SOLITONS*

By Th. W. Ruijgrok

Institute of Theoretical Physics, State University of Utrecht**

(Received June 15, 1982)

Solitons are solutions of nonlinear wave equations. A single soliton is just like a normal dispersionless wave in that it does not change its shape in the course of time. The soliton is known for its remarkable stability. When a soliton encounters another soliton of arbitrary size or velocity it can change beyond recognition for a short or long period, but ultimately it will revert to its original shape. This type of wave is observed in many fields of physics. In this paper I shall concentrate on giving a short explanation of the various ways in which mathematical physicists have tried to understand the amazing soliton. The reader is not expected to have studied advanced mathematics but he or she must be prepared to work through some of the algebra, which is sometimes rather tedious.

PACS numbers: 03.50.-z

1. The equation of Korteweg and de Vries (1895)

A fisherman who is paying more attention to the water than to his float will soon realise that a wave $u = f(x-ct)$ moving past him from left to right with velocity c is a solution of the partial differential equation

$$u_t + cu_x = 0. \tag{1}$$

A closer examination reveals however that higher waves move more quickly and it is obvious that this dispersion should be taken into account by replacing (1) by

$$u_t + uu_x = 0. \tag{2}$$

We guess that the solution of this equation, with the initial condition $u(x, 0) = f(x)$ at $t = 0$, is given by the solution of the implicit equation

$$u = f(x-ut). \tag{3}$$

* Presented at the XXII Cracow School of Theoretical Physics, Zakopane, May 30 — June 9, 1982.

** Address: Institute of Theoretical Physics, Rijksuniversiteit, Princetonplein, 5, P. O. Box 80006, 3508 TA Utrecht, The Netherlands.

From (3) it follows that $u_t+uu_x = -t(u_t+uu_x)f'$, where f' denotes the derivative of f with respect to its argument. Or equally

$$(1+tf')(u_t+uu_x) = 0. \tag{4}$$

For all x and t for which $1+tf' \neq 0$, it turns out therefore that we have in fact found the solution of equation (2). Let us take the special case where $f(x) = \cos \pi x$; then the equation for $u(x, t)$ becomes

$$u = \cos \pi(x-ut). \tag{5}$$

The condition $1+tf' \neq 0$ becomes

$$1-\pi t \sin \pi(x-ut) \neq 0. \tag{6}$$

It is obvious that this condition is satisfied for all $t < \frac{1}{\pi} \equiv t_B$. Fig. 1 shows the solution of Eq. (5) for a number of values of t. In the course of time the wave becomes steeper and at $t = t_B$ it is about to break. This occurs when $\sin \pi(x-ut_B) = 1$, therefore when

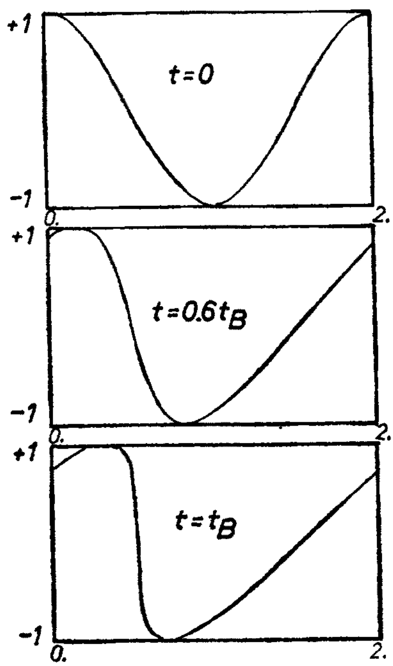


Fig. 1

$\cos \pi(x-ut_B) = 0$, thus when $u = 0$ (Eq. (5)), and so when $x = \frac{1}{2}$. For larger values of t there are more solutions, as can be seen in Fig. 2. This multivaluedness is a consequence of the hypothesis that even the strongly curved part of the wave can still be described adequately by Eq. (2). The extra effects of strong curvature can be taken into account by adding a term that includes a higher derivative. This then leads to the KdV-equation

$$u_t+uu_x+\delta^2 u_{xxx} = 0, \tag{7}$$

where $\delta \ll 1$, so that the added term only has an effect in places where the wave surface is strongly curved. This equation does not only have periodic solutions, which are given by the elliptical function $cn(z)$ of Jacobi, but it also allows functions of the form $u = f(x-ct)$. The propagation velocity c can be chosen arbitrarily. The function $f(x)$

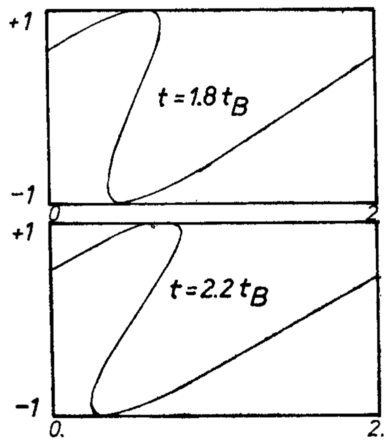


Fig. 2

however must satisfy Eq. (7). If we also want $f(x)$ to become 0 for $|x| \to \infty$, then the only possibility left is

$$u(x, t) = 3c \cdot \cosh^{-2}\left[\frac{\sqrt{c}}{2\delta}(x-ct)\right]. \tag{8}$$

A wave of this type was first observed and described in 1834 by Scott Russell when he saw a boat stop suddenly as he was riding his horse along the tow-path of a canal. Instead of delighting in the attempts of the crew to get the boat moving — it had probably gone aground — he followed the solitary wave that had come from the boat's prow, until after a chase of one or two miles he lost it in the windings of the canal.

