# TIME EVOLUTION IN FERMION PATH INTEGRALS*

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The formulation of fermion path integrals as transition amplitudes is discussed for gauge theories. Fermion states are specified by products of Grassmann variables, which form an orthogonal and complete basis. The expression for the Hamiltonian operating on the products is derived. In an application to Green functions, the fermion propagator is derived for a variety of boundary conditions at $t = \pm\infty$.

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## *1. Introduction*

We live in a world made out of quarks and leptons, interacting via gauge fields. Clearly a proper understanding of the fermion interactions is a prerequisite for describing the physical phenomena. However, beyond perturbation theory that becomes a highly non-trivial task. The fermion degrees of freedom manifest themselves via a determinant, which is a non-local functional of the gauge field [1, 2]. Even an approximate evaluation of the determinant is very difficult. This has recently been emphasized by numerical calculations with lattice gauge theories [3].

The problem posed by fermion interactions can be seen most simply in QED. (The situation is quite analogous in non-abelian gauge theories like QCD.) The generating functional $Z$ of QED Green functions is [4]

$$Z[J, \bar{\chi}, \chi] = \int \mathscr{D}(A)\mathscr{D}(\bar{\psi}, \psi) \exp\left[i \int d^4x(\mathscr{L} + J \cdot A + \bar{\chi}\psi + \bar{\psi}\chi)\right], \tag{1}$$

where $J(x)$, $\chi(x)$ and $\bar{\chi}(x)$ are sources and the Lagrangian is

$$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - e\gamma^\lambda A_\lambda - m)\psi, \tag{2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Any Green function can be obtained by differentiating $Z$ w.r.t. the source of each external leg, and then putting $J = \chi = \bar{\chi} = 0$. Thus an $N$-point Green function is just an expec-

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tation value of the product of $N$ fields,

$$G_N = \frac{1}{Z[0]} \int \mathscr{D}(A)\mathscr{D}(\bar{\psi}, \psi) \underbrace{A \ldots \bar{\psi} \ldots \psi \ldots}_{N \text{ fields}} \exp\left[ i \int d^4x \mathscr{L} \right]. \tag{3}$$

The path integral in (1) and (3) is a product of integrals over each field component at all space-time points:

$$\int \mathscr{D}(A)\mathscr{D}(\bar{\psi}, \psi) = \prod_x \prod_{\mu=0}^{3} \prod_{\alpha=1}^{4} \int_{-\infty}^{\infty} dA_\mu(x) \int d\bar{\psi}_\alpha(x) d\psi_\alpha(x). \tag{4}$$

The crucial difference between Bose and Fermi fields now appears. The photon field $A_\mu(x)$ is an ordinary number, whereas the electron fields $\psi, \bar{\psi}$ are anticommuting *Grassmann* numbers,

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0. \tag{5}$$

The "integrals" over Grassmann variables in (4) have little to do with ordinary integrals. The rules are

$$\int d\psi \psi = 1, \quad \int d\psi 1 = 0. \tag{6}$$

Thus the integral removes the integration variable (once it has been anticommuted to the left). If the variable does not appear in the integrand, the integral is zero. Since $\psi^2 = 0$ by (5), the rules (6) are sufficient for calculating any Grassmann integral. A further discussion of Grassmann calculus can be found in [1].

The usual procedure for evaluating Green functions in QED is to expand $Z$ in powers of the coupling $e$. The Grassmann integration rules (6) give rise to the same perturbation expansion as obtained in the operator formalism. This verifies the path integral representation (1) of $Z$.

Ordinary perturbation theory can be regarded as an expansion of the fields around the vacuum (zero-field) configuration [2, 4]. Other field configurations which minimize the total action can equally be used as the starting point of a perturbative expansion. An example of such a non-trivial classical solution is the instanton in (the pure gluon sector of Euclidean) QCD [5].

In QED, the classical field equations for photons are the Maxwell equations. Because photons carry no charge, the photon sector of QED is trivially solvable in terms of freely propagating plane waves.

The inclusion of electrons, on the other hand, introduces qualitatively new features. A naive minimization of the action $I = \int d^4x \mathscr{L}$ in (2),

$$\frac{\delta I}{\delta A_\nu(x)} = \partial_\mu F^{\mu\nu} - e\bar{\psi}\gamma^\nu\psi = 0 \tag{7}$$

evidently does not make sense, as (7) mixes ordinary ($A_\mu$) and Grassmann numbers ($\psi, \bar{\psi}$). The mistake in (7) was to apply standard differential calculus procedures to the non-standard Grassmann integration (6).

In order to find the minimum (w.r.t. $A_\mu$) of the effective action in the presence of electrons, the electron fields must be integrated out. This gives [1]

$$Z[J, \bar{\chi}, \chi] = \int \mathscr{D}(A)\, \mathrm{Det}\,(\partial\!\!\!/ + ie\gamma^\lambda A_\lambda + im) \exp\left[i \int d^4x(-\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + J \cdot A)\right]$$
$$\times \exp\left[-i \int d^4x d^4y\, \bar{\chi}(x)\,(i\partial\!\!\!/ - e\gamma^\lambda A_\lambda - m)^{-1}\chi(y)\right]. \tag{8}$$

When $\bar{\chi} = \chi = 0$, the integrand is an ordinary functional of $A_\mu$, which can be minimized using the rules of standard calculus.

