A RELATIONSHIP BETWEEN THE ELECTROVACUUM ERNST EQUATIONS AND NONLINEAR σ-MODEL

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The Ernst equations for the stationary Einstein-Maxwell fields are shown to be connected with the field theory of harmonic map from the Riemann three-space (M, h_{ik}) into a Kähler symmetric space SU(2,1)/S(U(2) × U(1) i.e. the non-linear σ-model on symmetric space of the covariance group of these equations; similarly, the vacuum Ernst equations can be interpreted as the σ-model on SU(1,1)/U(1). The formulation of Ernst's equations in terms of SU(2,1)-valued field subject to a quadratic constraint, which realizes an embedding of a Kähler symmetric space into the covariance group is given. For the axially symmetric fields it is shown that Ernst's equations are integrable. The Zakharov-Shabat null-curvature representation of these equations and the corresponding linear system of equations is presented. It is conjectured that an infinite-dimensional group of transformations acting on the solution space of the Ernst equations, which appears naturally in this approach, is connected with the Geroch-Kinnersley group.

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1. Introduction

The highly nonlinear character of the Einstein equations makes it difficult to obtain a sufficiently general class of exact solutions in General Relativity. However, the physical applications of the theory depend largely on the possibilities to find such a class of solutions to the field equations. During the last decade a substantial progress has been made in establishing the structure of vacuum and electrovacuum fields which have Killing vectors. It has been found for this case that field equations have a large hidden symmetry group [5, 8, 16–18, 30]. The stationary and axisymmetric vacuum and electrovacuum solutions to the field equations aroused much of interest, because of their relevance in astrophysics. A large class of transformations which permit to generate a new stationary and axisymmetric solutions from a given one is now known [16, 17, 24–27, 13–15, 34].

Ehlers and Harrison [8, 18] were the first to discover such transformations for vacuum

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and electrovacuum solutions, respectively. The complex potential formulation [12, 13] of stationary and axisymmetric field problem introduced by Ernst made the covariance of the field equations under the internal group G of transformations transparent. Geroch in his excellent paper [16] has shown that the hidden G = SO(2,1) = SL(2, **R**) = SU(1,1) symmetry is a remarkable feature of all stationary fields. Geroch noted also, that for the stationary and axisymmetric fields there appears the group A = SL(2, **R**) of linear transformations which acts in a two-dimensional vector space of commuting Killing vectors. This created a possibility of extending the internal covariance group G = SL(2, **R**) to the infinite-dimensional Geroch group K [17]. Kinnersley [24] has shown that the electrovacuum Ernst equations are covariant under the G = SU(2, 1) symmetry group. It turned out that for the stationary, axisymmetric electrovacuums an infinite-dimensional Kinnersley K′ group may be introduced, having the Geroch group K as a subgroup [25–27]. The appearance of such a rich internal structure of the field equations has been systematically explored to produce solution generating methods [22, 25–27]. The structure of K(K′) group was examined in detail by constructing a convenient representation for it and its Lie algebra [12–15, 25–27]. Currently, the effort of research on stationary and axisymmetric vacuum and electrovacuum solutions concentrates on two lines. One approach is that of Ernst, Geroch and Kinnersley. It tends to find all (asymptotically flat, at least) solutions to the field equations by using the K(K′) group representation which has been worked out in the complex Ernst potential approach [12–15, 25–27]. Numerous interesting solutions to the field equations have been found in this approach [6, 22, 25–27]. A second approach is based on the inverse method of Shabat and Zakharov [2, 3, 1, 32, 37] and the Backlund transformation method [20, 29, 34]. It seems that all these methods are somehow related [6, 28]. As it is well known, a number of two-dimensional field theories can be integrated by means of the inverse method [31]. This is the case when field equations can be represented as compatibility conditions of a certain pair of linear equations. Zakharov and Mikhailov [31] and Zakharov and Shabat [37] have shown that very general class of integrable models appears when one considers a null-curvature conditions for affine connections in certain fiber bundles. The Einstein and Einstein-Maxwell equations for stationary, axisymmetric fields have been integrated in this scheme [1–3].

The purpose of this paper is to show that the Ernst equations for vacuum and electrovacuum fields may be interpreted as σ-model equations on the symmetric spaces of noncompact type SU(1,1)/U(1) and SU(2,1)/S(U(2)×U(1)), respectively. We will develope the G (G′) group covariant formalism for the Einstein and Einstein-Maxwell equations for stationary fields. The effective Lagrangians covariant under G (G′) groups describing three-dimensional gravity coupled to nonlinear σ-models will be presented. The Ernst equations can be derived from these effective Lagrangians. First of all, we observe that these effective Lagrangians are covariant under the nonlinear homographic representations of G and G′ groups. This reminds us σ-models on the complex projective spaces CP(1) and CP(2) defined in terms of nonhomogeneous complex coordinates. Recently the vacuum Ernst equations were considered in the context of σ-model connected with the pseudosphere (Lobaczewski geometry) [4, 21]. However, this result is contained implicite in the Matzner and Misner paper [30]. They have shown that the stationary, axisymmetric vacuum Einstein field equations are connected with the harmonic mapping between $\mathbf{R}^3/SO(2)$ and the hyperboloid in Minkowski 3-space (for the definition of the harmonic mapping we reffer the reader to the paper by Misner [33]). We will show that the Ernst equations can be described as the principal σ-model equations restricted to invariant submanifolds of their covariance groups SU(1,1) and SU(2,1) for vacuum and electrovacuum case respectively. We introduce also the projector formalism developed previously for Grassmannian σ-models [9, 31], for σ-models on the hermitian hyperbolic spaces SU(1,1)/U(1) and SU(2,1)/S(U(2) × U(1)). The Shabat-Zakharov pair of linear equations for the Ernst equations will be introduced The Ernst equations for stationary and axisymmetric fields are represented as compatibility conditions of these linear equations. It turns out that one has to solve the homogeneous Riemann-Hilbert problem on a two-fold Riemann surface in order to generate solutions for the Ernst equations. The solutions of this problem present an infinite-dimensional representation of a hidden symmetry group of the field equations [31, 37, 38].