Zabusky and Kruskal gave the name "solitons" to solutions of the type mentioned above. They discovered a number of properties of these solitons by studying the numerical solution of Eq. (7) with $\delta = 0.022$ and with the initial condition $u(x, 0) = \cos \pi x$. Their work can be reproduced quickly with modern computers (see Fig. 3), but does give rise to some problems for large times [25]. The conclusions to be drawn from these calculations can be summarised as follows:

a) Up to $t = t_B$ the wave is described reasonably well by the solution of Eq. (2).

b) For larger times the wave is broken up into a number of solitons which all seem to originate from the breaking point $x = \frac{1}{2}$.

c) These solitons move with different velocities and their width is inversely proportional to the root of the velocity (Eq. (8)).

d) Two solitons moving with different velocities do not become deformed when they pass each other, but there is no superposition of solutions. Solitons are therefore very stable.

e) After a recurrence time $T_R = 30.4t_B$ all the solitons overlap and a large part of the original wave $u = \cos \pi x$ is reconstructed.

The last conclusion also explains why Fermi, Pasta and Ulam in their study of a one--dimensional anharmonic crystal did not find equipartition of energy over all possible harmonic vibrations.

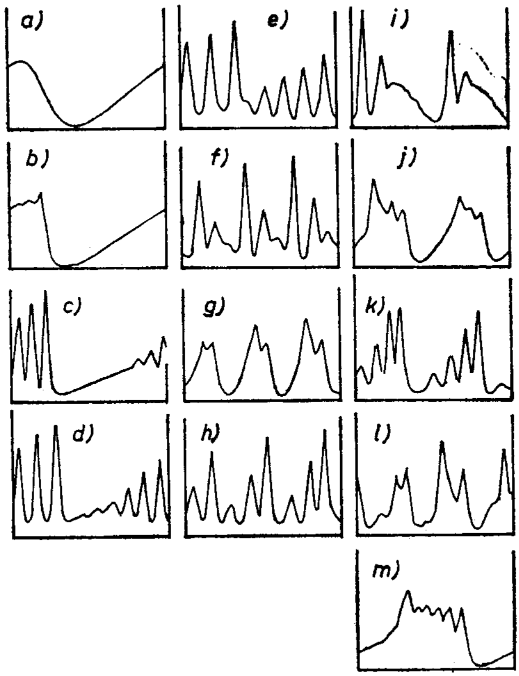


Fig. 3. Numerical solutions of the KdV equation for various values of time: a) $t = 0.5\,t_B$; b) $t = t_B$; c) $t = 2t_B$; d) $t = 3t_B$; *e)* $t = 4t_B$; *f)* $t = 9t_B$; g) $t = 10t_B$; h) $t = 11t_B$; i) $t = 14t_B$; j) $t = 15t_B$; k) $t = 16t_B$; l) $t = 20t_B$; m) $t = 30.4t_B$

The soliton phenomenon is not restricted to the KdV-equation but also occurs in a number of exactly integrable non-linear differential equations which have been constructed in the last few years. In view of the fact that several thousands of articles on solitons have been published since 1965 I shall restrict myself to the discussion of a few examples and some of the methods which have been introduced.

2. *The method of inverse scattering*

Consider the operators

$$L = p^2 - \tfrac{1}{6}u(x,t) \quad \text{and} \quad B = -4p^3 + \tfrac{1}{2}(up+pu). \tag{9}$$

In the function space where L and B operate, u is a multiplication operator and $p = -i\dfrac{\partial}{\partial x}$, so that $[p, u] = -iu_x$. The quantity t is a parameter which is not varied for the time being. $L_t = \partial L/\partial t = -\frac{1}{6} u_t$ is also a multiplication operator. If we now calculate the commutator $[B, L]$ from this so-called Lax-pair, then we find that

$$[B, L] = \frac{i}{6}(uu_x + u_{xxx}).$$

We see therefore that if $u(x, t)$ is a solution of the KdV-equation (7), with $\delta = 1$, then the following relation holds for L and B

$$iL_t = [B, L]. \tag{10}$$

This relation allows us to calculate what an arbitrary initial function $u(x, t = 0)$ will look like at a later time t, regardless of the magitude of t. This could never be achieved by a direct numerical integration of equation (7) because errors due to rounding up or down and frequent instabilities would quickly render the result unreliable. It will become obvious that only two *linear* equations have to be solved: a linear eigenvalue equation for the so--called "direct" problem at $t = 0$ and a linear integral equation for the "inverse" problem at time t. Because the reasoning is rather complicated, it is divided into three parts.
I. The direct problem. Consider at $t = 0$ the eigenvalue equation

$$L\bar{\psi} = E\bar{\psi}. \tag{11}$$

After substituting expression (9) for L, Eq. (11) acquires the form of a Schrödinger equation:

$$-\frac{d^2\bar{\psi}}{dx^2} - \tfrac{1}{6} u(x, 0)\bar{\psi} = E\bar{\psi}, \tag{12}$$

where the arbitrarily prescribed function $-\frac{1}{6} u(x, 0)$ plays the role of the potential. This is the first linear problem. Generally the spectrum will consist of discrete eigenvalues $E_n = -\kappa_n^2 (n = 1, \ldots, N)$ for the bound states $\bar{\psi}_n(x)$ and a continuous part $E = k^2$ for the scattering states $\bar{\psi}_k(x)$. The bar over $\bar{\psi}(x)$ indicates that the asymptotic behaviour of the wave functions is chosen in a special way, namely:

$$\bar{\psi}_n(x) \simeq \begin{cases} e^{-\kappa_n x} & \text{for} \quad x \to +\infty \\ C_n(0)e^{+\kappa_n x} & \text{for} \quad x \to -\infty \end{cases} \tag{13}$$

and

$$\bar{\psi}_k(x) \simeq e^{-ikx} + R(k, 0)e^{ikx} \quad \text{for} \quad x \to +\infty. \tag{14}$$