The catch is, however, that we have traded the (Gaussian) Grassmann integrals for a very complicated functional of $A_\mu$. The determinant in (8) should be taken over space-time as well as Dirac indices. Hence we are dealing with a determinant of an infinite dimensional matrix, whose functional dependence on $A_\mu$ is hard to visualize. Even a numerical evaluation, for a given field configuration $A_\mu(x)$, of the finite determinant corresponding to a discrete space-time is by no means trivial [3].

The purpose of these lectures is to show that the determinant, or more precisely the generalized determinants corresponding to expectation values of products of Grassmann variables as in (3), nonetheless have some simple properties. In particular, we shall work out the dependence of the determinant on the time interval covered by the path integral in (4) (for ordinary Green functions one integrates over the fields at all times, $-\infty < t < \infty$). We arrive at a picture of the determinant as a transition amplitude from one fermion configuration to another in the given time interval, and in the presence of an arbitrary background field $A_\mu(x)$.

The transition amplitude formulation for fermions differs in an important respect from that for bosons. The initial and final states for bosons are specified by the functional form (over space) of the boson field (at the corresponding times) [4]. This cannot be done for fermion fields, which are Grassmann variables. As we shall see, fermion states are specified by products of field variables, i.e., by the form of the Green function (3). This also ensures that the occupation number of fermions can be only zero or one, as required by the Pauli principle.

The fermion transition amplitudes satisfy a "time evolution" equation, which can be expressed in terms of a Hamiltonian operating on the product of Grassmann fields (fermion states). In an application of this equation, we show how it implies the usual equation of motion for the fermion propagator. The form of the propagator solution depends also on the boundary conditions imposed on the fermion states at $t = \pm\infty$.

A shorter version of the calculations to be presented below has already been published elsewhere [6].

## 2. *The determinant in* $0+1$ *dimensions*

The technique we shall use is most simply illustrated by considering the time dependence of a system defined at a single space point. The fermion part of the QED Lagrangian (2) then reduces to

$$\mathscr{L}_f(t) = \bar{\psi}[i\partial_t - eA(t)]\psi, \tag{9}$$

where $\psi(t)$, $\bar{\psi}(t)$ have only one component (there is no spin degree of freedom). The mass term has been absorbed in $eA(t)$.

We begin by considering the determinant

$$\langle 1, t_b|1, t_a\rangle \equiv \int_a^b \mathscr{D}(\bar{\psi}, \psi) \exp\{-\int_{t_a}^{t_b} dt\bar{\psi}(t)[\partial_t + ieA(t)]\psi(t)\}$$

$$= \mathrm{Det}[\partial_t + ieA(t)]. \tag{10}$$

The bracket notation in (10) is used to emphasize our interpretation of the determinant as a transition amplitude. The convenience of this notation will soon become apparent. The "1" appearing in $|1, t_a\rangle$ and $\langle 1, t_b|$ refers to the boundary conditions at the initial and final times $t_a$, $t_b$. (It is the factor multiplying the exponent in the integrand of (10), see below.)

The path integral in (10) extends over the fields $\bar{\psi}(t)$ and $\psi(t)$ for $t_a \leqslant t \leqslant t_b$. To define it properly we must first make the time interval discrete then consider the limit of continuous time development. Thus we divide the interval $\Delta t = t_b - t_a$ into $n$ parts, separated by the discrete times $t_0 = t_a$, $t_1 = t_a + \Delta t/n$, ..., $t_n = t_b$.

There are several possible choices for the differential operation $\partial_t$ when time is discrete. We use the "central derivative"

$$(\partial_t)_{k,l} = \frac{n}{2\Delta t}(\delta_{k+1,l} - \delta_{k,l+1}) \quad (k, l = 0, \ldots, n), \tag{11}$$

which has the virtue of keeping the discrete Lagrangian hermitian. Other choices, such as the "forward derivative",

$$(\partial_t)_{k,l} = \frac{n}{\Delta t}(\delta_{k+1,l} - \delta_{k,l}) \tag{12}$$

would lead to a different evolution equation. The difference presumably resides in the formalism rather than in the physics, however.

The determinant (10) is thus defined as

$$\langle 1, t_b|1, t_a\rangle = \int (\prod_{k=0}^{n} d\bar{\psi}_k d\psi_k)$$

$$\times \exp\left[-\sum_{k,l=0}^{n} \bar{\psi}_k \left(\delta_{k+1,l} - \delta_{k,l+1} + \frac{2\Delta t}{n} ieA_k\delta_{k,l}\right)\psi_l\right], \tag{13}$$

where $A_k \equiv A(t_k)$, etc. For finite $n$ this is a well-defined function of the gauge field $A_k$, $k = 0, \ldots, n$.

