LETTERS TO THE EDITOR

INFINITE SEQUENCES OF CONSERVATION LAWS FOR THE KADOMTSEV-PETVIASHVILI EQUATION

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Three infinite sequences of local conservation laws are found for the Kadomtsev-Petviashvili and 2+1 dimensional Korteweg-de Vries equations by a new method. These laws are then reobtained by an extension of the Noether theorem and appropriate symmetries are found. The new method should simplify general Lie and Lie-Bäcklund calculations by abolishing the need to include explicit x, t dependence in the generating functions.

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Recently Zakharov and Shulman found an infinity of nonlocal conservation laws for the Kadomtsev-Petviashvili equation (KP) in x, y, t space [1]. The n'th conservation law involves n-1 integrations in x, and the whole set generalizes that of Miura et al. for K-de V found twelve years earlier [2]. In a recent paper by one of the present authors eleven local constants of motion were found [3]. Here a refinement of the method proposed in [3] will be used to obtain three infinite sequences of local conservation laws in differential form.

The equation in question is

$$\dot{\phi}_x + \phi_x \phi_{xx} + \phi_{xxxx} + e\phi_{yy} = 0, \tag{1}$$

where $e = -1, 0, 1$ for KP, K-de V, and 2+1 dimensional K-de V respectively. This equation appears in one form or the other in plasma physics, fluid dynamics, and solid state physics [4–8]. It can be derived from the Lagrangian

$$L = 3\dot{\phi}\phi_x + \phi_x^3 + 3\chi^2 + 6\phi_x \chi_x + 3e\phi_y^2, \tag{2}$$

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as the $\delta\phi$ Euler-Lagrange equation, the $\delta\chi$ E-L equation yielding

$$\chi = \phi_{xx}. \tag{3}$$

Symmetries of the Lagrangian yield, via Noether's theorem,

$$t\colon\ (\phi_x^3-3\phi_{xx}^2+3e\phi_y^2)_t+(6\dot{\phi}_x\phi_{xx}-3\dot{\phi}^2-6\phi_{xxx}\dot{\phi}-3\phi_x^2\dot{\phi}_x)_x-(6e\dot{\phi}\phi_y)_y = 0, \tag{4}$$

$$x\colon\ (3\phi_x^2)_t+(2\phi_x^3+6\phi_x\phi_{xxx}-3\phi_{xx}^2-3e\phi_y^2)_x+(6e\phi_x\phi_y)_y = 0, \tag{5}$$

$$y\colon\ (3\phi_x\phi_y)_t+(3\dot{\phi}\phi_y+3\phi_x^2\phi_y+6\phi_{xxx}\phi_y-6\phi_{xy}\phi_{xx})_x+(3e\phi_y^2-3\dot{\phi}\phi_x-\phi_x^3+3\phi_{xx}^2)_y = 0, \tag{6}$$

$$\phi\colon\ (2\phi_x)_t+(\phi_x^2+2\phi_{xxx})_x+(2e\phi_y)_y = 0. \tag{7}$$

Only the first three laws are irreducible (the time derivative term cannot be incorporated in the spatial gradient).

Now introduce the notation

$$\left[{0 \atop t}\right]\colon\ T^0_{t|t}+X^0_{t|x}+Y^0_{t|y} = 0, \tag{8}$$

$$T_t^0 = \phi_x^3-3\phi_{xx}^2+3e\phi_y^2 \text{ etc.} \tag{9}$$

The subscript after a capital letter thus indicates the symmetry and the superscript the highest power of t appearing in a given conservation law explicitly (now a stroke denotes partial differentiation to avoid confusion). Suppose a vector quasipotential $\Pi^1_{t\alpha}$, $\alpha = x, y, t$ can be found such that

$$T_t^0+\Pi^1_{t\alpha|\alpha} = 0 \tag{10}$$

(summation over α understood). Then a new conservation law is obtained from Eqs (4), (9) and (10):

$$(T_t^0 t+\Pi^1_{tt})_t+(X_t^0 t+\Pi^1_{tx})_x+(Y_t^0 t+\Pi^1_{ty})_y = 0$$

and, similarly, the n'th conservation law will generate an $n+1$'st if a $\Pi^{n+1}_{t\alpha}$ exists such that

