{\hbar}\alpha\lambda\right) d\mu(\lambda)$, which being a Fourier transform of a finite measure on $\boldsymbol{R}$ is continuous and $f(0) = 1$. Zero points of such function include a nearest to zero one. Proof is completed.

We know by now that physical particles have sharply defined charge which can take values only from a denumerable set σ_e, and differences in charge are bounded from below; in particular, there exists charge $e_0 \neq 0$ nearest to zero — the minimal charge. As no other physical states are allowed there is a superselection rule in action [3, 4] which forbids forming the superpositions of different charge states. The unphysical nature of variable $\hat{\lambda}$ is confirmed (it does not commute with $\hat{e}$).

The discreteness of charge and the superselection rule being established the following third assumption suggests itself:

3) The set of allowed charge values is closed with respect to addition.

This is justified if only the Gauss law holds. The charge of a system of particles can then be determined by measurements in spacial infinity. When viewed from that distance the system can be regarded as one (possibly unstable) particle with charge being the sum of constituent charges.

We can now easily prove our main assertion:

Theorem. The electric charge is quantized: $e = ne_0$, $n = 0, \pm 1, \pm 2, \ldots$; e_0 is the minimal charge.

Proof. Let $e \in \sigma_e$. If $-e \notin \sigma_e$ then due to 3) for every $e', e'' \in \sigma_e$ there is

$$\left(\exp\left(-\frac{i}{\hbar}e\lambda\right),\quad \exp\left(\frac{i}{\hbar}e'\lambda\right)\right)_{L^2} = \int \exp\left(\frac{i}{\hbar}(e+e')\lambda\right) d\mu(\lambda)$$

$$= \int \overline{\exp\left(\frac{i}{\hbar}e''\lambda\right)} \exp\left(\frac{i}{\hbar}(e+e'+e'')\lambda\right) d\mu(\lambda)$$

$$= \left(\exp\left(\frac{i}{\hbar}e''\lambda\right),\quad \exp\left(\frac{i}{\hbar}(e+e'+e'')\lambda\right)\right)_{L^2} = 0,$$

which contradicts the completeness of $\left\{\exp\left(\frac{i}{\hbar}e\lambda\right)\right\}_{e\in\sigma_e}$. Hence σ_e is symmetric with respect to the change of sign. If e_0 is the minimal charge then from 3) and symmetry of σ_e we infer that $ne_0 \in \sigma_e$, $n = 0, \pm 1, \ldots$ This set of points exhausts the spectrum. Indeed, if $e' \in \sigma_e$ and $|ke_0| < |e'| < |(k+1)e_0|$ then $|e'|-|ke_0| \in \sigma_e$ and $0 < |e'|-|ke_0| < |e_0|$, which contradicts the minimality of e_0. This ends the proof.

The consistence of our assumptions 1)–3) which led us to charge quantization can be proved by presenting a model which satisfies them. Namely, let us put $d\mu(\lambda) = \frac{1}{\lambda_0} d\lambda$ on $\langle 0, \lambda_0\rangle$ and $d\mu(\lambda) = 0$ outside $\langle 0, \lambda_0\rangle$. The representants of elements of $L^2(\boldsymbol{R}, d\mu)$ can be chosen as periodic functions: $\psi(\lambda+\lambda_0) = \psi(\lambda)$. $U(\varepsilon)$ is defined as acting on these representants according to $(U(\varepsilon)\psi)(\lambda) = \psi(\varepsilon+\lambda)$. $U(\varepsilon)$ can be easily proved to be strongly continuous unitary group. Hence according to the Stone theorem $U(\varepsilon) = \exp\left(\frac{i}{\hbar}\varepsilon\hat{e}\right)$, where $\hat{e}$ is a selfadjoint operator. The functions $\left\{\exp 2\pi ik\frac{\lambda}{\lambda_0}\right\}$ form an orthonormal basis of L^2 and are periodic; $U(\varepsilon)$ acts on them as translation and $\hat{e}\exp 2\pi ik\frac{\lambda}{\lambda_0} = k\frac{2\pi\hbar}{\lambda_0}\exp 2\pi ik\frac{\lambda}{\lambda_0}$. The consistence is proved.

Finally let us observe that the discreteness of charge ensures the full gauge invariance of physical states (the operator $\exp\left(\frac{i}{\hbar}\hat{e}\Lambda(\hat{z})\right)$ generating the canonical transformation (4.2)–(4.5) reduces to the usual $\exp\left(\frac{i}{\hbar}e\Lambda(\hat{z})\right)$.

5. *Conclusions*

We started our treatment of mass and charge of a particle in quite symmetric way. Both mass and charge became variables correlated in like way to additional variables, each controlling one type of gauge symmetry. In course of elaboration, however, the theory reveals their essentially different nature. The mass becomes the value taken on by the generator of evolution, whereas charge is one of independent canonical variables. Upon quantization it turns out that mass remains undetermined by external interaction and moreover the consistence of the theory demands that it has some nonzero spread (see [1] and [2]). Both position and shape of mass distribution must be determined by internal dynamics of the particle or some other additional assumptions. On the other hand the results of the last section reproduce the observed universality of charge quantization and confirm its independence of internal dynamics. Our derivation of the charge superselection rule does not involve the Gauss law, which is however crucial in the derivation of the charge quantization rule — this is quite contrary to the usual arguments for charge superselection rule and quantization (cf. [4] and [5]). The scale of the elementary charge remains undetermined by our arguments.

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APPENDIX

The quantum nonequivalence of classically equivalent parametrizations

There is a distinguished class of choices of u in (3.9) leading to a class of different (but classically equivalent) Hamiltonians. Namely, if $\dot{u} = \left.\frac{dF(x)}{dx}\right|_{x=\mathscr{H}(\xi,\eta)}$ is chosen, then the equations (3.9) take on the form

$$\dot{\xi} = \frac{\partial F(\mathscr{H})}{\partial \eta}, \qquad \dot{\eta} = -\frac{\partial F(\mathscr{H})}{\partial \xi},$$

$F(\mathscr{H})$ taking the role of the Hamiltonian. Let us consider a simple model with one pair of canonical variables $[z, p] = i\hbar$ and the Hamiltonian $\mathscr{H} = p$. In this model a wave function has a fixed shape changing only its position. The square of $\mathscr{H}$, however, generates the usual spreading which cannot be eliminated by any change of parameter.

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