TENSOR FORM OF THE BREIT EQUATION

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The Breit equation for a system of two Dirac particles is represented in the scalar⊕vector⊕tensor form. Then, in the case of equal masses, the internal motion satisfies a generalized Klein-Gordon equation for spin $s = 0$ (parafermionium) and generalized Proca equations for spin $s = 1$ (orthofermionium). If the potential is central, one gets for orthofermionium radial wave functions being analogues of electric and magnetic multipole radial fields (but with $m \neq 0$ and $V \not\equiv 0$).

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As is well known, the Breit equation for a system of two Dirac particles [1],

$$[E-V-(\vec{\alpha}_1 \cdot \vec{p}+\beta_1 m_1)-(-\vec{\alpha}_2 \cdot \vec{p}+\beta_2 m_2)]\psi(\vec{r}) = 0 \tag{1}$$

(where $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ and $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$), is usually handled in the spinor ⊗ spinor representation [2]. In the present note we write it down in the scalar ⊕ vector ⊕ tensor representation which may be convenient for some purposes, giving directly relativistic equations for parafermionium and orthofermionium.

To start with we write Eq. (1) in the Dirac representation of matrices $\vec{\alpha}_i = \gamma_i^5 \vec{\sigma}_i$ and $\beta_i (i = 1, 2)$ and then combine the wave-function components $\psi_{\beta_1\beta_2}$ (where $\beta_i = \pm 1$ are eigenvalues of β_i) into the new components

$$f^{\pm} = \frac{\psi_{++} \pm \psi_{--}}{\sqrt{2}}, \quad g^{\pm} = \frac{\psi_{+-} \pm \psi_{-+}}{\sqrt{2}}. \tag{2}$$

In this calculation we assume that

$$V = V^I + \gamma_1^5 \gamma_2^5 V^R, \tag{3}$$

where $V^{I,R}$ depend on $\vec{r}$ and $\vec{\sigma}_i$ $(i = 1, 2)$ only. In general,

$$V^{I,R} = P_0 V_0^{I,R} + P_1 V_1^{I,R} \tag{4}$$

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with P_s ($s = 0, 1$) being the projection operators,

$$P_0 = \tfrac{1}{4}(1-\vec{\sigma}_1 \cdot \vec{\sigma}_2), \quad P_1 = \tfrac{1}{4}(3+\vec{\sigma}_1 \cdot \vec{\sigma}_2) = \tfrac{1}{8}(\vec{\sigma}_1+\vec{\sigma}_2)^2, \tag{5}$$

on states with spin $s = 0$ and $s = 1$, respectively. For instance, if the potential is of the Breit type

$$V = V_C(r)+\tfrac{1}{2}\left[\vec{\alpha}_1 \cdot \vec{\alpha}_2 + \frac{(\vec{\alpha}_1 \cdot \vec{r})(\vec{\alpha}_2 \cdot \vec{r})}{r^2}\right] V_B(r), \tag{6}$$

we have

$$V^I = V_C(r), \quad V^R = \tfrac{1}{2}\left[\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \frac{(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r})}{r^2}\right] V_B(r) \tag{7}$$

and hence

$$V_0^I = V_1^I = V_C(r), \quad V_0^R = -2V_B(r), \quad V_1^R = \tfrac{1}{4}\left[\frac{(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{r}}{r}\right]^2 V_B(r). \tag{8}$$

In this way we can represent Eq. (1) by the following system of equations:

$$\begin{cases} (E-V^I \mp V^R)f^{\pm} - (m_1+m_2)f^{\mp} = \pm(\vec{\sigma}_1 \mp \vec{\sigma}_2)\cdot\vec{p}g^{\pm}, \\ (E-V^I \mp V^R)g^{\pm} - (m_1-m_2)g^{\mp} = \pm(\vec{\sigma}_1 \mp \vec{\sigma}_2)\cdot\vec{p}f^{\pm}. \end{cases} \tag{9}$$