We wish to find the change in the determinant induced by a small variation of $t_a$ (or $t_b$). Hence we expand it into subdeterminants, which implies doing some of the Grassmann integrals in (13). Remembering that the exponent can be written as a product of

factors,

$$\exp\left(\sum_{k,l} \bar{\psi}_k B_{kl} \psi_l\right) = \prod_{k,l} (1 + \bar{\psi}_k B_{kl} \psi_l) \tag{14}$$

it is easy to see that the $k = 0$ integration in (13) gives

$$\langle 1, t_b | 1, t_a \rangle = \int \left( \prod_{k=1}^{n} d\bar{\psi}_k d\psi_k \right) \left( \psi_1 \bar{\psi}_1 + \frac{2\Delta t}{n} ieA_0 \right)$$

$$\times \exp\left[ - \sum_{k,l=1}^{n} \bar{\psi}_k (\ldots) \psi_l \right]. \tag{15}$$

Doing now also the $k = 1$ integral we obtain

$$\langle 1, t_b | 1, t_a \rangle = \int \left( \prod_{k=2}^{n} d\bar{\psi}_k d\psi_k \right) \left[ 1 - \frac{2\Delta t}{n} ieA_0 \bar{\psi}_2 \psi_2 + \left( \frac{2\Delta t}{n} ie \right)^2 A_0 A_1 \right]$$

$$\times \exp\left[ - \sum_{k,l=2}^{n} \bar{\psi}_k (\ldots) \psi_l \right]. \tag{16}$$

At this point we observe that the first term in the brackets of (16) is nothing but $\left\langle 1, t_b | 1, t_a + \frac{2\Delta t}{n} \right\rangle$, the original determinant with a shifted initial time. Bringing this term to the l.h.s. and letting $n \to \infty$ we have the evolution equation

$$\frac{d}{dt_a} \langle 1, t_b | 1, t_a \rangle = ieA(t_a) \langle 1, t_b | \bar{\psi}\psi, t_a \rangle. \tag{17}$$

The last term in (16) can be neglected, being $\mathcal{O}(1/n^2)$. The initial state $|\bar{\psi}\psi, t_a\rangle$ in (17) implies of factor $\bar{\psi}(t_a)\psi(t_a)$ in front of the exponent, as seen from (16).

An analogous calculation for the amplitude $\langle 1, t_b | \bar{\psi}\psi, t_a \rangle$ gives

$$\frac{d}{dt_a} \langle 1, t_b | \bar{\psi}\psi, t_a \rangle = ieA(t_a) \langle 1, t_b | 1, t_a \rangle. \tag{18}$$

Together with (17), this gives

$$\langle 1, t_b | 1, t_a \rangle = \tfrac{1}{2}(z + z^{-1}),$$

$$z = \exp\left[ -ie \int_{t_a}^{t_b} dt A(t) \right], \tag{19}$$

$$\langle 1, t_b | \bar{\psi}\psi, t_a \rangle = \tfrac{1}{2}(z - z^{-1}),$$

where we anticipated the initial value (23) for $t_a = t_b$. Hence the determinant can be calculated exactly for arbitrary $A(t)$ in $0+1$ dimensions.[1] Needless to say, this will not be the case in more realistic situations.

Anticipating the case with several space dimensions, let us introduce the following

[1] This has been noted before. See, e.g., Ref. [7].

notations. A general amplitude will be denoted $\langle C_b, t_b | C_a, t_a \rangle$. Here $C_a(C_b)$ specify the initial (final) fermion configurations, and are products of $\psi, \bar{\psi}$ fields at $t = t_a$ ($t = t_b$). In the above examples we encountered $C_a = 1$ and $\bar{\psi}\psi$, and $C_b = 1$. In 0+1 dimensions there are actually only four possible products,

$$C = 1, \psi, \bar{\psi} \quad \text{or} \quad \bar{\psi}\psi. \tag{20}$$

Any linear combination of these basis states with complex coefficients also defines a configuration.

The amplitude is defined as the $n \to \infty$ limit

$$\langle C_b, t_b | C_a, t_a \rangle = \lim_{\text{odd } n \to \infty} \int \Big( \prod_{k=0}^{n} d\bar{\psi}_k d\psi_k \Big) C_b^* C_a$$

$$\times \exp\left[ - \sum_{k,l=0}^{n} \bar{\psi}_k \left( \delta_{k+1,l} - \delta_{k,l+1} + \frac{2\Delta t}{n} ieA_k \delta_{k,l} \right) \psi_l \right], \tag{21}$$

where $n$ takes odd values. $C^*$ is obtained from $C$ by "Grassmann conjugation" or "involution" [1], which interchanges[2] $\psi$ and $\bar{\psi}$ ($\psi^* = \bar{\psi}$, $\bar{\psi}^* = \psi$) and reverses the ordering of the fields (like hermitian conjugation). Any numerical coefficients in $C$ appear complex conjugated in $C^*$. For example, $(i + \bar{\psi}\psi)^* = -i + \bar{\psi}\psi$.

The $C_b^*$ convention in (21) makes the initial value of the amplitude for $t_b \to t_a$ assume a simple form. When $\Delta t = t_b - t_a \to 0$ we may drop the last term in the exponent of (21). But then the evolution is trivial: The effect of an *even* number $k$ of integrations is to replace $C_a(\bar{\psi}_0, \psi_0)$ by $C_a(\bar{\psi}_k, \psi_k)$ (Eq. (16) was an example of this). After $n-1$ integrals we thus have

$$\langle C_b, t | C_a, t \rangle = \int \Big( \prod_{k=n-1}^{n} d\bar{\psi}_k d\psi_k \Big) C_b^* C_a \exp\left( \psi_n \bar{\psi}_{n-1} + \bar{\psi}_n \psi_{n-1} \right). \tag{22}$$

Now the $k = n-1$ integral equals the coefficient of $\psi_{n-1}\bar{\psi}_{n-1}$ in the integrand. If $\psi_{n-1}$ and/or $\bar{\psi}_{n-1}$ is missing from $C_a(\bar{\psi}_{n-1}, \psi_{n-1})$, it must be taken from the exponent, and brings with it a $\bar{\psi}_n$ and/or $\psi_n$. The $k = n$ integral is nonzero only provided $C_b^*(\bar{\psi}_n, \psi_n)$ times the previously obtained factors $\bar{\psi}_n$ and/or $\psi_n$ equals $\pm\psi_n\bar{\psi}_n$, i.e., $C_b = \pm C_a$. Because of the reversal of the ordering of the fields in $C_b^*$ the sign is positive,

$$\langle C_b, t | C_a, t \rangle = \delta(C_a, C_b), \tag{23}$$

where $C_a$ and $C_b$ are assumed to be any of the basis states (20) (single products with unit coefficient).