$$\Pi^n_{tt}+\Pi^{n+1}_{t\alpha|\alpha} = 0. \tag{11}$$

For Eq. (1) $\Pi^n_{\alpha\beta}$ and $\Pi^n_{\varphi\beta}$ exist to all orders, though the ϕ laws so obtained are all reducible to divergences. The general n law derivable from energy conservation and the $\Pi^n_{t\alpha}$, for example, will be

$$\left[{n \atop t}\right]\colon \left(T_t^0\,\frac{t^n}{n!}+\sum_{k=1}^{n}\Pi^k_{tt}\,\frac{t^{n-k}}{(n-k)!}\right)_t+\left(X_t^0\,\frac{t^n}{n!}+\sum_{k=1}^{n}\Pi^k_{tx}\,\frac{t^{n-k}}{(n-k)!}\right)_x$$

$$+\left(Y_t^0\,\frac{t^n}{n!}+\sum_{k=1}^{n}\Pi^k_{ty}\,\frac{t^{n-1}}{(n-1)!}\right) = 0, \tag{12}$$

and similarly for the x and y laws. The quasipotentials are:

$$t:\ \Pi^1_{tt} = -x\phi_x^2 - 2y\phi_x\phi_y,$$

$$\Pi^1_{tx} = x(\phi_{xx}^2 + e\phi_y^2 - 2\phi_x\phi_{xx} - \phi_{xx}^3) + y(4\phi_{xy}\phi_{xx}$$

$$-4\phi_{xxx}\phi_y - 2\phi_x^2\phi_y - 2\dot\phi\phi_y) - 2\phi\dot\phi - 2\phi\phi_{xxx} - \phi\phi_x^2 + 4\phi_x\phi_{xx},$$

$$\Pi^1_{ty} = -2ex\phi_x\phi_y - 2y(e\phi_y^2 - \dot\phi\phi_x - \tfrac{1}{3}\phi_x^3 + \phi_{xx}^2) - 2e\phi\phi_y;$$

$$\Pi^2_{tt} = (x^2 + \tfrac{1}{2}ey^2\phi_x)\phi_x,$$

$$\Pi^2_{tx} = x^2(\tfrac{1}{2}\phi_x^2 + \phi_{xxx}) + y^2(\tfrac{1}{3}e\phi_x^3 + e\phi_x\phi_{xxx} - \tfrac{1}{2}e\phi_{xx}^2 - \tfrac{1}{2}\phi_y^2) - 2x\phi_{xx} + 2\phi_x,$$

$$\Pi^2_{ty} = ex^2\phi_y + y^2\phi_x\phi_y;$$

$$\Pi^3_{tt} = -exy^2\phi_x, \quad \Pi^3_{tx} = -\tfrac{1}{2}xy^2\phi_x^2 - exy^2\phi_{xxx} - x^2\phi + ey^2\phi_{xx},$$

$$\Pi^3_{ty} = -xy^2\phi_y + 2xy\phi;$$

$$\Pi^4_{tt} = 0, \quad \Pi^4_{tx} = \tfrac{1}{12}y^4(\dot\phi + \tfrac{1}{2}\phi_x^2 + \phi_{xxx}) + exy^2\phi,$$

$$\Pi^4_{ty} = \tfrac{1}{12}ey^4\phi_y - \tfrac{1}{3}ey^3\phi;$$

$$\Pi^k_{t\alpha} = 0, \quad k \geqslant 5, \quad \alpha = x,\ y,\ t. \tag{13}$$

$$x:\ \Pi^1_{xt} = -2_x\phi_x, \quad \Pi^1_{xx} = -x(\phi_x^2 + 2\phi_{xxx}) + 2\phi_{xx}, \quad \Pi^1_{xy} = -2ex\phi_y; \tag{14}$$

$$\Pi^2_{xt} = 0, \quad \Pi^2_{xx} = 2ex\phi + ey^2(\dot\phi + \tfrac{1}{2}\phi_x^2 + \phi_{xxx}), \quad \Pi^2_{xy} = y^2\phi_y - 2y\phi;$$