Applying to Eqs. (9) the spin projection operators (5) we obtain the system of equations for the wave-function components $f_s^{\pm} = P_s f^{\pm}$ and $g_s^{\pm} = P_s g^{\pm}$ ($s = 0, 1$):

$$\begin{cases} (E-V_0^I \mp V_0^R)f_0^{\pm} - (m_1+m_2)f_0^{\mp} = \begin{cases}(\vec{\sigma}_1-\vec{\sigma}_2)\cdot\vec{p}g_1^{+} \\ 0\end{cases}, \\ (E-V_0^I \mp V_0^R)g_0^{\pm} - (m_1-m_2)g_0^{\mp} = \begin{cases}(\vec{\sigma}_1-\vec{\sigma}_2)\cdot\vec{p}f_1^{+} \\ 0\end{cases}, \\ (E-V_1^I \mp V_1^R)f_1^{\pm} - (m_1+m_2)f_1^{\mp} = \pm(\vec{\sigma}_1 \mp \vec{\sigma}_2)\cdot\vec{p}\begin{cases}g_0^{+} \\ g_1^{-}\end{cases}, \\ (E-V_1^I \mp V_1^R)g_1^{\pm} - (m_1-m_2)g_1^{\mp} = \pm(\vec{\sigma}_1 \mp \vec{\sigma}_2)\cdot\vec{p}\begin{cases}f_0^{+} \\ f_1^{-}\end{cases}. \end{cases} \tag{10}$$

Note that the "large-large" components ψ_{++}'s are included only in f's.

In Eqs. (10) the matrices $\vec{\sigma}_i$ can be easily absorbed into the wave function components *if* V_s^I and V_s^R do not depend on $\vec{\sigma}_i$ (this condition means practically that the retardation is neglected: $V_s^R \equiv 0$, see Eq. (8)). In fact, defining the new components by the formulae

$$\phi = f_0^-, \quad \phi^0 = f_0^+, \quad \vec{\phi} = \frac{\vec{\sigma}_1-\vec{\sigma}_2}{2}\, g_1^+, \quad \vec{\phi}^0 = \frac{\vec{\sigma}_1-\vec{\sigma}_2}{2}\, g_1^-,$$

$$\chi = g_0^-, \quad \chi^0 = g_0^+, \quad \vec{\chi} = \frac{\vec{\sigma}_1-\vec{\sigma}_2}{2}\, f_1^+, \quad \vec{\chi}^0 = \frac{\vec{\sigma}_1-\vec{\sigma}_2}{2}\, f_1^-, \tag{11a}$$

we are able to rewrite the system of equations (10) in the following form independent of $\vec{\sigma}_i$:

$$\begin{cases} \frac{1}{2}(E-V_0^I-V_0^R)\phi^0-\vec{p}\cdot\vec{\phi} = \frac{1}{2}(m_1+m_2)\phi, \\ \frac{1}{2}(E-V_0^I+V_0^R)\phi = \frac{1}{2}(m_1+m_2)\phi^0, \\ \frac{1}{2}(E-V_1^I-V_1^R)\vec{\phi}-\vec{p}\phi^0 = \frac{1}{2}(m_1-m_2)\vec{\phi}^0, \\ \frac{1}{2}(E-V_1^I+V_1^R)\vec{\phi}^0+i\vec{p}\times\vec{\chi}^0 = \frac{1}{2}(m_1-m_2)\vec{\phi} \end{cases} \tag{12a}$$

and

$$\begin{cases} \frac{1}{2}(E-V_0^I-V_0^R)\chi^0-\vec{p}\cdot\vec{\chi} = \frac{1}{2}(m_1-m_2)\chi, \\ \frac{1}{2}(E-V_0^I+V_0^R)\chi = \frac{1}{2}(m_1-m_2)\chi^0, \\ \frac{1}{2}(E-V_1^I-V_1^R)\vec{\chi}-\vec{p}\chi^0 = \frac{1}{2}(m_1+m_2)\vec{\chi}^0, \\ \frac{1}{2}(E-V_1^I+V_1^R)\vec{\chi}^0+i\vec{p}\times\vec{\phi}^0 = \frac{1}{2}(m_1+m_2)\vec{\chi}. \end{cases} \tag{12b}$$