The evolution equation for a general amplitude can be concisely expressed in terms of a Hamiltonian defined by

$$\frac{d}{dt_a} \langle C_b, t_b | C_a, t_a \rangle = \langle C_b, t_b | iH(t_a) | C_a, t_a \rangle \equiv i \langle C_b, t_b | H(t_a) C_a, t_a \rangle. \tag{24}$$

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[2] The rule in 3+1 dimensions is given in Eq. (35).

In 0+1 dimensions,

$$H(t) = eA(t)[\bar{\psi}(t)\psi(t) + \partial(t)\bar{\partial}(t)], \tag{25}$$

where $\partial$ is the Grassmann derivative, which acts [1] like the integral (6):

$$\partial\psi = 1, \quad \partial 1 = 0. \tag{26}$$

From $H \cdot 1 = eA\bar{\psi}\psi$ and $H\bar{\psi}\psi = eA$ one varifies (17) and (18). Moreover, $H\psi = H\bar{\psi} = 0$ so that

$$\frac{d}{dt_a}\langle C_b, t_b|\overset{(-)}{\psi}, t_a\rangle = 0 \tag{27}$$

as can also be obtained by a direct calculation.

The Grassmann representation of the adjoint field $\psi^+$ is obtained by noting that

$$\langle C_b, t|\psi C_a, t\rangle = \langle \partial C_b, t|C_a, t\rangle$$

according to (23). This implies [1, 2]

$$\psi^+ = \partial \tag{28}$$

which is consistent with $\{\psi^+, \psi\} = 1$. Hence the Hamiltonian (25) is hermitian,

$$H^+ = H \tag{29}$$

and we have

$$\frac{d}{dt_b}\langle C_b, t_b|C_a, t_a\rangle = -i\langle H(t_b)C_b, t_b|C_a, t_a\rangle. \tag{30}$$

By direct inspection we also note, for Grassmann conjugation,

$$(HC)^* = HC^*. \tag{31}$$

### *3. Evolution in 3+1 dimensions*

In a system with many space points the fermion configuration is specified as in (20) by products $C$ of $\psi$, $\bar{\psi}$ fields, which now carry both space and Dirac indices. To keep the notation clear (and the products $C$ finite) it is convenient to assume space to be discrete. In applications involving only configurations that can be labelled by a finite number of points (e.g., the two-point Green functions of Section 4), the continuum limit can then be directly taken.

We shall here work with a naive discretization of the continuum Lagrangian (2). Hence gauge invariance will be lost but should, just as rotational invariance, be recovered in the continuum limit.

For the space derivative on a cubic lattice of spacing $a$ we take the "central" difference

$$\Delta_j(\vec{x}, \vec{y}) = \frac{1}{2a}[\delta(\vec{x}+\vec{a}_j, \vec{y}) - \delta(\vec{x}-\vec{a}_j, \vec{y})] \quad (j = 1, 2, 3), \tag{32}$$

where $\vec{a}_j$ is a vector of length $a$ pointing in the $j$-direction, and $\delta$ is the Kronecker delta function.

A general amplitude is now defined as the $n \to \infty$ (continuous time) limit

$$\langle C_b, t_b | C_a, t_a \rangle = \lim_{\text{odd } n \to \infty} \int_a^b \mathscr{D}(\bar{\psi}, \psi) C_b^* C_a \exp\left\{ - \sum_{k,l=0}^{n} \sum_{\vec{x},\vec{y}} \sum_{\alpha,\beta=1}^{4} \bar{\psi}_\alpha(t_k, \vec{x}) \right.$$

$$\times \left[ (\delta_{k+1,l} - \delta_{k,l+1}) \delta(\vec{x}, \vec{y}) \gamma^0 - \frac{2\Delta t}{n} \delta_{k,l} \vec{\gamma} \cdot \vec{\Delta}(\vec{x}, \vec{y}) \right.$$

$$\left. \left. + \frac{2\Delta t}{n} i(m + e\gamma^\lambda A_\lambda(t_k, \vec{x})) \delta_{k,l} \delta(\vec{x}, \vec{y}) \right]^{\alpha\beta} \psi_\beta(t_l, \vec{y}) \right\}, \tag{33}$$

where $n$ is odd and

$$\int_a^b \mathscr{D}(\bar{\psi}, \psi) = \int \prod_{k=0}^{n} \prod_{\vec{x}} \prod_{\alpha=1}^{4} d\bar{\psi}_\alpha(t_k, \vec{x}) d\psi_\alpha(t_k, \vec{x}). \tag{34}$$

As in (21), the Grassmann conjugated $C^*$ is obtained from $C$ by reversing the field ordering and identifying

$$\psi^*(t, \vec{x}) = \bar{\psi}(t, \vec{x}) \gamma^0, \qquad [\bar{\psi}(t, \vec{x})]^* = \gamma^0 \psi(t, \vec{x}). \tag{35}$$

Ordinary numbers in $C$ appear complex conjugated in $C^*$. All $\bar{\psi}$ and $\psi$ variables appearing in the products $C_a(C_b)$ of (33) are evaluated at $t = t_a$ ($t = t_b$).