The paper is organized as follows: in Sect. 2 we briefly introduce the Ernst equations for electrovacuum case. We present the effective action for these equations. Subsect. 2.1 presents the formulation of Ernst's equations as the Kähler σ-model. We discuss shortly the isometry group G [7] of the Kähler manifold connected with this model. It turns out that the group G is a covariance group SU(2,1) of the Ernst equations. The Ernst potentials may be regarded as a certain map on the Kähler manifold $\mathscr{H}$ [33], which is proved to be a homogeneous space of the isometry group G. In Subsect. 2.2 we discuss Ernst's equations in the framework of theory of symmetric spaces. We show that the Kähler σ-model discussed in Subsect. 2.1 can be described in terms of a single G-valued field g which is subject to a quadratic constraint. This constraint realize an embedding of symmetric space $\mathscr{H}$ into the group G. In Sect. 3 we consider the case of axially symmetric fields. The Shabat-Zakharov null-curvature representation has been found for the Ernst equations. We present the solution generating for Ernst's equations using their equivalence to the σ-model on a symmetric space. This method is essentially due to Zakharov and Shabat [37] and Ward [38] in another context, also. An infinite-dimensional group acting on the space of solutions arises naturally. In Sect. 4 we conclude our discussion of the Ernst equations.

2. *The Lagrangian formulation of the field equations*

We consider the Einstein and Einstein-Maxwell equations for stationary fields. These equations have a large hidden symmetry group [16, 17, 24]. Using the formulation of these equations given by Ernst [10, 11] and Israel and Wilson [23] we show that there exists a Lagrangian formulation of the Ernst equations, which makes their inner symmetry transparent. As it is well known, these equations may be written as equations in a three-dimensional space (M, h_{ik}) for the Ernst potentials [10, 11, 23, 24]. The Ernst potentials appear as an immediate consequence of the fact, that a part of the Einstein and Maxwell equations can be interpreted as integrability conditions.

Below we consider the electrovacuum case, considering the vacuum case as a special case of the former, by setting the complex electromagnetic potential ψ equal to 0. The

Einstein-Maxwell equations written in terms of the Ernst potentials ε and ψ yield [11, 24]

$$f\nabla^2\varepsilon = (\nabla\varepsilon+2\bar{\psi}\nabla\psi)\cdot\nabla\varepsilon, \tag{2.1}$$

$$f\nabla^2\psi = (\nabla\varepsilon+2\bar{\psi}\nabla\psi)\cdot\nabla\psi, \tag{2.2}$$

$$f^2R_{ik} = \tfrac{1}{2}\,\varepsilon_{,(i}\bar{\varepsilon}_{,k)}+\psi\varepsilon_{,(i}\bar{\psi}_{,k)}+\bar{\psi}\bar{\varepsilon}_{,(i}\psi_{,k)}-2Re\varepsilon\psi_{,(i}\bar{\psi}_{,k)}, \tag{2.3}$$

where ∇ and "$\cdot$" denote the covariant derivative and scalar product in the three-space (M, h_{ik}) respectively. f is given by the formula

$$f = \mathrm{Re}\,\varepsilon+\psi\bar{\psi}. \tag{2.4}$$

These equations can be derived from a variational principle[1]. Consider the following action integral

$$S = S_{\mathrm{E}}+S_{\sigma}, \tag{2.5}$$

where S_E is the Einstein-Hilbert action for the three-dimensional gravity

$$S_{\mathrm{E}} = \int\sqrt{h}\,d^3xR \tag{2.6}$$

and S_σ is the action of a certain field theory model of purely geometrical nature, namely the Kähler σ-model [27]

$$S = -\tfrac{1}{2}\int\sqrt{h}\,d^3xf^{-2}(\nabla\varepsilon\cdot\nabla\bar{\varepsilon}+2\psi\nabla\varepsilon\cdot\nabla\bar{\psi}+2\bar{\psi}\nabla\bar{\varepsilon}\cdot\nabla\psi-4\,\mathrm{Re}\,\varepsilon\nabla\psi\cdot\nabla\bar{\psi}). \tag{2.7}$$

One can easily check that Eqs. (2.1) and (2.2) follow from the action S_σ, because S_E does not depend on ε and ψ. Eq. (2.3) can be derived from the action S. Taking variations of S with respect to h^{ik} one gets the three-dimensional Einstein equations with the "energy-momentum tensor" of the "matter" fields ε and ψ on the right-hand side

$$R_{ik}-\tfrac{1}{2}\,h_{ik}R = -\left(\frac{\delta L}{\delta h^{ik}}-\tfrac{1}{2}\,h_{ik}L_\sigma\right). \tag{2.8}$$

Contracting Eq. (2.8) with h^{ik} and using homogeneity of L_σ i.e. $h^{ik}\dfrac{\delta L_\sigma}{\delta h^{ik}} = L_\sigma$ we have

$$R+L_\sigma = 0. \tag{2.9}$$

Using this in (2.8) we obtain Eq. (2.3). Thus we have proved that the stationary Einstein-Maxwell equations are equivalent to the three-dimensional gravity coupled to certain nonlinear fields. Similar results hold for the vacuum case. Eq. (2.3) may be considered as a constraint for solutions of Eqs. (2.1) and (2.2). In general this constraint is too complicated to be solved explicitly apart from the axisymmetric case when it becomes trivial in this case Eq. (2.3) turns out to be integrable by quadratures.

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[1] The Lagrangian formulations of vacuum and electrovacuum cases were studied by Ernst [10] and Carter [39], respectively.