It can be seen from Eqs. (12) that (at least in the free case of $V \equiv 0$) ϕ and χ are two scalars, $(\phi^\mu) = (\phi^0, \vec{\phi})$ and $(\chi^\mu) = (\chi^0, \vec{\chi})$ form two four-vectors, and $\vec{\phi}^0 = (\phi^{0l})$ and $\vec{\chi}^0 = (\chi^{0l})$ are the time-space parts of two skew-symmetric tensors. Instead of $\vec{\phi}^0$ and $\vec{\chi}^0$, respectively, one can operate with the space-space parts

$$\left.\begin{matrix}\vec{\vec{\chi}} \\ \vec{\vec{\phi}}\end{matrix}\right\} = \left(\frac{\vec{\sigma}_1-\vec{\sigma}_2}{2}\,\frac{\vec{\sigma}_1+\vec{\sigma}_2}{2} - \frac{\vec{\sigma}_1+\vec{\sigma}_2}{2}\,\frac{\vec{\sigma}_1-\vec{\sigma}_2}{2}\right)\left\{\begin{matrix}g_1^- \\ f_1^-\end{matrix}\right. \tag{11b}$$

of two skew-symmetric tensors $\chi^{\mu\nu}$ and $\phi^{\mu\nu}$ (in Eq. (11b) the dyadic product is used). Then $\chi^{kl} = i\varepsilon^{klm}\phi^{om}$ and $\phi^{kl} = i\varepsilon^{klm}\chi^{om}$ so that in Eqs. (12) one can write $\vec{p}\cdot\vec{\vec{\chi}} = -i\vec{p}\times\vec{\phi}^0$ and/or $\vec{p}\cdot\vec{\vec{\phi}} = -i\vec{p}\times\vec{\chi}^0$. Obviously, the tensors $\chi^{\mu\nu}$ and $\phi^{\mu\nu}$ are mutually dual. Note, however, that Eqs. (12), even in the free case of $V \equiv 0$, are not evidently covariant in consequence of the lack of evident covariance of the relativistic Breit equation (1).

In the case of equal masses $m_1 = m_2 \equiv m$, the system (12) splits into three independent parts:

$$\begin{cases} \frac{1}{2}(E-V_0^I-V_0^R)\phi^0-\vec{p}\cdot\vec{\phi} = m\phi, \\ \frac{1}{2}(E-V_0^I+V_0^R)\phi = m\phi^0, \\ \frac{1}{2}(E-V_1^I-V_1^R)\vec{\phi}-\vec{p}\phi^0 = 0 \end{cases} \tag{13}$$

and

$$\begin{cases} \frac{1}{2}(E-V_0^I-V_0^R)\chi^0-\vec{p}\cdot\vec{\chi} = 0, \\ \frac{1}{2}(E-V_1^I-V_1^R)\vec{\chi}-\vec{p}\chi^0 = m\vec{\chi}^0, \\ \frac{1}{2}(E-V_1^I+V_1^R)\vec{\chi}^0+i\vec{p}\times\vec{\phi}^0 = m\vec{\chi}, \\ \frac{1}{2}(E-V_1^I+V_1^R)\vec{\phi}^0+i\vec{p}\times\vec{\chi}^0 = 0 \end{cases} \tag{14}$$

and one trivial equation implying $\chi \equiv 0$ i.e. $g_0^- \equiv 0$. The systems of equations (13) and (14) include the "large-large" components for spin $s = 0$ and $s = 1$, respectively, and so describe in a relativistic way states with $s = 0$ (parafermionium) and $s = 1$ (orthofermionium), separately.