The evolution equation is obtained just as in the 0+1-dimensional case by explicitly doing the $k = 0, 1$ integrals in (33). Keeping track of the space and Dirac indices one finds (24), where the Hamiltonian is now

$$H(t) = \sum_{\vec{x},\vec{y}} [i\vec{\gamma} \cdot \vec{\Delta}(\vec{x}, \vec{y}) + (m + e\gamma^\lambda A_\lambda(t, \vec{x})) \delta(\vec{x}, \vec{y})]^{\alpha\beta}$$

$$\times [\bar{\psi}_\alpha(t, \vec{x}) \psi_\beta(t, \vec{y}) + \partial_{\alpha'}(t, \vec{x}) \gamma_0^{\alpha'\alpha} \gamma_0^{\beta\beta'} \bar{\partial}_{\beta'}(t, \vec{y})] \tag{36}$$

(a sum over repeated Dirac indices is implied).

It is straightforward to show that just as in 0+1 dimensions $H$ is hermitian, $H^+ = H$, and real under Grassmann conjugation, $(HC)^* = HC^*$.

The value of $\langle C_b, t_b | C_a, t_a \rangle$ in (33) for $t_b \to t_a$ is also obtained as in the 0+1-dimensional case. The result (23) shows that the basis states $|C, t\rangle$ are orthonormal, where $C$ is any single product of $\psi, \bar{\psi}$ variables with unit coefficient. The relation

$$\sum_C \langle C_b, t_b | C, t \rangle \langle C, t | C_a, t_a \rangle = \langle C_b, t_b | C_a, t_a \rangle \tag{37}$$

shows that the basis states also form a complete set. To prove (37), we show first that the l.h.s. is independent of $t$, i.e., its time derivative vanishes:

$$i \sum_C \langle C_b, t_b | HC, t \rangle \langle C, t | C_a, t_a \rangle - i \sum_{C'} \langle C_b, t_b | C', t \rangle \langle HC', t | C_a, t_a \rangle = 0. \tag{38}$$

(Here we used also the evolution equation w.r.t. the final time, Eq. (30).) There is a pairwise cancellation of terms in (38). Corresponding to an arbitrary term in $\langle C_b, t_b | HC, t\rangle$ of the form

$$[i\vec{\gamma}\cdot\vec{\Delta}(\vec{x},\vec{y})+(m+e\gamma^\lambda A_\lambda)\delta(\vec{x},\vec{y})]^{\alpha\beta}\langle C_b, t_b|\bar{\psi}_\alpha(t,\vec{x})\psi_\beta(t,\vec{y})C, t\rangle$$

there is a term for $C' \equiv \bar{\psi}_\alpha(t,\vec{x})\psi_\beta(t,\vec{y})C$ in $\langle HC', t|C_a, t_a\rangle$ which cancels it:

$$[-i\vec{\gamma}^*\cdot\vec{\Delta}(\vec{y},\vec{x})+(m+e(\gamma^\lambda A_\lambda)^*)\delta(\vec{y},\vec{x})]^{\beta\alpha}\langle\partial_{\beta'}(t,\vec{y})\gamma_0^{\beta'\beta}\gamma^{\alpha\alpha'}\partial_{\alpha'}(t,\vec{x})C', t|C_a, t_a\rangle$$
$$= [i\vec{\gamma}\cdot\vec{\Delta}(\vec{x},\vec{y})+(m+e\gamma^\lambda A_\lambda)\delta(\vec{x},\vec{y})]^{\alpha\beta}\langle C, t|C_a, t_a\rangle.$$

Thus the l.h.s of (37) is independent of $t$. Now we may choose $t = t_a$ and use the orthonormality (23) to prove the completeness relation (37).

We conclude this section by showing that the standard relation

$$\langle C_b, t_b|C_a, t_a\rangle = \langle C_a, t_a|C_b, t_b\rangle^* \tag{39}$$

applies to our amplitudes. Because the amplitude is an ordinary number Grassmann conjugation equals complex conjugation,

$$\langle C_a, t_a|C_b, t_b\rangle^*$$
$$= \lim_{\text{odd } n\to\infty}\int\mathscr{D}(\bar{\psi},\psi)C_b^*C_a\exp\left\{+\sum\bar{\psi}_{\beta'}(t_l,\vec{y})\gamma^0_{\beta'\beta}\left[(\delta_{k+1,l}-\delta_{k,l+1})\delta(\vec{x},\vec{y})\gamma^0\right.\right.$$
$$\left.\left.-\frac{2\Delta t}{n}\delta_{k,l}\vec{\gamma}^*\cdot\vec{\Delta}(\vec{x},\vec{y})-\frac{2\Delta t}{n}i(m+e(\gamma^\lambda A_\lambda)^*)\delta_{k,l}\delta(\vec{x},\vec{y})\right]^{\alpha\beta}\gamma^0_{\alpha\alpha'}\psi_{\alpha'}(t_k,x)\right\}, \tag{40}$$

where the overall sign change in the exponent was due to interchanging the limits in the time integral. The expression (40) is equivalent to (33).