We shall prove now that the action integral (2.7) is that of the nonlinear σ-model on the symmetric space $\mathscr{H} = SU(2,1)/S(U(2)\times U(1))$. For the vacuum case the symmetric space $\mathscr{H}$ turns out to be the $SU(1,1)/U(1)$ symmetric space. We shall consider the model (2.7) from several different points of view, taking into account the following formulations:

2.1. The Ernst equations as equations for the Kähler σ-model

Let us consider an almost complex manifold $\mathscr{H}$ with a Hermitian metric $k_{\alpha\bar{\beta}}$. The metric on $\mathscr{H}$ may be written in local coordinates as follows

$$ds^2 = k_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta. \tag{2.10}$$

A complex Hermitian manifold is a Kähler manifold [7] if it admits a closed nondegenerate 2-form ω related to the imaginary part of $k_{\alpha\bar{\beta}}$

$$\omega = \frac{i}{2} k_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \tag{2.11}$$

$$d\omega = 0. \tag{2.12}$$

The condition (2.12) is equivalent to the equations

$$\frac{\partial k_{\alpha\bar{\beta}}}{\partial z^\gamma} = \frac{\partial k_{\gamma\bar{\beta}}}{\partial z^\alpha}, \qquad \frac{\partial k_{\alpha\bar{\beta}}}{\partial \bar{z}^\gamma} = \frac{\partial k_{\alpha\bar{\gamma}}}{\partial \bar{z}^\beta}. \tag{2.13}$$

Then the Kähler σ-model is determined by the action integral [35, 36]

$$S = c \int \sqrt{h}\, d^3x k_{\alpha\bar{\beta}} \nabla z^\alpha \cdot \nabla \bar{z}^\beta, \tag{2.14}$$

where c is a real constant and "$\cdot$" denotes the scalar product in 3-space (M, h_{ik}). We shall prove that (2.7) describes the σ-model on a Kähler manifold $\mathscr{H}$ of a noncompact type. The Ernst potentials ε and ψ may be interpreted as local coordinates on $\mathscr{H}$: $z_1 = \varepsilon$, $z_2 = \psi$. Putting in Eq. (2.14) $c = -1/2$ we can see that the metric $k_{\alpha\bar{\beta}}$ related to (2.7) has the form

$$(k_{\alpha\bar{\beta}}) = f^{-2}\begin{pmatrix} 1 & 2\psi \\ 2\bar{\psi} & -2(\varepsilon+\bar{\varepsilon}) \end{pmatrix} = \left(\frac{z^1+\bar{z}^1}{2} + z^2\bar{z}^2\right)^{-2}\begin{pmatrix} 1 & 2z^2 \\ 2\bar{z}^2 & -2(z^1+\bar{z}^1) \end{pmatrix}. \tag{2.15}$$

One can show that the metric (2.15) is a Kähler metric. The metric $k_{\alpha\bar{\beta}}$ can be represented in a more symmetric form by means of a holomorphic transformation of local coordinates $z^\alpha (\alpha = 1,2)$. This form enables us to identify the isometry group of the manifold $\mathscr{H}$ as the SU(2,1) group. The new coordinates are introduced by the following holomorphic map $z^\alpha = z^\alpha(w^\beta)$, $\alpha = 1,2$

$$z^1 = \frac{w^1-1}{w^1+1}, \qquad z^2 = \frac{w^2}{w^1+1}. \tag{2.16}$$

They are related to the field variables studied by Kinnersley [24]. We denote the coordinates w^1, w^2 by ξ and η respectively. The action integral (2.7) in terms of ξ and η takes on the particularly symmetric form

$$S = -2\int \sqrt{h}\, d^3x(\xi\bar{\xi}+\eta\bar{\eta}-1)^{-2}(\nabla\xi\cdot\nabla\bar{\xi}+\nabla\eta\cdot\nabla\bar{\eta}$$
$$-(\xi\nabla\eta-\eta\nabla\xi)\cdot(\bar{\xi}\nabla\bar{\eta}-\bar{\eta}\nabla\bar{\xi}). \qquad (2.17)$$

Hence, the Kähler metric $k_{\alpha\bar{\beta}}$ on $\mathscr{H}$ in the new coordinates w^α has the form

$$(k_{\alpha\bar{\beta}}) = (\xi\bar{\xi}+\eta\bar{\eta}-1)^{-2}\begin{pmatrix}1-\eta\bar{\eta} & \eta\bar{\xi}\\ \xi\bar{\eta} & 1-\xi\bar{\xi}\end{pmatrix}. \qquad (2.18)$$

Now, we will study the isometry group G of the comlex Kähler manifold $\mathscr{H}$ endowed with the metric (2.18).

Holomorphic transformations of local coordinates w^α which leave the metric invariant, form an isometry group G of the manifold $\mathscr{H}$. Here a composition rule of mappings of $\mathscr{H}$ onto itself plays a role of a group product. It turns out that the metric (2.18) is invariant under the rational, homographic maps of $\mathscr{H}$ onto itself

$$w^{1'} = \frac{u^1_1w^1+u^1_2w^2+u^1_3}{u^3_1w^1+u^3_2w^2+u^3_3}, \qquad w^{2'} = \frac{u^2_1w^1+u^2_2w^2+u^2_3}{u^3_1w^1+u^3_2w^2+u^3_3}, \qquad (2.19)$$

where the matrix $u = (u^\mu_\nu)$, $\mu, \nu = 1, 2, 3$ satisfies the following condition

$$u^+\eta u = \eta, \qquad (2.20)$$

"+" denotes hermitian conjugation of a matrix and $\eta = (\eta_{\mu\nu})$ is the diagonal matrix $\eta = \text{diag}(1, 1, -1)$. The transformations (2.19) present a nonlinear homographic realization of the isometry group G. It is clear that the phase factor in u is immaterial and we can choose u to satisfy the condition: $\det u = 1$. The conditions stated above imply that the isometry group G of $\mathscr{H}$ is isomorphic to SU(2,1). It should be noted that the action integral (2.17) is determined by the SU(2,1)-invariant Kähler structure on $\mathscr{H}$.

