ELEMENTARY MODEL REFLECTING SOME CONTROVERSIAL POINTS IN THE THEORY OF HEAVY QUARKONIA*

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Some unusual features of the theory of heavy quarkonia interacting with the QCD vacuum gluon condensate are reproduced using a generalized harmonic oscillator model. The model, however, suggests further results contradicting the usual approach.

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1. Introduction

It should be possible to get information about the chromoelectric field in the QCD vacuum from spectroscopic data for heavy quarkonia. The idea is [1, 2] as follows. For very large quark masses (m) it is consistent to describe the quarkonium by analogy with a nonrelativistic positronium with $\beta = 4\alpha_s/3$ replacing the electromagnetic fine structure constant α. In this limit textbook formulae can be used to obtain energy levels, transition probabilities etc. For instance for the energy (mass) and electronic width of the $1S$ ground state one finds

$$E_{\text{Coul}} = 2m - \tfrac{1}{4} m\beta^2, \quad \Gamma_{\text{Coul}} = \tfrac{1}{2} Q^2\alpha^2 m\beta^3, \tag{1}$$

where Q is the quark electric charge in units of the electron charge. The crucial assumption is that the leading corrections to the Coulombic formulae are due to the interaction with the low frequency and long wave length chromoelectric field of the gluon condensate filling the QCD vacuum. Using perturbation theory one can evaluate these corrections [1, 2]. In particular, for the energy and electronic width of the ground state

$$E = E_{\text{Coul}} + 1.648 m\beta^2\gamma, \quad \Gamma = \Gamma_{\text{Coul}}(1+a\gamma), \tag{2}$$

where

$$\gamma = -\frac{\langle E^2 \rangle}{m^4\beta^6}, \quad \langle E^2 \rangle \approx -0.1 \text{ GeV}^4. \tag{3}$$

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The constant a is discussed below. Since $\langle E^2\rangle < 0$, analogies with standard perturbation theory may be misleading [3]. The second order shift of the ground state energy is positive, contrary to a well-known theorem. The calculation of the electronic width is still controversial [4]. The problem has been clearly stated by Voloshin [3]: The physical quarkonium has a component $|1\rangle$, where the $q\bar{q}$ is a colour singlet, and a component $|8\rangle$, where the $q\bar{q}$ is a colour octet with an additional gluon making the quarkonium colourless. Leutwyler [2] normalized the component $|1\rangle$ to unity and found $a = 14$. Voloshin, on the other hand, found for component $|1\rangle$ a norm exceeding unity [3] and obtained $a = 19$. Voloshin explains that his approach probably does not violate unitarity, because his norm for component $|8\rangle$ is negative and the physical quarkonium state $|1\rangle + |8\rangle$ is normalized to unity as it should.

In the present paper we describe an elementary, exactly soluble, quantum mechanical model, which illustrates some features of the heavy quarkonium problem. The positive energy shift of the ground state and the unusual norms are reproduced. It is not clear, whether it is justified to push the analogy further. We show however that, if this is done, results very different from the usual ones are obtained for heavy quarkonia.

2. *The model*

In the heavy quarkonium problem there is a singlet qq Hamiltonian H_1 with eigenstates $|1\rangle$, an octet $q\bar{q}$ Hamiltonian H_8 with eigenstates $|8\rangle$, and an interaction V with the chromoelectric vacuum field, which mixes the states $|1\rangle$ and $|8\rangle$. By analogy we propose to consider a model with the Hamiltonian

$$H = \begin{pmatrix} H_1 & V \\ V & H_8 \end{pmatrix} = -\tfrac{1}{2}\nabla^2 + \tfrac{1}{2}\omega_0^2 x^2 - \tfrac{1}{2}\omega_2^2 x^2 \sigma_z - \lambda\omega_2^2 x^2 \sigma_x, \tag{4}$$

where σ_z and σ_x are the standard Pauli matrices. For $\lambda = 0$ the eigenstates $|1\rangle$ and $|8\rangle$ are the eigenstates of harmonic oscillators with frequencies

$$\omega_1 = \sqrt{\omega_0^2 - \omega_2^2}, \qquad \omega_8 = \sqrt{\omega_0^2 + \omega_2^2}\,. \tag{5}$$

