THE ACTION PRINCIPLE FOR THE LONGITUDINAL ELECTROMAGNETIC FIELD II

By A. Staruszkiewicz

Institute of Physics, Jagellonian University, Cracow*

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An improved argument is presented which allows to determine the numerical coefficient with which the longitudinal part of the electromagnetic field should enter into the total action.

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1. A comment on the previously considered theory of free phase

In the previous paper [1] we were led to consider a theory of a scalar field S called phase, for which the total action has the form

$$-\frac{1}{32\pi^2}\int d^4x(\Box S)^2. \tag{1}$$

The canonical commutation relation has the form

$$[S(x), S(y)] = \frac{2\pi}{i}\operatorname{sign}(x^0-y^0)\theta[(x-y)(x-y)],$$

where θ is the Heaviside step function. Using the well known operator identity one has

$$e^{iS(x)}e^{iS(y)}-e^{iS(y)}e^{iS(x)} = 0. \tag{2}$$

There is no obvious reason why the above commutator should vanish. Therefore we wish to indicate another, more compelling argument, which might serve as a justification for the particular choice of the dimensionless coupling constant implicit in (1).

It follows from (2) that

$$\lim_{x\to y}\{e^{iS(x)}e^{iS(y)}-e^{iS(y)}e^{iS(x)}\} = 0;$$

* Address: Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

this means that the field $\exp[iS(x)]$ is continuous. Continuity of the operator valued field $\exp[iS(x)]$ is certainly a valid regularity condition to be imposed upon the theory. We thus conclude that the dimensionless coupling constant in (1) can be thought of as selected by the condition of continuity of the field $\exp[iS(x)]$.

2. *The phase associated in an invariant way with the electromagnetic field*

Let $A_\mu(x)$ be the electromagnetic vector potential and e the elementary charge. A scalar field $S(x)$ is called phase if the vector $eA_\mu(x)+\partial_\mu S(x)$ is gauge invariant. It is possible to associate a phase with a given electromagnetic field in a Poincaré invariant way:

$$S(x) = \frac{e}{4\pi}\int d^4y A_\mu(x+y)\partial^\mu\delta(yy). \tag{3}$$

If one performs a gauge transformation

$$\delta A_\mu(x) = \partial_\mu f(x),$$

where $f(x)$ is an arbitrary function of compact support, then

$$\delta S(x) = -ef(x),$$

which shows that $S(x)$ is indeed a phase.

In what follows we shall have to deal with the case when the function $f(x)$ has to satisfy the wave equation and therefore cannot be a function of compact support. In this case the formula (3) has to be modified to the form

$$S(x) = 2\cdot\frac{e}{4\pi}\int d^4y A_\mu(x+y)\partial^\mu\delta(yy). \tag{4}$$

The correctness of this definition is checked in the Appendix.

3. *The commutator of two phases and the condition of continuity*

Consider now the electromagnetic theory based on the action integral

$$-\frac{1}{16\pi}\int d^4x\{F^{\mu\nu}F_{\mu\nu}+2\gamma(\partial^\mu A_\mu)^2\},$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and γ is a dimensionless constant. Canonical commutation relations are [2]

$$[A_\mu(x), A_\nu(y)] = \frac{i}{2}\left\{g_{\mu\nu}\Box+\left(\frac{1}{\gamma}-1\right)\partial_{\mu\nu}\right\}\operatorname{sign}(x^0-y^0)\theta[(x-y)(x-y)].$$

Introducing the phase as in (4),

$$S(x) = \frac{e}{2\pi}\int d^4y A_\mu(x+y)\partial^\mu\delta(yy),$$

one finds the commutator of two phases

$$[S(x), S(y)] = i\frac{e^2}{8\pi}\int d^4\xi \int d^4\zeta\partial^\mu\delta(\xi\xi)\partial^\nu\delta(\zeta\zeta)$$

$$\times\left\{g_{\mu\nu}\Box + \left(\frac{1}{\gamma}-1\right)\partial_{\mu\nu}\right\}\text{sign}\,(x^0+\xi^0-y^0-\zeta^0)\theta[(x+\xi-y-\zeta)^2].$$

Computing the integral on the right hand side one should not drop surface integrals because they do not vanish. Introducing the spherical coordinates and integrating over angular variables first one has a well defined integral to take at every step of integration and one obtains

$$[S(x), S(y)] = i\frac{e^2}{2}\left(1-\frac{1}{\gamma}\right)\text{sign}\,(x^0-y^0)\theta[(x-y)(x-y)].$$

This will coincide with the free phase case described in the first section if

$$\frac{e^2}{2}\left(\frac{1}{\gamma}-1\right) = 2\pi$$

i.e. if

$$\gamma = \frac{e^2}{4\pi+e^2}.$$

For small e^2 this result coincides with that obtained in a different way in [3].

APPENDIX

Let

$$\delta S(x) = \frac{e}{4\pi}\int d^4y\partial_\mu f(x+y)\cdot\partial^\mu\delta(yy).$$

Then

$$\delta S(x) = \begin{cases} -ef(x) & \text{if } f(x) \text{ is of compact support,} \\ -\dfrac{e}{2}f(x) & \text{if} \quad \Box f(x) = 0. \end{cases}$$

The first statement is easily proved by application of the Gauss-Ostrogradski theorem:

$$\delta S(x) = \frac{e}{4\pi}\int d^4y\{\partial_\mu[f(x+y)\partial^\mu\delta(yy)] - f(x+y)\Box\delta(yy)\}.$$

The first term vanishes because $f(x)$ is of compact support; the second term equals $-ef(x)$ because $\Box\delta(yy) = 4\pi\delta(y^0)\delta(y^1)\delta(y^2)\delta(y^3)$.

To prove the second statement it is enough to consider the spherically symmetric case

$$f(x) = \frac{1}{r}[f(t+r)-f(t-r)],$$

where $t = x^0$, $r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$, $f(t)$ is an arbitrary function of one real variable; in what follows we assume that the support of $f(t)$ is compact.

$$\delta S(x) = \frac{e}{4\pi}\int d^4y\partial^\mu[\delta(yy)\partial_\mu f(x+y)]$$

because $\Box f = 0$. Applying again the Gauss-Ostrogradski theorem we have

$$\delta S(x) = \lim_{T\to\infty}\frac{e}{4\pi}\int\limits_{|y^0|\leq T} d^4y\partial^\mu[\delta(yy)\partial_\mu f(x+y)]$$

$$= \lim_{T\to\infty}\frac{e}{4\pi}\left\{\int\limits_{y^0=T} d^3y\delta(yy)\partial_0 f(x+y)-\int\limits_{y^0=-T} d^3y\delta(yy)\partial_0 f(x+y)\right\}$$

$$= -\frac{e}{2r}[f(t+r)-f(t-r)] = -\frac{e}{2}f(x).$$

REFERENCES

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