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A NONLINEAR MODIFICATION OF THE SCHRÖDINGER EQUATION

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It is pointed out that the previously proposed theory of free phase leads to a uniquely determined nonlinear modification of the Schrödinger equation. Expressions for probability density and energy density appropriate for the modified equation are given.

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1. The Galilean limit of the total action of free phase

In Ref. [1] we were led to consider the action

$$-\frac{1}{32\pi^2}\int d^4x(\Box S)^2, \tag{1}$$

where S is a field physically identical with the phase of a wave function. The integral (1) is Poincaré invariant if S is a scalar, which is indeed the case in the relativistic mechanics. In the Galilean limit the second time derivative is small when compared with the second space derivatives and (1) goes over into the expression

$$-\frac{1}{32\pi^2}\int dtdV(\Delta S)^2. \tag{2}$$

Here $dV = dxdydz$, Δ is the Laplace operator; we use units such that $h = c = 1$. The above action is invariant under Galilean transformation

$$t = t',$$

$$\boldsymbol{x} = \boldsymbol{x}'+\boldsymbol{v}t', \tag{3}$$

if S transforms as the phase of a nonrelativistic wave function i.e. if

$$S(t, \boldsymbol{x}) = S'(t', \boldsymbol{x}')+m\boldsymbol{v}\cdot\boldsymbol{x}'+\tfrac{1}{2}mv^2t', \tag{4}$$

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m being the mass of a particle. One has from (3) and (4)

$$\nabla S = \nabla' S' + m\boldsymbol{v},$$

$$\Delta S = \Delta' S',$$

which shows that the integral (2) is indeed invariant under Galilean transformations.

2. *The modified Schrödinger equation*

Since the field S is physically identical with the phase of a wave function, we can add the action (2) to the action of a free particle of mass m, thus obtaining the modified action

$$-\int dt dV \left\{ R^2 \frac{\partial S}{\partial t} + \frac{1}{2m} [(\nabla R)^2 + R^2 (\nabla S)^2] + \frac{1}{32\pi^2} (\Delta S)^2 \right\}. \tag{5}$$

Here R is the amplitude and S is the phase; thus $\psi = R \exp(iS)$ is the Schrödinger wave function.

It is difficult to say without further investigations if there are physical phenomena described by the above action. We find it remarkable, however, that this action is uniquely determined by electromagnetic rather than quantum-mechanical considerations. Several authors [2, 3] introduced nonlinear modifications of the Schrödinger equation; the equations so far proposed contain dimensional constants which is obviously a Schönheitsfehler. The nonlinear modification presented above contains no dimensional constants; there does appear a dimensionless constant apparently determined by arguments given in Ref. [1].

3. *The energy density and the probability density*

Equation of motion for the phase S has the form

$$\frac{\partial}{\partial t}(R^2) + \frac{1}{m} \nabla (R^2 \nabla S) - \frac{1}{16\pi^2} \Delta\Delta S = 0$$

which shows that R^2 is the probability density and

$$\boldsymbol{j} = \frac{1}{m} R^2 \nabla S - \frac{1}{16\pi^2} \nabla (\Delta S)$$

is the current. The energy density is easily found to be

$$\frac{1}{2m} [(\nabla R)^2 + R^2 (\nabla S)^2] + \frac{1}{32\pi^2} (\Delta S)^2.$$

It seems reasonable to assume that a physically acceptable solution should be normalized, $\int R^2 dV = 1$, and should have finite total energy:

$$\int dV \left\{ \frac{1}{2m} [(\nabla R)^2 + R^2 (\nabla S)^2] + \frac{1}{32\pi^2} (\Delta S)^2 \right\} < \infty.$$

The relevance of the last remark may be seen from the following example. The function

$$\psi(t, \mathbf{x}) = \left(\frac{mt_0}{\pi}\right)^{3/4} \frac{1}{(t-it_0)^{3/2}} \exp\left[i\frac{m|\mathbf{x}|^2}{2(t-it_0)}\right]$$

is an exact, normalized solution of the Schrödinger equation. It is also an exact solution of the modified equation, because

$$S = -\frac{mt|\mathbf{x}|^2}{2(t^2+t_0^2)} - \frac{3}{4i}\ln\frac{t-it_0}{t+it_0}$$

and

$$\Delta S = -\frac{3mt}{t^2+t_0^2}, \qquad \Delta\Delta S = 0.$$

One sees that the total energy associated with the solution is infinite; apparently the phase should be much smaller at spacial infinity than is the case for the Gaussian wave packet.

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