$|R(k, 0)|^2$ is the reflection coefficient. It is only the quantities κ_n, $C_n(0)$ and $R(k, 0)$ which will play a role in the story from now on.
II. Development in time. We again consider Eq. (11) and therefore (12), but this time $u(x, 0)$ is replaced by $u(x, t)$. The only thing we know about this function is that it satisfies the KdV-equation. However, it is possible to indicate how the new $\bar{\psi}(x, t)$ for large x is linked to $\bar{\psi}(x, 0)$. In this way the problem is reduced to the problem of how, given

the asymptotic behaviour of $\bar{\psi}(x, t)$, the function $u(x, t)$ can be solved from the equation

$$-\frac{d^2\bar{\psi}(x,t)}{dx^2} - \tfrac{1}{6}u(x,t)\bar{\psi}(x,t) = E\bar{\psi}(x,t). \tag{15}$$

Again we write the eigenvalue problem as

$$L\psi(x,t) = E\psi(x,t), \tag{16}$$

but for the time being do not specify the asymptotic behaviour of this $\psi(x, t)$ without a bar.

The t-dependence of $\psi(x, t)$ is chosen in such a way that $\psi(x, t)$ satisfies the equation

$$i\frac{\partial\psi}{\partial t} = B\psi(x,t). \tag{17}$$

Because equation (16) is also valid for all t, we shall have to prove that the last two equations do not contradict each other. By differentiating (16) with respect to t we see that this is indeed the case, provided E is independent of t:

$$iL_t\psi + iL\frac{\partial\psi}{\partial t} = iE\frac{\partial\psi}{\partial t}. \tag{18}$$

By using the essential relation (10) and by substituting (16) and (17) we prove that (18) is indeed an identity. The eigenvalues $-\kappa_n^2$ are therefore independent of t. The functions $\bar{\psi}(x, t)$ are again defined as solutions of (16) with the special asymptotic behaviour

$$\bar{\psi}_n(x,t) \simeq \begin{cases} e^{-\kappa_n x} & \text{for} \quad x \to +\infty \\ C_n(t)e^{+\kappa_n x} & \text{for} \quad x \to -\infty \end{cases} \tag{19}$$

and

$$\bar{\psi}_k(x,t) \simeq e^{-ikx} + R(k,t)^{ikx} \quad \text{for} \quad x \to +\infty. \tag{20}$$

It is easy to prove the following surprisingly simple but important formulas for the time dependence of C_n and R

$$C_n(t) = C_n(0)e^{-8\kappa_n{}^3 t} \quad \text{and} \quad R(k,t) = R(k,0)e^{8ik^3 t}. \tag{21}$$

The only hypothesis concerning $u(x, t)$ here is that $u(x, t) \to 0$ for $|x| \to \infty$. From Eq. (19) it then follows that

$$B \simeq -4p^3 = -4i\frac{\partial^3}{\partial x^3} \quad \text{for} \quad |x| \to \infty. \tag{22}$$

If the asymptotic behaviour of the solutions of Eq. (17), which correspond to bound states is written as

$$\psi_n(x,t) \simeq \begin{cases} A_n(t)e^{-\kappa_n x} & \text{for} \quad x \to +\infty \\ D_n(t)e^{+\kappa_n x} & \text{for} \quad x \to -\infty, \end{cases} \tag{23}$$

then it follows from (22) that the coefficients $A_n(t)$ and $D_n(t)$ must satisfy the equations

$$\frac{dA_n}{dt} = 4\varkappa_n^3 A_n(t) \quad \text{and} \quad \frac{dD_n}{dt} = -4\varkappa_n^3 D_n(t), \tag{24}$$

which lead to the solutions

$$A_n(t) = A_n(0)e^{4\kappa_n{}^3 t} \quad \text{and} \quad D_n(t) = D_n(0)e^{-4\kappa_n{}^3 t}. \tag{25}$$

Now the functions $\psi_n(x, t)$ and $\bar{\psi}_n(x, t)$ for a given t are both solutions of Eq. (16). Their ratio is therefore independent of x. By comparing (19) and (23) one comes to the conclusion that $C_n(t) = D_n(t)/A_n(t)$, from which the first relation of (21) immediately follows. The second relation can be derived in a similar way.

It should be noted that Eq. (17) has the form of a time-dependent Schrödinger equation, in which B, which is hermitian here too, plays the role of the Hamiltonian. It follows that the norm of the wave functions $\psi_n(x, t)$ is independent of t. But this is no longer true of the functions $\bar{\psi}_n(x, t)$. The reader however can easily prove that the time dependence of the quantity $\gamma_n(t)$, which is linked to the norm of $\bar{\psi}_n(x, t)$ in the following way

$$\gamma_n^{-1}(t) = \int_{-\infty}^{+\infty} |\bar{\psi}_n(x, t)|^2 dx, \tag{26}$$

is given by

$$\gamma_n(t) = \gamma_n(0)e^{8\kappa_n{}^3 t} \tag{27}$$

III. The inverse problem. The question is now whether knowledge of the asymptotic behaviour of $\bar{\psi}(x, t)$ given by the quantities κ_n, $\gamma_n(t)$ and $R(k, t)$, enables one to derive the potential $V = -\frac{1}{6}u(x, t)$ from equation (15), which can now be written as

$$-\frac{d^2\bar{\psi}}{dx^2} + V(x)\bar{\psi}(x) = E\bar{\psi}(x) \tag{28}$$

from which, for simplicity's sake, the t-dependence has been omitted. This inverse problem has been solved by Gel'fand, Levitan and Marchenko, who have shown that

$$V(x) = -2\frac{d}{dx}K(x, x), \tag{29}$$

where $K(x, y)$ is the solution of the *linear* integral equation

$$K(x, y) + B(x+y) + \int_x^{\infty} K(x, z)B(z+y)dz = 0 \quad (y > x). \tag{30}$$

In this equation $B(x+y)$ is given by

$$B(x+y) = \sum_{n=1}^{N} \gamma_n(t)e^{-\kappa_n(x+y)} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(k, t)e^{ik(x+y)}dk, \tag{31}$$

where $\gamma_n(t)$ and $R(k, t)$ are as in Eq. (27) and Eq. (21). The proof of this Marchenko equation is too complicated to reproduce here.