The isometry group SU(2,1) acts on $\mathscr{H}$ transitively i.e. it can transform a point on $\mathscr{H}$ into an arbitrary point of $\mathscr{H}$. We take a fixed point p_0 of $\mathscr{H}$ with the coordinates $w^\alpha(p_0) = w^\alpha_0 = (0, 0)$. Then an arbitrary point p of $\mathscr{H}$ can be represented by $p = up_0$, where u is an element of G = SU(2,1), or in the local coordinates w^α

$$w^1 = \frac{u^1_3}{u^3_3}, \qquad w^2 = \frac{u^2_3}{u^3_3}. \qquad (2.21)$$

Hence, $\mathscr{H}$ is a connected homogeneous space of a noncompact Lie group G = SU(2,1). Consider the isotropy subgroup H of the point p_0. H is a subgroup of G which leaves p_0 fixed i.e. H = $(h : hp_0 = p_0)$. Two elements u and u' of G represent the same point of $\mathscr{H}$ if $u' = uh$, where h belongs to the subgroups H. Then the space $\mathscr{H}$ is diffeomorphic to the left coset space G/H. One can show that the isotropy subgroup H of the point p_0 is isomorphic to S(U(2)×U(1)) e.g. H $\ni h_2h_1$, where $h_2 \in$ U(2), $h_1 \in$ U(1) and $\det h = 1$. Let us note, that the global group G acts linearly on suitable coordinates on G/H. We could regard u^μ_3, $\mu = 1, 2, 3$ as the homogeneous coordinates on G/H = $\mathscr{H}$.

Thus we have proved that the action integral (2.17) describes the nonlinear σ-model on the homogeneous space $\mathscr{H} = SU(2,1)/S(U(2)\times U(1))$. Similarly, the analogous result holds for the vacuum case when we have $\eta = 0$, and the isometry group of $\mathscr{H}$ is isomorphic to SU(1,1); $\mathscr{H}$ is the homogeneous space SU(1,1)/U(1) for this case.

Finally, it should be noted that $\mathscr{H}$ is not only a homogeneous space but a symmetric space also. One could in principle study the model under investigation formulated in terms of another coordinates on G/H, in which the group G acts linearly on the coset space G/H.

2.2. The σ-model on a symmetric space of the Kinnersley group G

The theory of symmetric spaces provides a convenient framework for disscussing the inner symmetries of the σ-models. Description of the model (2.17) in the language of symmetric spaces has many attractive features. It enables us to apply the results obtained recently for the classical two-dimensional σ-models [9, 31] to an analysis of the problem of complete integrability of the Ernst equations for the axisymmetric fields.

We shall study the action integral (2.17) in the new coordinates v^μ, $\mu = 1, 2, 3$ on $\mathscr{H}$ defined by

$$w^1 = \frac{v^1}{v^3}, \quad w^2 = \frac{v^2}{v^3}. \tag{2.22}$$

w^α are the standard coordinates on the space of equivalence classes of complex vectors $\boldsymbol{v} = (v^1, v^2, v^3) \neq 0$. We say that two vectors $\boldsymbol{v}, \boldsymbol{v}'$ are equivalent if there exists a complex number c such that $\boldsymbol{v}' = c\boldsymbol{v}$. The action of isometry group G on $\mathscr{H}$ is now linearized by means of introduction of the homogeneous coordinates v^μ on $\mathscr{H}$: $\boldsymbol{v}' = u\boldsymbol{v}$, where $\boldsymbol{u}$ belongs to the linear matrix representation of the group G = SU(2,1). A homogeneous space $\mathscr{H} = G/H$ is diffeomorphic to the space of equivalence classes of complex vectors with the SU(2,1)-invariant metric structure. The action integral (2.17) takes on the manifestly SU(2,1)-invariant form

$$S = -4c \int \sqrt{h}\, d^3x \langle \boldsymbol{v}, \boldsymbol{v}\rangle^{-2}(\langle \boldsymbol{v}, \boldsymbol{v}\rangle \langle \nabla\boldsymbol{v}, \nabla\boldsymbol{v}\rangle - \langle \boldsymbol{v}, \nabla\boldsymbol{v}\rangle \langle \nabla\boldsymbol{v}, \boldsymbol{v}\rangle), \tag{2.23}$$

where $\langle \boldsymbol{u}, \boldsymbol{v}\rangle$ denotes the SU(2,1)-invariant scalar product in the space of complex vectors $\boldsymbol{v} = (v^1, v^2, v^3) \neq 0$

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \bar{u}^\mu \eta_{\mu\nu} v^\nu, \quad \eta = (\eta_{\mu\nu}) = \operatorname{diag}(1, 1, -1). \tag{2.24}$$

A complex vector $\boldsymbol{v}$ determines the projector P

$$P = \boldsymbol{v} \otimes \bar{\tilde{\boldsymbol{v}}}/\langle \boldsymbol{v}, \boldsymbol{v}\rangle, \tag{2.25}$$

where

$$\tilde{\boldsymbol{v}} = \eta \boldsymbol{v}, \quad \tilde{v}_\alpha = \eta_{\alpha\beta} v^\beta, \quad (\boldsymbol{v} \otimes \bar{\tilde{\boldsymbol{v}}})^\alpha_\beta = v^\alpha \bar{\tilde{v}}_\beta. \tag{2.26}$$

In virtue of (2.25) P satisfies

$$\operatorname{Tr} P = P^\alpha_\alpha = 1, \quad (\eta P)^+ = \eta P, \quad P^2 = P. \tag{2.27}$$

P transforms under the SU(2,1) linear transformation as follows

$$P' = uPu^{-1} \quad \text{if} \quad v' = uv. \tag{2.28}$$

Using the identity

$$\tfrac{1}{2}\operatorname{Tr}(\nabla P \cdot \nabla P) = \langle v, v\rangle^{-2} (\langle v, v\rangle \langle \nabla v, \nabla v\rangle - \langle v, \nabla v\rangle \langle \nabla v, v\rangle), \tag{2.29}$$

one can write Eq. (2.23) in the form

$$S = -2c \int \sqrt{h}\, d^3x \operatorname{Tr}(\nabla P \cdot \nabla P). \tag{2.30}$$

We introduce the matrix g [9, 31]

$$g = \mathrm{I} - 2P, \tag{2.31}$$

where I is the unit matrix. It follows from (2.27) that the matrix g should satisfy the conditions

$$g^2 = \mathrm{I}, \quad g^+\eta g = \eta, \quad g^+ = \eta g \eta^{-1}. \tag{2.32}$$

One can see now that g is an element of G = SU(2,1) satisfying a quadratic constraint. This leads us to the conclusion that the σ-model (2.17) may be described in terms of matrix G-valued field g which is subject to a quadratic constraint. Substituting g to Eq. (2 30) one obtains an action integral for the σ-model on the symmetric space SU(2,1)/S(U(2) × U(1))

$$S = -1/2c \int \sqrt{h}\, d^3x \operatorname{Tr}(\nabla g \cdot \nabla g^{-1}). \tag{2.33}$$

To conclude this section let us describe the relationship between the present formulation of the σ-model (2.33) and the previous one (2.17). As it was shown in Subsect. 2.1, the action integral (2.17) describes the nonlinear σ-model on $\mathscr{H}$ = SU(2,1)/S(U(2) × U(1)). The constraint (2.32) realizes an embedding of symmetric space into the group SU(2,1) as a closed totally geodesic submanifold of the later. As an immediate consequence of this we have that the space of solutions of the σ-model on the symmetric space of G is a subspace of the space of solutions of the principal σ-model on G. In other words the constraint (2.32) is compatible with the Euler-Lagrange equations of the action integral (2.33) [9, 31].