### *4. Green functions*

Green functions are of the form (3). Here we shall be concerned only with the Grassmann part of the path integral. Thus the fermion propagator in an arbitrary background field $A_\mu$ is, in our notation,

$$D_{\alpha\beta}(x,y) = \langle C_0, +\infty|\psi_\alpha(x)\bar{\psi}_\beta(y)|C_0, -\infty\rangle/\langle C_0, +\infty|C_0, -\infty\rangle$$
$$\equiv\int\mathscr{D}(\bar{\psi},\psi)C_0^*(+\infty)\psi_\alpha(x)\bar{\psi}_\beta(y)C_0(-\infty)\exp\{-\sum\bar{\psi}[\ldots]\psi\}$$
$$\times\langle C_0, +\infty|C_0, -\infty\rangle^{-1}\equiv D(C_0), \tag{41}$$

where the exponent is as in (33).

It is clear from (41) that the propagator can only be calculated given the boundary configurations $C_0$ at $t = \pm\infty$. This is analogous to the situation for bosons, where the boundary conditions [2] specify the type of propagator (e.g., retarded, advanced or Feynman

propagators). In the path integral formulation little attention is usually paid to the boundary conditions for fermions, however.

We have seen above that for fermions the boundary conditions are specified by products $C$ of the Grassmann variables. Now it is not immediately obvious which "vacuum" configuration $C_0$ should be used in (41). However, we would expect any bare vacuum to satisfy ($H_0 = H(A_\mu = 0)$),

(*i*) $H_0 C_0 = E_0 C_0$ (time translation invariance). (42a)

(*ii*) Space translation invariance (on the lattice). (42b)

Since the vacuum should be "neutral", it is furthermore natural to impose

(*iii*) $C_0^* = C_0$. (42c)

Consider first the 0+1-dimensional case. The eigenstates of the Hamiltonian (25) with $eA(t) = m$ are $1 \pm \bar{\psi}\psi$, $\psi$ and $\bar{\psi}$:

$$H_0(1 \pm \bar{\psi}\psi) = \pm m(1 \pm \bar{\psi}\psi), \quad H_0\psi = H_0\bar{\psi} = 0. \tag{43}$$

Note, in particular, that $C = 1$ is not an eigenstate.

It is straightforward to calculate from (41) the free propagators in 0+1-dimensions, corresponding to the states $C_0$ in (43). For $C_0 = \bar{\psi}$ we have

$$D(\bar{\psi}) = \theta(x-y)\,\langle \bar{\psi}, x|\psi(x)\bar{\psi}(y)|\bar{\psi}, y\rangle - \theta(y-x)\,\langle \bar{\psi}, y|\bar{\psi}(y)\psi(x)|\bar{\psi}, x\rangle \tag{44}$$

since $\langle \bar{\psi}, +\infty|\bar{\psi}, -\infty\rangle = 1$ and the vacuum develops trivially until perturbed by $\psi(x)\bar{\psi}(y)$. The first term in (44) vanishes since $\bar{\psi}^2(y) = 0$. From

$$H_0\bar{\psi}\psi = m, \quad H_0^2\bar{\psi}\psi = m^2\bar{\psi}\psi$$

we find, when the number of time intervals in $y-x$ is odd,

$$D(\bar{\psi}) = -\theta(y-x)\,\langle \psi\bar{\psi}, y| \exp[-i(y-x)H_0]|\psi\bar{\psi}, y\rangle = -\theta(y-x)\cos m(x-y). \tag{45}$$

Thus the boundary condition $C_0 = \bar{\psi}$ leads to an advanced propagator. Similarly one finds a retarded propagator for $C_0 = \psi$,

$$D(\psi) = \theta(x-y)\cos m(x-y). \tag{46}$$

On the other hand, when $C_0 = 1 \pm \bar{\psi}\psi$ the propagator is nonzero for both $x > y$ and $x < y$:

$$D(1 \pm \bar{\psi}\psi) = \theta(x-y)\,\langle \bar{\psi}, x|\bar{\psi}, y\rangle / \langle 1 \pm \bar{\psi}\psi, x|1 \pm \bar{\psi}\psi, y\rangle$$
$$-\theta(y-x)\langle \psi, y|\psi, x\rangle / \langle 1 \pm \bar{\psi}\psi, y|1 \pm \bar{\psi}\psi, x\rangle. \tag{47}$$

Using $\langle \bar{\psi}, x|\bar{\psi}, y\rangle = \langle \psi, y|\psi, x\rangle = 1$ and $\langle 1 \pm \bar{\psi}\psi, x|1 \pm \bar{\psi}\psi, y\rangle = 2\exp[\mp im(x-y)]$ we get

$$D(1 \pm \bar{\psi}\psi) = \tfrac{1}{2}\,\varepsilon(x-y)\exp[\pm im|x-y|]. \tag{48}$$

The change of sign at equal times ($x = y$) is a characteristic of Feynman-type propagators.

The eigenstates $\psi, \bar{\psi}$ can be trivially generalized to 3+1-dimesions,

$$\prod_{\vec{x},\alpha} \psi_\alpha(t, \vec{x}) \quad \text{and} \quad \coprod_{\vec{x},\alpha} \bar{\psi}_\alpha(t, \vec{x}), \tag{49}$$

where the products run over all space points and Dirac components As seen from the Hamiltonian (36), these states have zero eigenvalue They again give rise to retarded and advanced propagators. We shall not consider them further here.