A simple case occurs when the initial potential happens to be chosen in such a way that no reflection occurs for any value of k and there is only one bound state. Then $B(x+y) = g(x)g(y)$ with $g(x) = \sqrt{\gamma(t)}e^{-\kappa x}$. The integral equation (30) then becomes

$$K(x, y)+g(x)g(y)+g(y)\int_x^\infty K(x, z)g(z)dz = 0 \quad (y > x). \tag{32}$$

The solution is obviously of the form $K(x, y) = w(x)g(y)$. Substitution in (32) gives for $w(x)$

$$w(x) = \frac{-g(x)}{1+\int_x^\infty g^2(z)dz}, \tag{33}$$

where the integral is elementary. The function $K(x, x)$ now becomes

$$K(x, x) = \frac{-\kappa e^{-z}}{\cosh z},$$

with $z = \kappa(x-x_0)-4\kappa^3 t$ and $\gamma(0) = 2\kappa e^{2\kappa x_0}$. Via (29) the solution of the original problem is then finally given by

$$u(x, t) = \frac{12\kappa^2}{\cosh^2 z}. \tag{34}$$

With the identification $c = 4\kappa^2$ this solution becomes identical to the one-soliton solution (8).

For N bound states, but still for a reflectionless potential, the integral equation (32) reduces to an algebraic problem which can be solved exactly. For large times, both positive and negative, this solution is the sum of N solitons of the type (34) which move from left to right with different velocities. During the time when two or more of these solitons overlap, the solution is certainly not a superposition of individual solitons. After the complicated overtaking manoeuvre they reappear intact and have merely pushed each other forwards or backwards.

For an arbitrary initial function $u(x, 0)$ the reflection term in (31) will also make a contribution. However, it has been proved that the effect of this on $u(x, t)$ for $t \to \infty$ approaches zero, so only those solitons can be seen in which the original wave has broken up. For a high wave this number is given by

$$N \simeq \frac{1}{\pi}\int_{-\infty}^{+\infty} \sqrt{|u(x, 0)|}\, dx.$$

3. *The Bäcklund transformation*

Consider a number of identical pendulums which are hanging from a horizontal torsion wire and which due to the pull of gravity can move in vertical planes that are perpendicular to the wire. The potential energy of this system is

$$V = -\sum_{k=1}^{N} mgl\cos\phi_k+\tfrac{1}{2}m\omega^2 l^2\sum_{k=1}^{N}(\phi_{k+1}-\phi_k)^2, \tag{35}$$

where the angular deviation of the k-th pendulum from the rest position is given by ϕ_k. With the kinetic energy $T = \sum_{k=1}^{N} \frac{1}{2} ml^2\dot{\phi}_k^2$ the Lagrange equations can be derived quite simply, resulting in

$$\ddot{\phi}_k = -\frac{g}{l}\sin\phi_k + \omega^2(\phi_{k+1} - 2\phi_k + \phi_{k-1}). \tag{36}$$

If the distance Δ between the pendulums is small the equations valid in the continuum limit can be obtained by making the following replacements

$$\phi_k \to \phi(x, t) \quad \text{and} \quad \phi_{k\pm 1} \to \phi(x, t) \pm \Delta \frac{\partial\phi}{\partial x} + \frac{1}{2}\Delta^2 \frac{\partial^2\phi}{\partial x^2}.$$

If $\sqrt{l/g}$ and $\Delta\omega\sqrt{l/g}$ are now introduced as new units for time and length, then finally one obtains the following partial differential equation:

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = \sin\phi(x, t). \tag{37}$$

This is the so-called sine-Gordon equation which has found innumerable applications ranging from Bloch walls to liquid crystals but which will be used here only for the purpose of illustrating a new integration method. First of all Eq. (37) is transformed into

$$\frac{\partial^2\phi}{\partial\xi\partial\eta} = \sin\phi(\xi, \eta) \tag{38}$$

by switching over to light cone coordinates $\xi = \frac{x-t}{2}$ and $\eta = \frac{x+t}{2}$. We now define a function $\bar{\Phi}(\xi, \eta)$ by the following so-called Bäcklund transformation [26]

$$\frac{\partial}{\partial\xi}\left(\frac{\bar{\Phi}+\phi}{2}\right) = \frac{1}{\lambda}\sin\left(\frac{\bar{\Phi}-\phi}{2}\right) \tag{39}$$

and

$$\frac{\partial}{\partial\eta}\left(\frac{\bar{\Phi}-\phi}{2}\right) = \lambda\sin\left(\frac{\bar{\Phi}+\phi}{2}\right), \tag{40}$$

where $\phi(\xi, \eta)$ is a given solution of (38) and λ an arbitrary real parameter. These two equations are not contradictory because the η-derivative of (39) minus the ξ-derivative of (40) again gives Eq. (38). The sum of these two derivatives gives