Define the symmetric space of G as the coset space G/H, where H is a fixed subgroup of an involutive authomorphism of group G. Then the embedding of G/H into G is realized by the map

$$\varphi : \mathrm{G/H} \to \mathrm{G}, \quad \mathrm{G} \ni g = \varphi(\mathrm{H}u) = \varrho(u^{-1}) g_0 u, \tag{2.34}$$

where ϱ is an involutive automorphism of G and g_0 is a fixed element of G satisfying

$$g_0 \varrho(g_0) = \lambda \mathrm{I}, \tag{2.35}$$

where λ is a real or complex number. From Eqs. (2.34) and (2.35) it follows that g satisfies the quadratic constraint

$$g \varrho(g) = \lambda \mathrm{I}. \tag{2.36}$$

H is that subgroup of G which leaves g_0 invariant

$$g_0 = \varrho(h^{-1})g_0 h, \quad k \in \mathrm{H}, \tag{2.37}$$

or H is a fixed subgroup of G invariant with respect to the involutive automorphism

$$\sigma : u \to \sigma(u) = g_0^{-1}\varrho(u)g_0. \tag{2.38}$$

In the case under consideration an involutive automorphism ϱ is trivial i.e. $\varrho = \mathrm{id}$, ϱ is the identity map. The following choice of g_0 defines an involutive automorphism such that the subgroup H of G = SU (2, 1) is isomorphic to S(U(2) × U(1))

$$g_0 = \mathrm{diag}\,(-1, 1, 1). \tag{2.39}$$

This proves that the quadratic constraint $g^2 = \mathrm{I}$ realizes the embedding of $\mathscr{H}$ into G = SU(2,1). Finally, it should be noted that the choice of g_0 (Eq. (2.39)) corresponds, by Eqs. (2.22), (2.25), (2.31), to the choice of a fixed point p_0 on $\mathscr{H}$ with the coordinates $w_0^\alpha = (0, 0)$.

3. *The null-curvature representation of the Ernst equations for the axisymmetric fields*

We shall study the Ernst equations for the stationary and axisymmetric Einstein--Maxwell fields. One could consider the Ernst equations as the equations for the nonlinear σ-model on symmetric space SU(2,1)/S(U(2) × U(1)). As it is well known, the 2-dimensional principal σ-models as well as σ-models on symmetric spaces can be integrated by the method developed by Shabat and Zakharov [31, 32, 37]. It turns out that this method can be applied to the 3-dimensional σ-model with axial symmetry [3, 32].

In Sect. 2 we have shown that there exists a natural SU(2,1)-invariant formulation of the Ernst equations in terms of SU(2,1)-valued field g, restricted by a quadratic constraint.

The action integral for the Ernst equations takes on the particularly simple form

$$S = -\tfrac{1}{2}c \int \varrho d^2x \,\mathrm{Tr}\,(\partial_\mu g \partial_\mu g^{-1}), \quad \mu = 1, 2, \tag{3.1}$$

if one considers axisymmetric fields; g is SU(2,1)-valued field, which is subject to the quadratic constraint (2.32). The field ϱ satisfies the 2-dimensional Laplace equation

$$\partial_\mu \partial_\mu \varrho = 0, \tag{3.2}$$

where x^μ are local conformal coordinates on a 2-space of orbits of the isometry group generated by two Killing vectors [16, 17] i.e. the coordinates x^μ are chosen on a 2-space of orbits in such a way that the metric has the conformally Euclidean form. The action integral (3.1) leads to the field equations

$$\partial_\mu(\varrho \partial_\mu g)g^{-1} - \varrho \partial_\mu g g^{-1} \partial_\mu g g^{-1} = 0. \tag{3.3}$$

These equations can be written in the more concise form when one introduces the right--invariant current on G/H

$$A_\mu = \partial_\mu g g^{-1}. \tag{3.4}$$

A_μ may be considered to be an element of the Lie algebra of G. The "curvature" $F_{\mu\nu}$ of A_μ is equal to zero

$$F_{\mu\nu} \doteq \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = 0. \tag{3.5}$$

It follows from Eqs. (3.3) and (3.4) that the field equation expressed in terms of A_μ field takes a simple form

$$\partial_\mu(\varrho A_\mu) = 0. \tag{3.6}$$

One could solve Eq. (3.5) by the "inverse" method. The procedure is as follows:

(*i*) Construct a set of sufficiently locally regular G-valued fields g on a 2-space.

(*ii*) Insert the constructed field g into the formula for the current A_μ. The assumption of proper regularity of g ensures the solution of Eq. (3.5) to be sufficiently regular.

The point is that the Eqs. (3.5) and (3.6) may be solved by the same procedure if one is able to cast them into the null-curvature form. The idea is to find an appropriate system of linear equations with compatibility conditions equivalent to Eqs. (3.5) and (3.6).