The following states satisfy all the conditions (42), i.e., they are translationally invariant, "neutral" eigenstates of $H_0$:

$$C_\pm(t) = \exp\left[\pm \sum_{\vec{x}} \bar{\psi}(t, \vec{x})\gamma^0\psi(t, \vec{x})\right], \tag{50a}$$

$$C_P(t) = \exp\left[\sum_{\vec{x}} P(\vec{x})\bar{\psi}(t, \vec{x})\psi(t, \vec{x})\right]. \tag{50b}$$

In (50b), $P(\vec{x}) = \pm 1$ is a sign which alternates over the lattice points: $P(\vec{x}+\vec{a}_j) = -P(\vec{x})$.

From (36) we find, in the case of an arbitrary gauge field $A_\mu(t, \vec{x})$,

$$H(t)C_\pm(t) = \pm 4 \sum_{\vec{x}} eA^0(t, \vec{x})C_\pm(t), \tag{51a}$$

$$H(t)C_P(t) = \sum_{\vec{x},\alpha,\beta} [-2e\vec{A}(t, \vec{x}) \cdot \vec{\gamma}_{\alpha\beta}]\bar{\psi}_\alpha(t, \vec{x})\psi_\beta(t, \vec{x})C_P(t). \tag{51b}$$

Hence $C_P$ is an eigenstate of $H$ only when $\vec{A} = 0$.

Consider now the fermion propagator (41). To show that it satisfies the equations of motion we may differentiate it w.r.t. $x^0$ or $y^0$, and use (24) on the state $\bar{\psi}(x)C_0$ (if $x^0 < y^0$) or $\psi(y)C_0$ (for $y^0 < x^0$). The vacuum states (50) satisfy, for any $A_\mu$,

$$[H, \psi(\vec{z})]C_\pm = \mp \sum_{\vec{x}} \gamma^0 K_A(\vec{z}, \vec{x})\psi(\vec{x})C_\pm, \tag{52a}$$

$$[H, \bar{\psi}(\vec{z})]C_\pm = \mp \sum_{\vec{x}} \bar{\psi}(\vec{x})K_A(\vec{x}, \vec{z})\gamma^0 C_\pm, \tag{52b}$$

$$[H, \psi(\vec{z})]C_P = -\sum_{\vec{x}} \gamma^0 K_A(\vec{z}, \vec{x})\gamma^0 P(\vec{x})C_P, \tag{52c}$$

$$[H, \bar{\psi}(\vec{z})]C_P = -\sum_{\vec{x}} \bar{\psi}(\vec{x})P(\vec{x})\gamma^0 K_A(\vec{x}, \vec{z})\gamma^0 C_P, \tag{52d}$$

where

$$K_A(\vec{z}, \vec{x}) \equiv i\vec{\gamma} \cdot \vec{\Delta}(\vec{z}, \vec{x}) + [m + e\gamma^\lambda A_\lambda(\vec{z})]\delta(\vec{z}, \vec{x}). \tag{53}$$

We define the $y^0$ derivative as the discrete time difference (11) (which also appears in the Lagrangian of (33)):

$$\frac{d}{dy^0} D(y^0) \equiv \frac{n}{2\Delta t}\left[D\left(y^0 + \frac{\Delta t}{n}\right) - D\left(y^0 - \frac{\Delta t}{n}\right)\right]. \tag{54}$$

Then from (24) and (52b) we find that the free ($A_\mu = 0$) propagator (41) satisfies, for $C_0 = C_\pm$,

$$\frac{d}{dy^0} D_\pm(x, y) = i \sum_{\vec{z}} D_\pm(x, z) K_0(\vec{z}, \vec{y}) \gamma^0, \tag{55}$$

where $z^0 = y^0$. The $\mp$ sign of (52b) is compensated in (55) by the fact that $dD_\pm/dy^0$, according to (54), is evaluated at odd times compared to $D(y^0)$. From the Lagrangian of (33) one readily sees that a shift of one time interval implies

$$\bar{\psi}(y^0, \vec{y}) C_\pm(y^0) \to \mp \bar{\psi}\left(y^0 + \frac{\Delta t}{n}, \vec{y}\right) C_\pm\left(y^0 + \frac{\Delta t}{n}\right) + \mathcal{O}\left(\frac{1}{n}\right). \tag{56a}$$

For the $C_P$ vacuum of (50b) a shift of one time interval gives

$$\bar{\psi}(y^0, \vec{y}) C_P(y^0) \to -P(\vec{y}) \gamma^0 \bar{\psi}\left(y^0 + \frac{\Delta t}{n}, \vec{y}\right) C_P\left(y^0 + \frac{\Delta t}{n}\right) + \mathcal{O}\left(\frac{1}{n}\right). \tag{56b}$$

Thus using (52d) the free propagator with boundary condition $C_0 = C_P$ satisfies

$$\frac{d}{dy^0} D_P(x, y) = i \sum_{\vec{z}} D_P(x, z) \gamma^0 P(\vec{z}) K_0(\vec{z}, \vec{y}) P(\vec{y}) = i \sum_{z} D_P(x, z) K_0(\vec{z}, \vec{y}) \gamma^0 \tag{57}$$

i.e., $C_0 = C_\pm$ and $C_0 = C_P$ lead to the same equation of motion.