$$\frac{\partial^2\bar{\Phi}}{\partial\xi\partial\eta} = \sin\bar{\Phi}, \tag{41}$$

so that by means of the Bäcklund transformation a new solution $\bar{\phi}$ is obtained from a given solution ϕ. In particular $\phi \equiv 0$ leads to the equations

$$\frac{\partial}{\partial \xi}\left(\frac{\phi}{2}\right) = \frac{1}{\lambda} \sin\left(\frac{\phi}{2}\right) \quad \text{and} \quad \frac{\partial}{\partial \eta}\left(\frac{\phi}{2}\right) = \lambda \sin\left(\frac{\phi}{2}\right),$$

which with the help of the integral

$$\int \frac{du}{\sin u} = \log \operatorname{tg} \frac{u}{2}$$

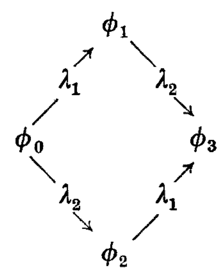
can be solved directly:

$$\operatorname{tg} \frac{\phi}{4} = \operatorname{tg} \frac{\phi(\xi_0, \eta_0)}{4} \exp\left[\frac{1}{\lambda}(\xi - \xi_0) + \lambda(\eta - \eta_0)\right].$$

By using the parameter $c = (1-\lambda^2)/(1+\lambda^2)$ (which lies between -1 and $+1$) instead of parameter λ one can write this solution as

$$\operatorname{tg} \frac{\phi(x, t)}{4} = \operatorname{tg} \frac{\phi(x_0, t_0)}{4} \cdot \exp\left[\pm \frac{(x-x_0)-c(t-t_0)}{\sqrt{1-c^2}}\right]. \tag{42}$$

These one-soliton solutions have the form of a kink or anti-kink each of which can move to the left or the right with velocity $|c|$. The closer $|c|$ is to the speed of light $|c| = 1$, the steeper the kink. Equation (42) shows that the angle ϕ between $x = -\infty$ and $x = +\infty$ increases or decreases by an amount 2π. This means that somewhere in between, the pendulums make a full swing round the torsion wire. By applying the Bäcklund transformation for a second time one obtains a two-soliton solution. This can be done in two ways, as is shown in the following diagram



An explicit formula for the two-soliton solution ϕ_3 can now be proved; it looks like this

$$\operatorname{tg}\left(\frac{\phi_3 - \phi_0}{4}\right) = \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \operatorname{tg}\left(\frac{\phi_1 - \phi_2}{4}\right).$$

The innumerable solutions of the sine-Gordon equation are not exhausted with these one-(and multi)-soliton states. There are also solutions where a kink and an anti-kink remain close to each other during their propagation as a bound state called a "breather". However, this phenomenon and the mutual annihilation of a kink or an anti-kink cannot be discussed here.

4. *Prolongation*

It must have become clear in Section 2 that the KdV-equation could be solved exactly as a result of the chance circumstance that there was a Lax-pair which satisfied equation (10). Hitting upon a Bäcklund transformation is also mainly a matter of ingenuity or luck and one therefore wonders whether there is not a systematic way of deciding whether a given partial differential equation can or cannot be integrated exactly. The so-called prolongation method is an attempt in this direction. This method will now be explained with the help of a simple example.

Consider the equation which is used to describe tidal waves and traffic waves

$$u_t + uu_x = Du_{xx}, \tag{43}$$

which is known as Burgers' equation and was studied by Bateman as early as 1915. In the following, use will be made of the Lie-bracket

$$[A, B] = \frac{dA}{dq} B - A \frac{dB}{dq} = A_q B - AB_q \tag{43a}$$

for two functions $A(q)$ and $B(q)$. This brackets-symbol does not only satisfy the usual relations

$$[A, A] = 0 \quad \text{and} \quad [A+B, C] = [A, C] + [B, C]$$

but it also satisfies the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{44}$$

In analogy with the inverse scattering method, where a function $\phi(x, t)$ was introduced which had to satisfy equations (16) nad (17), we now introduce a so-called pseudo-potential $q(x, t)$, which has to satisfy

$$q_x = A(q, u) \quad \text{and} \quad q_t = B(q, u, u_x). \tag{45}$$

The functions A and B must be chosen in such a way that the condition for integrability

$$\frac{\partial}{\partial t} q_x = \frac{\partial}{\partial x} q_t \tag{46}$$

is fulfilled for every $u(x, t)$ which satisfies (43). Because A and B are not defined unambiguously as a result of this, there is still in principle a possibility of imposing extra restrictions on these functions so that (45) is simplified to two linear or at least to two easily soluble equations. The method of finding $u(x, t)$ thereafter is like the inverse scattering method.

By writing out Eq. (46) and making use of Eq. (43) to eliminate u_t one obtains

$$(DA_u - B_{u_x})u_{xx} = (uA_u + B_u)u_x - [A, B], \tag{47}$$

where differentiation with respect to a variable is given by a subscript. This relation must hold for every function $u(x)$ (t is kept constant). However, since u_{xx} does not occur in the right-hand side it cannot appear in the left-hand side either, whence it follows that

$$B_{u_x} = DA_u. \tag{48}$$

Now A_u does not depend on u_x, so (48) can be integrated to

$$B = Du_x A_u + C(q, u) \tag{49}$$

and therefore it also holds that

$$B_u = Du_x A_{uu} + C_u(q, u). \tag{50}$$

Substitution of these two expressions into (47) gives

$$DA_{uu}u_x^2 + (uA_u + C_u - D[A, A_u])u_x - [A, C] = 0. \tag{51}$$

In this formula the quantities u_x and u_x^2 occur only in the places where they are written explicitly. The three coefficients on the left-hand side of (51) must therefore equal zero:

$$A_{uu} = 0, \tag{52}$$

$$uA_u + C_u - D[A, A_u] = 0, \tag{53}$$

$$[A, C] = 0. \tag{54}$$

From (52) it follows that

$$A = x_1 + ux_2, \tag{55}$$

where x_1 and x_2 (and later x_3 and x_4 as well) still depend only on q. Differentiation of A with respect to u gives

$$A_u = x_2 \quad \text{and therefore} \quad [A, A_u] = [x_1, x_2] \equiv x_3. \tag{56}$$