It is convenient to introduce complex coordinates ζ_1 and ζ_2

$$\zeta_1 \doteq x_1 + \dot{i}x_2, \quad \zeta_2 \doteq \bar{\xi}_1 = x_1 - \dot{i}x_2. \tag{3.7}$$

Then, Eqs. (3.5) and (3.6) can be written in the form

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 - [A_1, A_2] = 0, \tag{3.8}$$

$$\partial_1 A_2 + \partial_2 A_1 + \varrho^{-1}(\varrho_{,1} A_2 + \varrho_{,2} A_1) = 0, \tag{3.9}$$

where A_μ are components of the connection in the new coordinates. We can now introduce a one-parameter family of currents $A_\mu(\lambda)$ depending on a spectral parameter λ as follows

$$A_\mu(\lambda) = A_\mu^\nu(\lambda) A_\nu, \tag{3.10}$$

where the matrix $A(\lambda)$ is given by

$$A(\lambda) = \begin{pmatrix} (\mu+1)^{-1} & 0 \\ 0 & -(\mu-1)^{-1} \end{pmatrix}. \tag{3.11}$$

μ depends on the spectral parameter λ and the harmonic functions ϱ and z

$$\mu = \mu(\lambda) = \varrho^{-1}(\lambda + iz + \sqrt{(\lambda+iz)^2 - \varrho^2}). \tag{3.12}$$

where z is the harmonic conjugate function of ϱ

$$z_{,1} = -i\varrho_{,1}, \quad z_{,2} = i\varrho_{,2}. \tag{3.13}$$

The spectral parameter λ is defined on the two-fold Riemann surface $\mathscr{R}$ with branching points at $\lambda = -\xi_1$ and $\lambda = \xi_2$, where

$$\xi_1 \doteq \varrho + iz, \quad \xi_2 \doteq \bar{\xi}_1 = \varrho - iz. \tag{3.14}$$

It is not difficult to verify, using Eqs. (3.10), (3.11), (3.12) and (3.13), that the field equations (3.8) and (3.9) must be satisfied iff $A_\mu(\lambda)$ has a null curvature

$$F_{\mu\nu}(\lambda) \doteq \partial_\mu A_\nu(\lambda) - \partial_\nu A_\mu(\lambda) - [A_\mu(\lambda), \quad A_\nu(\lambda)]. \tag{3.15}$$

The partial-fraction decomposition of $F_{\mu\nu}(\lambda)$ with respect to μ yields two terms which must vanish if A_μ satisfies (3.8) and (3.9). It turns out that (3.15) is a compatibility condition for the system of linear equations

$$\partial_\alpha \psi(\lambda) = A_\alpha(\lambda)\psi(\lambda). \tag{3.16}$$

Thus we have found the Zakharov-Shabat null-curvature representation for the field equations of nonlinear σ-model (Eq. (3.1)) i.e. for the Ernst equations (2.1), (2.2).

It should be instructive to present another formulation of the Zakharov-Shabat pair of linear equations for Eq. (3.3). This formulation is similar to that presented by Bielinsky and Zakharov [3]. The compatibility condition of the following linear system is equivalent to the field equations (3.8) and (3.9)

$$L_\alpha \tilde{\psi}(\mu) = A_\alpha(\mu)\tilde{\psi}(\mu), \tag{3.17}$$

where μ plays now the role of a spectral parameter and the linear operators L_α are defined as follows

$$L_1 \doteq \partial_1 - \varrho^{-1}\varrho_{,1}\mu \frac{\mu-1}{\mu+1}\frac{\partial}{\partial\mu}, \tag{3.18}$$

$$L_2 \doteq \partial_2 - \varrho^{-1}\varrho_{,2}\mu \frac{\mu+1}{\mu-1}\frac{\partial}{\partial\mu}. \tag{3.19}$$

L_1 and L_2 commute if ϱ is a harmonic function

$$[L_1, L_2] = 0 \quad \text{if} \quad \varrho_{,12} = 0. \tag{3.20}$$

One can easily check that certain relationship between solutions to Eqs (3.16) and (3.17) holds

$$L_\alpha \tilde{\psi}(\mu) = \partial_\alpha \psi(\lambda), \quad \tilde{\psi}(\mu(\lambda)) = \psi(\lambda). \tag{3.21}$$

In general, one would expect the solution ψ of (3.16) or (3.17) to be an element of the general linear group GL(3,C). One should find restrictions on the solutions ψ of (3.16) implied by the conditions that g is an element of SU(2,1) group subject to a quadratic constraint. It follows immediately from (3.17) and (2.32) that the matrix function $\tilde{\psi}$ should satisfy the conditions

$$g\tilde{\psi}\left(\frac{1}{\mu}\right) = \tilde{\psi}(\mu)C(\lambda), \quad \eta^{-1}\tilde{\psi}^{+}(-\bar{\mu})\eta = \tilde{\psi}^{-1}(\mu), \tag{3.22}$$

where $C(\lambda)$ is a matrix subject to a certain constraints implied by Eqs. (3.12), (3.21) and (3.22). Let us note that the involution in the complex μ-plane

$$\mu \to \frac{1}{\mu}, \tag{3.23}$$

defines the involution $*$ on the two-fold Riemann surface

$$* : \lambda \to \lambda^{*}, \quad \mu(\lambda^{*}) = \frac{1}{\mu(\lambda)}. \tag{3.24}$$

One can see that the relationship between μ and λ holds

$$\lambda = \tfrac{1}{2}\varrho\left(\mu + \frac{1}{\mu}\right) - iz. \tag{3.25}$$

Then, from (3.12) and (3.25) it follows that * means the change of the sheet of $\mathcal{R}$. In virtue of this we have that the conditions on $\psi(\lambda)$ implied by (2.32) may be written in the form

$$g\psi(\lambda^{*}) = \psi(\lambda)C(\lambda), \quad \eta^{-1}\psi^{+}(-\bar{\lambda})\eta = \psi^{-1}(\lambda). \tag{3.26}$$

With each solution to the field equation (3.3) there is associated the solution $\psi(\lambda)$ of the linear system (3.16) satisfying the conditions (3.26). The remarkable property of the system (3.16) that one can obtain a solution of (3.3) having a solution of (3.16) is the basic point of procedure we describe below. In fact, we can show, using (3.17) and (3.22), that g is given by the formula

$$g = \tilde{\psi}(0), \quad g = \psi(\infty), \tag{3.27}$$

where we have chosen the appropriate sheet of $\mathcal{R}$, such that the point $\mu = 0$ is mapped at $\lambda = \infty$.