In deriving (55) and (57) we assumed $x^0 > y^0$ (the same equations result for $x^0 < y^0$). For $x^0 = y^0$ we have, using the orthogonality condition (23),

$$\frac{d}{dy^0} D(x, y) = \frac{n}{2\Delta t} \left\langle C_0, +\infty | \psi(x^0, \vec{x}) \left[ \bar{\psi}\left(x^0 + \frac{\Delta t}{n}, \vec{y}\right) - \bar{\psi}\left(x^0 - \frac{\Delta t}{n}, \vec{y}\right) \right] \middle| C_0, -\infty \right\rangle$$

$$\times \langle C_0, +\infty | C_0, -\infty \rangle^{-1} = -\frac{n}{2\Delta t} \gamma^0 \delta(\vec{x}, \vec{y}). \tag{58}$$

As $n \to \infty$, this becomes $-\gamma^0 \delta(x^0 - y^0) \delta(\vec{x}, \vec{y})$. Hence for arbitrary $x$ and $y$ the free lattice propagator satisfies

$$i \frac{d}{dy^0} D(x, y) \gamma^0 = -\sum_{\vec{z}} D(x, z) K_0(\vec{z}, \vec{y}) - i\delta(x^0 - y^0) \delta(\vec{x}, \vec{y}). \tag{59}$$

As is well-known, there is a fermion species multiplication on the lattice. This is also evident from (59). Due to the symmetry property

$$K_0(\vec{z}, \vec{y}) = P(\vec{z}) K_0(-\vec{z}, -\vec{y}) P(\vec{y}) \tag{60}$$

there is, corresponding to any "smooth" propagator $D(x, y)$ a related, "oscillating" one $P(\vec{x}) D(\tilde{x}, \tilde{y}) P(\vec{y})$, where $\tilde{x} = (x^0, -\vec{x})$. (Actually, there are $2^3$ degenerate solutions of (59),

corresponding to sign oscillations along any combination of lattice directions.) A unique propagator can be found in the continuum limit by imposing a momentum cut-off (smoothness in $\vec{x}$).

In the continuum limit the propagator equation (59) reduces to the standard one,

$$D(x, y)\,(i\overleftarrow{\partial}_y + m) = -i\delta^4(x-y). \tag{61}$$

To find out how our propagators (41) are related to the standard Feynman propagators,

$$D_F^{\pm}(x, y) = i\int \frac{d^4k}{(2\pi)^4}\,\frac{\exp\left[-ik\cdot(x-y)\right]}{k^2-m^2\pm i\varepsilon}\,(\not{k}+m) \tag{62}$$

we should study their behaviour for $x^0 \to y^0$. Using (23),

$$\lim_{x^0-y^0\to 0+} D(x, y) = \lim_{n\to\infty} \frac{\left\langle \gamma^0\bar{\psi}(x)C_0, x^0 | \bar{\psi}(y)C_0, x^0 - \dfrac{\Delta t}{n}\right\rangle}{\langle C_0, +\infty | C_0, -\infty\rangle} = \tfrac{1}{2}\gamma^0\delta(\vec{x}, \vec{y}),$$

$$\lim_{x^0-y^0\to 0-} D(x, y) = -\tfrac{1}{2}\gamma^0\delta(\vec{x}, \vec{y}) \tag{63}$$

for any choice of $C_0 = C_{\pm}, C_P$. Comparing this with

$$\lim_{x^0-y^0\to 0\pm} [D_F^+(x, y) + D_F^-(x, y)] = \pm\gamma^0\delta^3(\vec{x}-\vec{y}) \tag{64}$$

we conclude that in the continuum limit,

$$D(x, y) = \tfrac{1}{2}[D_F^+(x, y) + D_F^-(x, y)]. \tag{65}$$

It is interesting that a formalism based on this half retarded plus half advanced propagator was studied long ago by Wheeler and Feynman [8]. Their "absorber" could correspond to the nontrivial boundary condition $C_0 \neq 1$.

### 5. *Conclusions*

The path integral formulations for fermions and bosons are formally very similar. The fact that the fermion variables are Grassmann numbers leads, however, to qualitative differences in the interpretation of the integrals as transition amplitudes. The boundary conditions for bosons are specified by an ordinary function over space: the field configuration at a given time. For fermions, the occupation number (0 or 1) at each point should be given instead. With fermion states labeled by products of Grassmann variables (cf. Eq. (33)) this is indeed the case. The time evolution of a fermion state is completely determined by the background gauge field, and expressed in terms of the Hamiltonian (36) operating on the Grassmann product.

Just as for bosons, the boundary conditions imposed at $t = \pm\infty$ affect the type of Green functions one obtains. The requirements (42) of time and space translational in-

variance and $\psi \leftrightarrow \bar{\psi}$ symmetry severely restricts the possible choices. We found three "vacuum" states (50) satisfying these conditions. They all led to the same free propagator (65), which is the average of the Feynman propagators with $\pm i\varepsilon$ proscriptions. Hence it seems that perturbation theory with the above lattice regularization method will be different from the standard one ($+i\varepsilon$ prescription). Presumably the results for physical quantities ($S$-matrix elements) will be the same, however.

The evolution equation (24) expresses the dependence of the fermion determinant on the gauge field. Hence it may be useful in determining the extrema of the effective action, with the fermion degrees of freedom integrated out. In particular, the connection of such "classical" solutions to the fermion bound states should be investigated.

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