In the case of $V \equiv 0$, Eqs. (13) and (14) take the form

$$\begin{cases} p_\mu \phi^\mu = m\phi, \\ p_0 \phi = m\phi_0, \\ p_0 \vec{\phi} - \vec{p}\phi_0 = 0 \end{cases} \tag{15}$$

and

$$\begin{cases} p_\mu \chi^\mu = 0, \\ p_0 \vec{\chi} - \vec{p}\phi_0 = m\vec{\chi}_0, \\ p_0 \vec{\chi}_0 + i\vec{p} \times \vec{\phi}_0 = m\vec{\chi}, \\ p_0 \vec{\phi}_0 + i\vec{p} \times \vec{\chi}_0 = 0, \end{cases} \tag{16}$$

respectively, where $p_0 = \frac{1}{2} E$ and $(p_\mu) = (p_0, -\vec{p})$. Eqs. (15) lead to the free Klein-Gordon equation:

$$(p^2 - m^2)\phi = 0. \tag{17}$$

Denoting in Eqs. (16)

$$A^\mu = \chi^\mu, \quad \vec{E} = im\vec{\chi}^0, \quad \vec{B} = -m\vec{\phi}^0, \tag{18}$$

we can recognize this second system as a part of free Proca equations:

$$\begin{cases} \partial_\mu A^\mu = 0, \\ \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} - \operatorname{grad} A_0, \\ -\dfrac{\partial \vec{E}}{\partial t} + \operatorname{rot} \vec{B} = -m^2 \vec{A}, \\ \dfrac{\partial \vec{B}}{\partial t} + \operatorname{rot} \vec{E} = 0, \end{cases} \tag{19}$$

when $p_\mu = i\partial/\partial x^\mu$ and $(x^\mu) = (t, \vec{r})$. The rest of free Proca equations follows already from the former:

$$\begin{cases} \vec{B} = \operatorname{rot} \vec{A}, \\ \operatorname{div} \vec{E} = -m^2 \varphi. \end{cases} \tag{20}$$

Eqs. (17) and (18), if put together, are evidently covariant and imply that

$$(p^2-m^2)A^\mu = 0, \quad p_\mu A^\mu = 0, \tag{21}$$

where $p^2 = \Box$. Of course, all these equations describe the *internal* motion of a system of two Dirac particles, in contrast to the original Klein-Gordon and Proca equations referring to the external motion of one particle.

Turning back to the interaction case of $V \not\equiv 0$ let us assume that $V_0^I = V_0^I \equiv V$ and $V_0^R = V_1^R \equiv 0$ (it is the case of spin-independent static potential). Then the simplest equation we can obtain for *one* wave-function component from the system (13) is

$$\left\{\left(\frac{E-V}{2}\right)^2 - \vec{p}^2 - m^2 - \frac{[\vec{p}, V]}{E-V}\cdot\vec{p}\right\}\phi^0 = 0. \tag{22}$$

If the potential is central, $V = V(r)$, Eq. (22) gives the radial equation

$$\left[\left(\frac{E-V}{2}\right)^2 + \frac{1}{r}\frac{d^2}{dr^2}r - \frac{j(j+1)}{r^2} - m^2 + \frac{\frac{dV}{dr}}{E-V}\frac{d}{dr}\right]\phi^0(r) = 0, \tag{23}$$

where the radial wave function $\phi^0(r)$ is labelled by $n_r, j, l = j, s = 0$ and $P = (-1)^j$.

Similarly, the simplest equation following for *one* wave-function component from the system (14) is

$$\left\{\left(\frac{E-V}{2}\right)^2 - \vec{p}^2 - m^2 - \frac{[\vec{p}, V]}{E-V}\cdot\vec{p}\right\}\vec{\chi}^0 + \left(\frac{[p^k, V]}{E-V}\vec{p} + \vec{p}\frac{[p^k, V]}{E-V}\right)\chi^{0k} = 0, \tag{24}$$

where $\vec{\chi}^0 = (\chi^{0k})$. If the potential is central, Eq. (24) leads for the radial projection of $\vec{\chi}^0$, $\chi^0_{el} = \hat{r}\cdot\vec{\chi}^0\left(\text{with } \hat{r} = \frac{\vec{r}}{r}\right)$, to the radial equation