For $\lambda \neq 0$ it is easy to find the exact solutions by diagonalizing the Hamiltonian (4). For the analogy with quarkonia, however, it is better to use perturbative approximations. For the ground state

$$E = \tfrac{1}{2}\omega_1 - \frac{\lambda^2\omega_2^2}{2\omega_1}, \qquad |\psi\rangle = (1 - \tfrac{1}{2}\lambda^2)\,|1\rangle + \lambda|8\rangle. \tag{6}$$

Here the wave function corresponding to the state $|1\rangle$ ($|8\rangle$) is understood as a two-component object with the first (second) component normalized to unity and the second (first) vanishing.

In the heavy quarkonium problem one distinguishes the perturbative vacuum, which by definition contains no gluon condensate, and the physical vacuum. For each of these vacua $\langle E^2\rangle$ is given by an ultraviolet divergent integral and has to be regularized by subtraction. Since we are interested in the chromoelectric field of the gluon condensate, it is natural to interpret the $\langle E^2 \rangle$ from formula (3) as the $\langle E^2 \rangle$ for the physical vacuum minus the $\langle E^2 \rangle$ for the perturbative vacuum. Thus, $\langle E^2 \rangle < 0$ means that the presence of the condensate reduces the fluctuations of the chromoelectric field. In our model we use formulae (6) for both vacua, with $\lambda = \lambda_0$ for the perturbative one and $\lambda = \lambda_1$ for the physical one. Thus $\langle E^2 \rangle < 0$ corresponds to

$$\delta\lambda^2 = \lambda_0^2 - \lambda_1^2 > 0. \tag{7}$$

The calculations reported in Refs. [1] and [2] refer to changes of energy due to the presence of the gluon condensate. Therefore, we need for comparison expressions of quantities defined in the physical vacuum by quantities calculated for the perturbative vacuum. For the ground state

$$E_{\text{phys}} = E_{\text{pert}} + \delta\lambda^2 \frac{\omega_2^2}{2\omega_1}, \qquad \Gamma_{\text{phys}} = \Gamma_{\text{pert}} \left(1 + \delta\lambda^2 \frac{\omega_2^2}{2\omega_1^2}\right)(1 + \delta\lambda^2). \tag{8}$$

The electronic width has been calculated assuming that it is proportional to $(1-\lambda^2)\, |\langle x = 0|1\rangle|^2$.

Formulae (8) are close analogues of the formulae obtained by Voloshin [1] and Leutwyler [2]. The energy shift is positive. The electronic width is multiplied by two factors. The first, due to the perturbation of the wave function $\langle x|1\rangle$, is greater than one and has been included in both references. The second, due to the change in the norm of the singlet component of $|\psi\rangle$, has been included only by Voloshin and results in an increase of parameter a. In the model no problem with unitarity arises, because the increase of the norm of the singlet component is exactly compensated by the decrease of the norm of the octet component.

The model (4) is a representative of a broad class of models, which yield similar results. In particular, the n, pπ^- sector of the Lee model[1] (cf. e.g. [5]) can be used to draw qualitatively similar conclusions. The identification is: physical neutron—quarkonium, bare neutron state—state $|1\rangle$, bare proton state—state $|8\rangle$, pion—gluon. The gluon condensate is assumed to change the gluon propagator by a constant factor $f < 1$. This is equivalent to a change in the coupling constant from the perturbative value $g = \lambda_0$ to $\lambda_1 = \lambda_0\sqrt{f}$. The Coulomb limit corresponds to $\lambda_0 \to 0$. The negative norm of the state $|1\rangle$, occurring for large λ_0 in this model [5], is irrelevant for our analysis.