Substitution into (53) results in $C_u = ux_2 + Dx_3$ and therefore

$$C = -\tfrac{1}{2}u^2 x_2 + Dux_3 + x_4.$$

From the last condition (54) one now finds simply that

$$-\tfrac{1}{2}[x_1, x_2] + D[x_2, x_3] = 0; \quad D[x_1, x_3] + [x_2, x_4] = 0; \quad [x_1, x_4] = 0.$$

Summarising, the condition for integrability (46) therefore leads to the requirement that the functions $x_i(q)$, $i = 1,4$ must satisfy

$$[x_1, x_2] = x_3; \quad [x_1, x_3] = a(q); \quad [x_1, x_4] = 0;$$

$$[x_2, x_3] = \frac{1}{2D}x_3; [x_2, x_4] = -Da(q); [x_3, x_4] = b(q), \tag{57}$$

where $a(q)$ and $b(q)$ must be chosen in such a way that the Jacobi identity (44) is satisfied. If not, then $a(q)$ and $b(q)$ are regarded as new functions x_5 and x_6, the Lie-brackets of which must be well chosen with the other x_i. This prolongation is continued until a closed algebra is obtained for all x_i. From the exactly integrable equations known to date it would appear that this is not really necessary. In particular (57) can be satisfied by choosing

$$x_1 = x_3 = x_4 = a = b = 0$$

where only x_2 is a function of q requiring further specification. Equations (45) now acquire the following form

$$q_x = ux_2(q) \quad \text{and} \quad q_t = (Du_x - \tfrac{1}{2}u^2)x_2(q). \tag{58}$$

One finds in a simple way that Eq. (46) now becomes

$$\frac{\partial}{\partial t}q_x - \frac{\partial}{\partial x}q_t = (u_t + uu_x - Du_{xx})x_2(q) = 0,$$

which makes it clear that Burgers' equation can indeed be regarded as the integrability condition of equations (58). If x_2 is chosen as

$$x_2(q) = -\frac{q}{2D},$$

then equations (58) are equivalent to

$$q_t = Dq_{xx}, \tag{59}$$

and

$$u = -2D\frac{q_x}{q}. \tag{60}$$

Equation (59) is the diffusion equation, the general solution of which can be found. Ultimately equation (60) gives the desired solution of Burgers' equation.

This solution was found thirty years ago by Cole and Hopf, who produced the transformation (60) out of the blue. It was not until 1975 however that Estabrook and Wahlquist discovered the prolongation method.

5. The direct method

The direct method of Hirota is another systematic method of constructing solutions. In this method too use is made of the Lie-bracket (43a), which in this case however serves to define an antisymmetrical operator acting on an ordered product $a(x)b(x)$. The definition is as follows

$$D_x a \cdot b = \lim_{x' \to x}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)a(x)b(x') = a_x b - ab_x = [a, b]. \tag{61}$$

Powers of this operator are defined by

$$D_x^n a \cdot b = \lim_{x' \to x} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n a(x)b(x'). \tag{62}$$

In particular

$$D_x^2 a \cdot b = a_{xx}b - 2a_x b_x + ab_{xx} \quad \text{and} \quad D_x^2 a \cdot a = 2(a_{xx}a - a_x a_x). \tag{63}$$

In order to explain this method more clearly we shall now deal with the non-linear Schrödinger equation:

$$i\psi_t = -\psi_{xx} - |\psi|^2 \psi. \tag{64}$$

In this equation the function $-|\psi|^2$ plays the role of an attractive potential. In other words the wave function moves in the trough which it digs for itself. The idea of the direct method is now to replace $\psi(x, t)$ by

$$\psi = \frac{g(x, t)}{f(x, t)} \quad \text{with } f \text{ real} \tag{65}$$

and then to derive for g and f equations that are simple to solve. In so doing use is made of the easily demonstrable formulas

$$\frac{\partial}{\partial x}\left(\frac{g}{f}\right) = \frac{D_x g \cdot f}{f^2} \tag{66}$$

and

$$\left(\frac{\partial}{\partial x}\right)^2 \left(\frac{g}{f}\right) = \frac{D_x^2 g \cdot f}{f^2} - \frac{g}{f}\,\frac{D_x^2 f \cdot f}{f^2}. \tag{67}$$

By substituting (65) into (64) we now get

$$i\frac{D_t g \cdot f}{f^2} + \frac{D_x^2 g \cdot f}{f^2} - \frac{g}{f}\,\frac{D_x^2 f \cdot f}{f^2} + \frac{g|g|^2}{f^3} = 0. \tag{68}$$

The essential element in this bit of juggling is that (68) does not change if g and f are multiplied by the same factor. The result is that if $g(x, t)$ and $f(x, t)$ are now defined as a solution of the equations

$$iD_t g \cdot f + D_x^2 g \cdot f = 0 \tag{69}$$

and

$$D_x^2 f \cdot f = |g|^2, \tag{70}$$

these two equations are bilinear and therefore easier to solve than the original equation (64). A special solution is obtained if it is assumed that $|g|^2 = \text{constant} = 2\alpha^2$ and that