We shall describe, following Zakharov and Shabat [37], a method for generating exact solutions of the Ernst equations. We choose a closed analytic contour Γ on the Riemann surface $\mathcal{R}$. Hawing chosen a special solution g_0 of Eq. (3.3) we can find the "current" $A_{\alpha}^{0}(\lambda)$. $A_{\alpha}^{0}(\lambda)$ is singular on $\mathcal{R}$ at the branching points $\lambda = -\xi_1, \xi_2$. Consider a piecewise holomorphic gauge transformation of the current A_{α}^{0}, depending on a spectral parameter λ

$$A_{\alpha}^{1}(\lambda) = \psi_1^{-1}(\lambda)A_{\alpha}^{0}(\lambda)\psi_1(\lambda) - \psi_1^{-1}(\lambda)\partial_{\alpha}\psi_1(\lambda), \tag{3.28}$$

$$A_{\alpha}^{2}(\lambda) = \psi_2^{-1}(\lambda)A_{\alpha}^{0}(\lambda)\psi_2(\lambda) - \psi_2^{-1}(\lambda)\partial_{\alpha}\psi_2(\lambda), \tag{3.29}$$

where ψ_1 and ψ_2 are matrices holomorphic inside and outside the contour Γ respectively. Then, the new current $A_\alpha(\lambda)$

$$A_\alpha(\lambda) = \begin{cases} A^1_\alpha(\lambda) & \text{inside the contour } \Gamma \\ A^2_\alpha(\lambda) & \text{outside the contour } \Gamma, \end{cases} \tag{3.30}$$

is everywhere holomorphic on $\mathscr{R}$ apart of singularities at the branching points $\lambda = -\xi_1, \xi_2$. It is easy to see that the gauge transformed current $A_\alpha(\lambda)$ has a null curvature $F_{\alpha\beta}(\lambda)$. Hence, the gauge transformation generates a new solution of the field equations (3.8) and (3.9). We demand A_α to be defined on the contour Γ by

$$A_\alpha(\lambda) \stackrel{\Gamma}{=} A^1_\alpha(\lambda) \stackrel{\Gamma}{=} A^2_\alpha(\lambda), \tag{3.31}$$

where $\stackrel{\Gamma}{=}$ denotes equality on the contour Γ. It is seen from (3.16) that the formulas (3.28) and (3.29) for the current A_α can be written more consisely introducing two matrices ψ and ψ'

$$A^1_\alpha(\lambda) = \partial_\alpha\psi(\lambda)\psi^{-1}(\lambda), \tag{3.32}$$

$$A^2_\alpha(\lambda) = \partial_\alpha\psi'(\lambda)\psi'^{-1}(\lambda), \tag{3.33}$$

where

$$\psi(\lambda) = \psi_1^{-1}(\lambda)\psi_0(\lambda), \qquad \psi'(\lambda) = \psi_2^{-1}(\lambda)\psi_0(\lambda). \tag{3.34}$$

Since the current A_α satisfies (3.31) one would expect to obtain a certain relationship between the matrices ψ_1 and ψ_2 on the contour Γ. In fact, taking into account (3.31), (3.32) and (3.33) we have

$$\partial_\alpha(\psi'^{-1}\psi) \stackrel{\Gamma}{=} 0. \tag{3.35}$$

We define on the contour Γ a matrix G_0 which, in general, cannot be continued analytically outside a small vicinity of Γ on $\mathscr{R}$, as follows:

$$G_0(\lambda) \stackrel{\Gamma}{=} \psi'^{-1}(\lambda)\psi(\lambda), \qquad \partial_\alpha G_0(\lambda) \stackrel{\Gamma}{=} 0. \tag{3.36}$$

The homogeneous Riemann-Hilbert problem emerges: having a given contour Γ with a matrix G defined on it

$$G(\lambda) \stackrel{\Gamma}{=} \psi_0(\lambda)G_0(\lambda)\psi_0^{-1}(\lambda), \tag{3.37}$$

we have to "split" G

$$G(\lambda) \stackrel{\Gamma}{=} \psi_2(\lambda)\psi_1^{-1}(\lambda), \tag{3.38}$$

where ψ_1 and ψ_2 are matrices holomorphic inside and outside the contour Γ respectively. Now suppose we begin with the solution g_0 of (3.3). Then we have to solve the homogeneous Riemann-Hilbert problem (3.38) for a certain contour Γ endowed with the matrix G_0. The transformed solution g is given in terms of solution of the underlying homogeneous

Riemann-Hilbert problem

$$g = \psi_2^{-1}(\infty)g_0, \tag{3.39}$$

where the appropriate sheet of $\mathscr{R}$ should be chosen carefully.

We would like to be sure that the transformed solution g is an element of SU(2,1) group constrained by (2.32). The procedure described above produces a solution of (3.3) consistent with the constraint if the contour Γ and the matrix G_0 are chosen properly. A discussion of the freedom involved in the construction presented above shows that without loss of generality one can choose the matrix G_0 to satisfy

$$\tilde{G}_0^{-1}(\mu) \stackrel{\Gamma}{=} \eta^{-1}\tilde{G}_0^{+}(-\bar{\mu})\eta, \tag{3.40}$$

or

$$G_0^{-1}(\lambda) \stackrel{\Gamma}{=} \eta^{-1}G_0^{+}(-\bar{\lambda})\eta, \tag{3.41}$$

and

$$\tilde{G}_0\left(\frac{1}{\mu}\right) \stackrel{\Gamma}{=} \tilde{G}_0(\mu), \qquad G_0(\lambda^*) \stackrel{\Gamma}{=} G_0(\lambda), \tag{3.42}$$

where G_0 denotes the generating matrix for solutions of (3.17) and $\tilde{G}_0(\mu) = G_0(\lambda(\mu))$.

Finally, it should be noted that the contour Γ cannot be chosen arbitrarily. It must be preserved by two involutions in the complex μ-plane

$$\mu \to -\bar{\mu}, \qquad \mu \to \frac{1}{\mu}, \tag{3.43}$$

as one can see from (3.40) and (3.42). There is some freedom in the choice of matrix G_0, also

$$G_0 \to CG_0C', \tag{3.44}$$

where C and C' are SU(2,1) matrices depending on λ and are subject to a certain constraints implied by (3.41) and (3.42).