3. *Discussion*

At the formal level formulae (8) yield an instructive, though qualitative, illustration for the unusual features of formulae (2). There is an important physical difference, however. The Coulomb approximation for quarkonia corresponds to the approximation $\lambda_0 = 0$ in the model. However, the model makes sense only if

$$\delta\lambda^2 < \lambda_0^2, \tag{9}$$

[1] The author thanks dr. Th. Ruijgrok for the suggestion to use this model.

therefore, it is inconsistent to keep $\delta\lambda^2$ and neglect λ_0^2. On the other hand, the identification of the perturbative quantities in formula (8) with the Coulomb quantities in formula (2) corresponds to $\lambda_0 \to 0$. Only in this limit the increase of the norm of the singlet state and the decrease of the norm of the octet state means that the first becomes bigger than one and the second negative. According to the model, the correct relations between the physical and the Coulombic quantities are

$$E_{\text{phys}} = E_{\text{Coul}} - \frac{\lambda_1^2 \omega_2^2}{2\omega_1}, \qquad \Gamma_{\text{phys}} = \Gamma_{\text{Coul}} \left(1 - \lambda_1^2 \frac{\omega_2^2}{2\omega_1^2}\right)(1-\lambda_1^2). \tag{10}$$

Thus the shifts of both the energy and the electronic width are reversed compared to the quarkonium case (2).

The model can be also used to estimate the range of frequencies contributing to $\langle \boldsymbol{E}^2 \rangle$. Let us assume by analogy with (9)

$$|\langle \boldsymbol{E}^2(\omega) \rangle| < \langle \boldsymbol{E}^2_{\text{pert}}(\omega) \rangle \approx \frac{4\omega^3}{\pi^2}, \tag{11}$$

where the approximate equality is from a crude estimate obtained by multiplying by eight the estimate used for the electromagnetic field by Welton [6]. Welton was discussing the Lamb shift and assumed $\langle \boldsymbol{E}^2 \rangle = \langle \boldsymbol{B}^2 \rangle$. This contradicts the result $\langle \boldsymbol{E}^2 \rangle = -\langle \boldsymbol{B}^2 \rangle$ following from Lorentz invariance (cf. e.g. [1, 2]). For the Lamb shift, however, frequencies higher than the electron mass do not contribute. This explicitly breaks Lorentz invariance and the constraint $\langle \boldsymbol{E}^2 \rangle = -\langle \boldsymbol{B}^2 \rangle$ becomes irrelevant[2]. It is plausible that the same holds for quarkonia. Estimate (11) is only the zero order term of a perturbative expansion. We find, however that, even if it is uncertain by an order of magnitude, it leads to interesting implications. Using for the integral over frequencies of $\langle \boldsymbol{E}^2(\omega) \rangle$ estimate (3), we find for the upper limit of frequencies contributing to $\langle \boldsymbol{E}^2 \rangle$

$$\omega_{\max} \gtrsim (\pi^2 |\langle \boldsymbol{E}^2 \rangle|)^{1/4} \approx 1 \text{ GeV}. \tag{12}$$

This would make the low frequency assumption valid only for extremely heavy quarkonia [7, 8]. It should be stressed, however, that this conclusion follows from the model used beyond the range, where it agrees with the usual analysis.

The conclusion from the present work is that, either our model misses an important point in the theory of heavy quarkonia, and then a clear identification of this point seems necessary to get a physical understanding of this theory, or the usual method of analysis will have to be reconsidered.

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[2] This constraint it also used to obtain $\langle \boldsymbol{E}^2 \rangle$ from available estimates of $\langle G^a_{\mu\nu} G^{\mu\nu a} \rangle$ [1, 2]. Here one could perhaps look for a trivial possibility of eliminating, at least for the field felt by the quarkonium, of the unusual $\langle \boldsymbol{E}^2 \rangle < 0$.

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