$f = f(x-ct)$, which goes to infinity with $|x| \to \infty$, so that $\psi(x, t)$ then approaches zero. The solution of (70) is then $f(x, t) = \cosh \alpha(x-ct)$, after which (69) becomes a simple linear equation for $g(x, t)$. Ultimately the wave function becomes

$$\psi(x, t) = \frac{g}{f} = \frac{\alpha\sqrt{2}\exp\left[\frac{1}{2}ic\left(x-\left(\frac{c}{2}-\frac{2\alpha^2}{c}\right)t\right)\right]}{\cosh \alpha(x-ct)}. \tag{71}$$

This is a complex soliton, the envelope of which propagates with velocity c but with different phase velocity. Two- and multi-soliton solutions of (69) and (70) can be constructed as well. A more systematic way of working is to expand g and f in powers of a formal parameter ε, which is later made equal to one, and to equate terms of the same order in ε. In this way one obtains linear equations for the expansion coefficients. In some cases the two series will break off; then an exact solution has been found.

In addition to the transformation (65) there are two other transformations which are often used with success. The principle of the direct method however is the same in all cases.

6. *Moving poles*

In the systematic method described below an attempt is made to construct a solution of the partial differential equation as a finite sum of terms which are singular in the points $x = x_i(t)$. The t-dependence of the solution is entirely accounted for in the poles $x_i(t)$ The Ansatz for Burgers' equation is now taken as

$$u(x, t) = u_0 \sum_{i=1}^{n} \frac{1}{x-x_i(t)}. \tag{72}$$

It will not be difficult for the reader who has some dexterity in splitting partial fractions to show that after substitution into (43) and with $u_0 = -2D$ the equation becomes equivalent to the following series of ordinary differential equations:

$$\dot{x}_i = 2D \sum_{j=1}^{n} \frac{1}{x_j - x_i} \quad (i = 1, 2, \ldots, n). \tag{73}$$

The singular term must be omitted from the sum. Once more differentiating with respect to time gives

$$\ddot{x}_i = 8D^2 \sum_{j=1}^{n} \frac{1}{(x_j-x_i)^3} = -\frac{\partial V}{\partial x_i} \tag{74}$$

with

$$V(x_1, \ldots, x_n) = -2D^2 \sum_{i \neq j} \frac{1}{(x_i-x_j)^2}. \tag{75}$$

This shows that Burgers' equation, at least for solutions of the type (72), is equivalent to the problem of n particles on a line, the interaction of which is described by the potential (75). This problem from classical mechanics still looks daunting. Here too it is possible to linearise it and to transform it into an $n \times n$ eigenvalue problem. For this purpose two $n \times n$ matrices L and M are first defined by

$$L_{ii} = 2D \sum_j \frac{1}{x_j - x_i} = \dot{x}_i; \quad L_{ij} = \frac{-2D}{x_i - x_j} \quad (i \neq j), \tag{76}$$

$$M_{ii} = 2D \sum_j \frac{1}{(x_j - x_i)^2}; \quad M_{ij} = \frac{-2D}{(x_i - x_j)^2} \quad (i \neq j). \tag{77}$$

Some rather tedious but elementary calculation shows that L and B form a Lax-pair in the sense that

$$\dot{L} = [M, L]. \tag{78}$$

The problem of calculating $x_j(t)$ with given $x_j(0)$ is now formulated as follows. Define the $n \times n$ matrix

$$Y(t) = R^{-1}(t)X(t)R(t), \tag{79}$$

where $R(t)$ is a transformation of the diagonal matrix $X_{ij}(t) = \delta_{ij}x_j(t)$. This $R(t)$ is defined as a solution of the equation

$$\dot{R} = MR \quad \text{where} \quad R(0) = 1. \tag{80}$$

The sought-after quantities $x_j(t)$ are in fact the eigenvalues of $Y(t)$ and the original problem is therefore reduced to a (linear) eigenvalue problem. This method is only of real use if the matrix $Y(t)$ can be calculated for every t. Now this is just what is made possible by the existence of the Lax-pair. Differentiation of $Y(t)$, as given in (79), and substitution of (80) yields

$$\dot{Y}(t) = R^{-1}\{\dot{X} - [M, X]\}R. \tag{81}$$

Now it follows simply from (76) and (77) that

$$L(t) = \dot{X} - [M, X], \tag{82}$$

so that

$$\dot{Y} = R^{-1}LR. \tag{83}$$

Differentiating again leads to

$$\ddot{Y} = R^{-1}\{\dot{L} - [M, L]\}R = 0 \tag{84}$$

because of (78). The solution of this differential equation is the known matrix

$$Y(t) = Y(0) + t\dot{Y}(0) = X(0) + tL(0), \tag{85}$$

the eigenvalues of which are the desired functions $x_j(t)$.

7. What remains to be done

Readers whose hunger for solitons is not yet satisfied may be able to fill their bellies by reading the now famous article by Scott, Chu and McLaughlin [2] with its 267 references. The article deals with a large number of examples including the Toda lattice and contains a derivation of the equation for inverse scattering devised by Gel'fand, Levitan and Marchenko. The above-mentioned authors also discuss the stability of solitons. The dynamic reason for stability is linked with the mutual compensation of dispersion and non-linearity. In addition it holds that stability is always connected with the existence of a conserved quantity. This can be a "charge" which is coupled to a continuous symmetry via the Noether theorem, or it can be a topological charge like the swing in the line of pendulums described by the sine-Gordon equation. Nevertheless, a single soliton may be stable, and yet when two such solitons collide inelastic effects occur [3]. Stable solitons only occur in multiple dimensions in special circumstances. Generally they will be unstable, according to a theorem of Derrick [4] and Hobart [5]. For the time-dependent N-dimensional sine-Gordon equation

$$\Delta_N \phi(\vec{r}) = \sin \phi(\vec{r}) \tag{85}$$

this theorem can be proved quite simply. Eq. (85) is the Euler-Lagrange equation that follows from the action principle $\delta A = 0$ with

$$A[\phi] = \int \tfrac{1}{2} \{ \sum_{k=1}^{N} (\partial_k \phi)^2 + 4 \sin^2 \tfrac{1}{2} \phi \} d_N \vec{r} \equiv I_1 + I_2.$$