$$\Pi^k_{x\alpha} = 0, \quad k \geqslant 3,$$

$$y:\ \Pi^1_{yt} = -\tfrac{1}{2}ey\phi_x^2, \quad \Pi^1_{yx} = y(\tfrac{1}{2}\phi_y^2 - e\phi_x\phi_{xxx} + \tfrac{1}{2}e\phi_{xx}^2 - \tfrac{1}{3}e\phi_x^3),$$

$$\Pi^1_{yy} = -y\phi_x\phi_y;$$

$$\Pi^2_{yt} = exy\phi_x, \quad \Pi^2_{yx} = exy(\ \phi_x^2 + \phi_{xxx}) - ey\phi_{xx}, \quad \Pi^2_{yy} = xy\phi_y - x\phi;$$

$$\Pi^3_{yt} = 0, \quad \Pi^3_{yx} = -\tfrac{1}{12}y^3(2\dot\phi + \phi_x^2 + 2\phi_{xxx}) - exy\phi,$$

$$\Pi^3_{yy} = -\tfrac{1}{6}ey^3\phi_y + \tfrac{1}{2}ey^2\phi_y;$$

$$\Pi^k_{y\alpha} = 0, \quad k \geqslant 4. \tag{15}$$

These values give three infinities of conservation laws when used in Eq. (12) and its x, y counterparts. Note that a new, nontrivial n'th law is obtained even when all $\Pi^n_{\alpha\beta}$ are zero (*nonexistence* would stop the procedure). It is straight forward to show completeness in the sense that any polynomial conservation law (polynomial in x, y, t) can be generated by our method if the appropriate zero order law is known.

Although our method is extremely simple and requires no knowledge of group theory, it is natural to ask what symmetries the conservation laws obtained correspond to. Equation (1) expressed in terms of ϕ and χ is found to be invariant under the transformation

$$
\begin{aligned}
& x \to x+\varepsilon\xi, \quad y \to y+\varepsilon\sigma, \quad t \to t+\varepsilon\tau, \quad \phi \to \phi+\varepsilon\eta, \quad \chi \to \chi+\varepsilon\kappa, \\
& \xi = \dot{f}x-\tfrac{1}{2}ey^2\ddot{f}-\tfrac{1}{2}ey\dot{g}+h, \\
& \sigma = 2y\dot{f}+g, \quad \tau = 3f, \\
& \eta = -\dot{f}\phi-[\tfrac{1}{2}ey^2\dddot{f}+\tfrac{1}{2}ey\ddot{g}-\dot{h}]x+\tfrac{1}{2}\ddot{f}x^2t+\tfrac{1}{24}y^4\ddddot{f} \\
& \quad +\tfrac{1}{12}y^3\dddot{g}-\tfrac{1}{2}ey^2\ddot{h}+ym+n, \\
& \kappa = -3\dot{f}\chi+\ddot{f},
\end{aligned}
\tag{16}
$$

and f, g, h, m, n are general functions of time. Under the transformations the action element is transformed into ($d\Omega = dxdydt$):

$$
Ld\Omega \to Ld\Omega+W_{\alpha|\alpha}d\Omega, \tag{17}
$$

and so the action *integral* is still invariant. The transformations of Eq. (16) thus generate a generalized Lie group [9]. The irreducible conservation laws generated by f, g, h can be recreated by combining those obtained from Eq. (12) and vice versa. Thus the new method yields the generalized Lie group for Eq. (1) (this will of course contain the proper Lie group when $W_{\alpha|\alpha} = 0$: $[{0 \atop x}]$, $[{0 \atop y}]$, $[{1 \atop y}]$, $[{0 \atop t}]$, $[{1 \atop t}]$). For Maxwell's equations, however, it yields just the 15 invariants of the proper Lie group. The same is true of the cubic nonlinear Schrödinger equation in x, y, t and of the equations of ideal gas dynamics in D dimensions (9 and $D+1$ respectively).

In the Lie-Bäcklund method more general symmetries, in which the generating functions depend on derivatives of ϕ, are investigated [10]. The method is in general quite cumbersome and any simplification should be welcome. Now it will be possible to initially drop the explicit dependence on x, t in the generating functions, recreating conservation laws in which the independent variables appear explicitly by our method. This could perhaps encourage a wider application of Lie-Bäcklund.

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