To summarize, the choice of a closed contour Γ endowed with matrix G_0 determines completely a transformation of a given solution of (3.3) into a new one. The idea is to solve the Riemann-Hilbert problem. Then, the new solution is given by (3.39). The Ernst potentials are related to g by the formulas:

$$\xi = \frac{P_3^1}{P_3^3}, \qquad \eta = \frac{P_3^2}{P_3^3}, \qquad \text{for electrovacuum case} \tag{3.45}$$

and

$$\xi = \frac{P_2^1}{P_2^2}, \qquad \eta = 0, \qquad \text{for vacuum case.} \tag{3.46}$$

4. Conclusions

The equivalence of Einstein's equations for the stationary axially symmetric case to the two-dimensional nonlinear σ-model on a symmetric space $\mathscr{H}$ enables us to apply the solution generating techniques developed earlier [31, 32, 37]. We have succeeded in finding the null-curvature representation of the field equations. It is cosiderably easier to study the properties of solutions of (3.16) than those of (3.3). The reason is, as it was pointed out by Geroch [16], that the solutions of (3.3) are labeled by boundary conditions, because it is a system of an elliptic partial differential equations, while the solutions of (3.16) are labeled by contours on the Riemann surface $\mathscr{R}$, endowed with a matrix satisfying certain conditions. Furthermore, a very rich structure arises — an infinite-dimensional group of transformations acting on the space of solutions of (3.16). This group acts linearly on the space of solutions of (3.16) by means of matrix multiplication and is generated by the solutions of the Riemann-Hilbert problem. If one chooses the generating matrix G_0 and the contour Γ properly, the Riemann-Hilbert problem has a unique solution. We obtain in this way, that the elements of the hidden symmetry group of Eq. (3.3) can be represented by the matrices G_0, defined on the closed contour Γ. The difficult part of the above solution generating procedure is to find solutions of the homogeneous Riemann-Hilbert problem. In particular, the Riemann-Hilbert problem is equivalent to a certain singular equation with the Cauchy-type kernel [14, 31, 37] which, in general, is too difficult to be solved explicitly. It seems likely that the approach presented here is closely related to these which are presently known [6, 12, 14, 28].

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REFERENCES

[1] G. A. Alekseev, *Pisma Zh. Eksp. Teor. Fiz.* **32**, 301 (1980).
[2] V. A. Belinsky, *Pisma Zh. Eksp. Teor. Fiz.* **30**, 32 (1979).
[3] V. A. Belinsky, V. E. Zakharov, *Zh. Eksp. Teor. Fiz.* **75**, 1953 (1978).
[4] V. Benza et al., *Nuovo Cimento* **50B**, 1 (1979).
[5] H. Buchdahl, *Quart. J. Math.* **5** (1954).
[6] C. M. Cosgrove, *J. Math. Phys.* **21**, 2417 (1980).
[7] B. Dubrovin et al., *Sovremennaya geometriya*, Moscow 1979.
[8] J. Ehlers, in *Les theories relativistes de la gravitation*, CNRS, Paris 1959.
[9] H. Eichenherr, M. Forger, *Nucl. Phys.* **B164**, 528 (1980).
[10] F. J. Ernst, *Phys. Rev.* **167**, 1175 (1968).
[11] F. J. Ernst, *Phys. Rev.* **168**, 1415 (1968).
[12] F. J. Ernst, I. Hauser, *Phys. Rev.* **D20**, 1783 (1979).
[13] F. J. Ernst, I. Hauser, *Phys. Rev.* **D20**, 362 (1979).
[14] F. J. Ernst, I. Hauser, *J. Math. Phys.* **21**, 1126 (1980).
[15] F. J. Ernst, I. Hauser, *J. Math. Phys.* **21**, 1418 (1980).
[16] R. Geroch, *J. Math. Phys.* **12**, 918, (1971).
[17] R. Geroch, *J. Math. Phys.* **13**, 394 (1972).
[18] B. K. Harrison, *Phys. Rev.* **B138**, 488 (1965).
[19] B. K. Harrison, *J. Math. Phys.* **9**, 1744 (1968).
[20] B. K. Harrison, *Phys. Rev. Lett.* **41**, 1197 (1978).

[21] M. Hirayama et al., *Phys. Lett.* **66A**, 352 (1978).
[22] C. Hoenselaers, W. Kinnersley, B. C. Xanthopoulos, *J. Math. Phys.* **20**, 2530 (1972).
[23] W. Israel, G. A. Wilson, *J. Math. Phys.* **13**, 865 (1972).
[24] W. Kinnersley, *J. Math. Phys.* **14**, 651 (1973).
[25] W. Kinnersley, *J. Math. Phys.* **18**, 1529 (1977).
[26] W. Kinnersley, *J. Math. Phys.* **19**, 1926 (1978).
[27] W. Kinnersley, *J. Math. Phys.* **19**, 2037 (1978).
[28] D. Kramer, submitted for publication to *J. Phys. A* (1981).
[29] D. Kramer, G. Neugebauer, *J. Phys. A* **14**, L333 (1981).
[30] R. Matzner, C. Misner, *Phys. Rev.* **154**, 1229 (1967).
[31] A. V. Mikhailov, V. E. Zakharov, *Zh. Eksp. Teor. Fiz.* **74**, 1953 (1978).
[32] A. V. Mikhailov, A. I. Yarimchuk, CERN preprint, TH-3150 (1981).
[33] C. Misner, *Phys. Rev.* **D18**, 4510 (1978).
[34] G. Neugebauer, *J. Phys.* **A12**, L67 (1979).
[35] A. Perelomov, *Physica* **4D**, 1 (1981).
[36] B. A. Putko, *Teor. Mat. Fiz.* **50**, 108 (1982).
[37] A. B. Shabat, V. E. Zakharov, *Func. Anal. and Appl.* **13**, n.3, 13 (1979).
[38] R. S. Ward, *Commun. Math. Phys.* **80**, 563 (1980).
[39] B. Carter, in *Black Holes*, Eds. C. De Witt, B.S.De Witt, Gordon and Breach, New York 1973.