Neither I_1 nor I_2 is negative. Let $\phi(\vec{r})$ be a solution with finite energy. Now consider a class of trial functions $\phi_\lambda(\vec{r}) = \phi(\lambda \vec{r})$ which are obtained from the real solution by scale enlargement. The appropriate action is

$$A_\lambda = \lambda^{-N+2} I_1 + \lambda^{-N} I_2.$$

Then it must hold that $(dA_\lambda / d\lambda)_{\lambda=1} = 0$, or that

$$(N-2) I_1 + N I_2 = 0. \tag{86}$$

For $N \geqslant 2$ Eq. (86) can only be satisfied if $I_2 = 0$, which implies that $\phi(\vec{r}) \equiv 0$. There is therefore no solution with a finite energy.

There is no doubt that soliton theories can be applied in practically all areas of physics. There is no point in reproducing (reproduction is forbidden anyway) the review articles by Bullough [6, 7] concerning applications. Magnetic monopoles as solitons and the connection with instantons are discussed in a simple way in the articles by Parsa [8] and Rebbi [9]. The reader is also referred to the work of Schneider and Stoll [10] on the statistical mechanics of nonlinear lattices. The systematic methods which have been touched upon in the previous sections are discussed more fully in the following references.

Calogero [11] considers the method of inverse scattering as a generalisation of the Fourier transformation.

The authors of references [12–14] are strong supporters of the prolongation method.

Hirota [15] is the best prophet of his own direct method, whereas several persons [16–19] have devoted their efforts to moving poles.

The work of AKNS [20] has not been discussed at all here, although these authors give a large class of integrable equations.

Those who are interested in learning more about the prolongation method and about the link between solitons and the geometry of a space with a negative curvature [21] are advised to first master the formalism of the differential forms.

Finally the authors of this article should draw his own conclusions or at least give some encouragement to anyone who wants to plunge into the sea of solitons. Since the author does not have sufficient time, energy or space he hopes that readers will be content with what has been offered above. Those who persevere would do well to consult three books on solitons which have been written recently [22–24].

It is a pleasure to thank Mrs. S. M. McNab for translating this paper from Dutch into real English. I am grateful to Miss M. Hollander for the fast and accurate typing job she did.

REFERENCES

[1] N. J. Zabusky, M. D. Kruskal, *Phys. Rev. Lett.* **15**, 240 (1965).
[2] A. C. Scott, F. Y. F. Chu, D. W. McLaughlin, *Proc. IEEE* **61**, 1443 (1973).
[3] J. Courtenay Lewis, J. A. Tjon, *Phys. Lett.* **73A**, 275 (1979).
[4] G. H. Derrick, *J. Math. Phys.* **5**, 1252 (1964).
[5] R. H. Hobart, *Proc. Phys. Soc.* **82**, 201 (1963).
[6] R. K. Bullough, in A. O. Barut (ed)., *Nonlinear Equations in Physics and Mathematics*, Reidel Publ. Comp., Dordrecht 1978, pp. 99–141.
[7] R. K. Bullough, in *Interaction of Radiation with Condensed Matter*, Vol. I, IAEA-SMR-20/51, Vienna 1977, pp. 381–469.
[8] Z. Parsa, *Am. J. Phys.* **47**, 56 (1979).
[9] C. Rebbi, *Scientific American*, February 1979, p. 76.
[10] T. Schneider, in Ref. [6], p. 239.
[11] F. Calogero, *Lecture Notes*, Groningen 1979.
[12] H. D. Wahlquist, F. B. Estabrook, *J. Math. Phys.* **16**, 1 (1975); **17**, 1293 (1976).
[13] J. Corones, *J. Math. Phys.* **17**, 756 (1976).
[14] D. J. Kaup, *Lecture Notes*, MIT 1978.
[15] R. Hirota, in R. M. Miura (ed.), *Bäcklund Transformations*, Springer Lecture Notes in Mathematics **515** (1976).
[16] M. D. Kruskal, *Lectures in Applied Math.* **15**, Am. Math. Soc. 1974, p. 61.
[17] H. Airault, H. P. McKean, J. Moser, *Comm. Pure Appl. Math.* **30**, 95 (1977).
[18] D. V. Choodnovsky, G. V. Choodnovsky, *Nuovo Cimento* **40B**, 339 (1977).
[19] F. Calogero, Ref. [6], pp. 3–53.
[20] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur, *Phys. Rev. Lett.* **31**, 125 (1973).
[21] R. Sasaki, *Proc. R. Soc. London* **A373**, 373 (1980).
[22] R. K. Bullough, P. J. Caudrey (eds.), *Solitons*, Springer Topics in Current Physics **17** (1980).
[23] G. L. Lamb, Jr., *Elements of Soliton Theory*, J. Wiley, 1980.
[24] G. Eilenberger, *Solitons*, Springer Series in Solid-State Sciences **19** (1981).
[25] K. Abe, T. Abe, *Phys. Fluids* **22**, 1644 (1979).
[26] R. M. Miura (ed.), *Bäcklund Transformations*, The inverse scattering method, solitons and their applications, Springer Lecture Notes in Mathematics **515** (1976).