$$\left[\left(\frac{E-V}{2}\right)^2 + \frac{1}{r}\frac{d^2}{dr^2}r - \frac{j(j+1)}{r^2} - m^2 - \frac{d}{dr}\frac{\frac{dV}{dr}}{E-V}\right]\chi^0_{el}(r) = 0, \tag{25}$$

where the radial wave function $\chi^0_{el}(r)$ is labelled by $n_r, j, s = 1$ and $P = (-1)^j$, corresponding to a mixture of $l = j-1$ and $l = j+1$ if $l > 0$ ($j = 1$ if $l = 0$). Since $\vec{\chi}^0 \sim \vec{E}$ is analogous to the electric field (see Eqs. (18)–(20)), $\chi^0_{el}(r)$ is an analogue of the electric 2^j-pole radial field (remember, however, that here $m \neq 0$ and $V \not\equiv 0$). Two other radial equations follow from Eq. (24) for two perpendicular-to-radial projections of $\vec{\chi}^0$, $\chi^0_{lon} = \frac{\partial}{\partial\hat{r}}\cdot\vec{\chi}^0$ and $\chi^0_{mag} = \left(\hat{r}\times\frac{\partial}{\partial\hat{r}}\right)\cdot\vec{\chi}^0 = \hat{r}\cdot\left(\frac{\partial}{\partial\hat{r}}\times\vec{\chi}^0\right)$. Making use of the relations $\frac{\partial}{\partial\vec{r}} = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\hat{r}}$, $\hat{r}\cdot\frac{\partial}{\partial\hat{r}} = 0, \frac{\partial}{\partial\hat{r}}\cdot\hat{r} = 2, \left(\frac{\partial}{\partial\hat{r}}\right)^2 = -\vec{L}^2$ and $i\vec{L} = \vec{r}\times\frac{\partial}{\partial\vec{r}} = \hat{r}\times\frac{\partial}{\partial\hat{r}}$ one obtains for these projections the radial equations

$$\left[\left(\frac{E-V}{2}\right)^2+\frac{1}{r}\frac{d^2}{dr^2}r-\frac{j(j+1)}{r^2}-m^2+\frac{\frac{dV}{dr}}{E-V}\left(\frac{d}{dr}-\frac{3}{r}\right)\right]\chi^0_{lon}(r)$$

$$=2\left[\frac{\frac{dV}{dr}}{E-V}\frac{d}{dr}+\frac{d}{dr}\frac{\frac{dV}{dr}}{E-V}-\frac{\frac{dV}{dr}}{E-V}\frac{j(j+1)+1}{r}\right]\chi^0_{el}(r) \tag{26}$$

and

$$\left[\left(\frac{E-V}{2}\right)^2+\frac{1}{r}\frac{d^2}{dr^2}r-\frac{j(j+1)}{r^2}-m^2+\frac{\frac{dV}{dr}}{E-V}\left(\frac{d}{dr}+\frac{1}{r}\right)\right]\chi^0_{mag}(r)=0. \tag{27}$$

The radial wave function $\chi^0_{lon}(r)$ or $\chi^0_{mag}(r)$ is labelled by n_r, j, $s=1$ and $P=(-1)^j$ or by n_r, j, $s=1$ and $P=(-1)^{j+1}$, corresponding to a mixture of $l=j-1$ and $l=j+1$ or to $l=j$. It is an analogue of the longitudinal electric radial field or magnetic 2^j-pole radial field, respectively (but with $m\neq 0$ and $V\not\equiv 0$). It can be seen that $\chi^0_{el}(r)=\frac{1}{2}\chi^0_j(r)$ and $\chi^0_{mag}(r)=0$ if $l=0$. If $l>0$, all three radial wave functions appear independently, describing relativistically the orbital-angular-momentum triplet with the radial quantum number n_r, total angular momentum j and spin $s=1$. Note that the definitions of "electric" and "magnetic" multipoles interchange when the function $\vec{\phi}^0\sim\vec{B}$ is considered in place $\vec{\chi}^0\sim\vec{E}$ (see Eqs.(18)–(20)).

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