MATHEMATICAL ASPECTS OF FIELD QUANTIZATION. QUANTUM ELECTRODYNAMICS*

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In these lectures fundamental mathematical aspects of quantum field theory are discussed. A brief review of various recent approaches to mathematical problems of quantum electrodynamics is given, preceded by a more extensive account of the development of ideas on the mathematical nature of quantum fields in general, providing an appropriate historical context.

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1. *Introduction*

In these lectures fundamental mathematical aspects of quantum field theory will be discussed. It is a basic fact, of which, in the present excitement, not everybody in particle physics is aware, that general quantum field theory as a rigorous mathematical theory does not yet exist. It is an intriguing body of in some cases astonishingly effective heuristic procedures of which formal perturbative series, with ill-defined terms, and not convergent in any known sense, form the main ingredients. It does not exist in the same sense as classical mechanics, classical electro magnetism or non-relativistic quantum mechanics: as mathematically well-formulated models for parts of the physical world.

At present almost nothing is rigorously known or understood of the fundamental mathematical structure of general quantum gauge fields. The recent clarification of the structure of classical gauge fields by means of concepts from modern differential geometry has so far not been of much help in understanding the corresponding quantum situation. There is therefore not much to say on the mathematics of the general quantum gauge fields that have come to dominate particle physics, except that they present a major challenge to mathematical physics, and even to pure mathematics. Something can be said however on the simpler but still typical special case of quantum electrodynamics.

The lectures will contain a brief review of various approaches to the basic mathematical

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problems of quantum electrodynamics, developed in the last decade. It will be preceded by a more extensive account of the development of ideas on the mathematics of quantum fields in general, which started much earlier. This will provide an appropriate historical context.

Section 2 will have some general historical remarks, in Section 3 early work, in particular that of Friedrichs and Segal will be sketched, and in Section 4 the axiomatic approaches of Wightman and Haag-Kastler. The discussion will be brought up to the present time with constructive quantum field theory in Section 5. Section 6 will contain the material connected with quantum electrodynamics together with a few final remarks on general non abelian gauge theories.

2. *Remarks on the history of quantum field theory*

Field quantization is almost as old as quantum mechanics. Its starting point can be taken to be Dirac's 1927 paper on the quantization of electromagnetic radiation [1]. From this there was a long and complicated development, in which the ideas involved acquired an ever widening scope and were transformed accordingly, not without a lot of conceptual confusion. For a brief review, see [2]. A first definitive and very successful stage was reached in quantum electrodynamics, in the late fourties and early fifties, when through the work of Feynman, Schwinger, Dyson and others, the removal of annoying infinities became possible, at least at the practical level of explicit perturbative calculations. Further exploration of the world of sub-nuclear particles led to attempts to extend the ideas of field quantization to non-electromagnetic interactions. The success of this was limited. (The Yukawa theory for the strong and the Fermi theory for the weak interactions.) During the sixties there was a long period of stagnation in quantum field theory. Various other ideas, now long forgotten, such as „bootstrapping" and „pure S-matrix theory", were tried out for the description of fundamental interactions. A revival of quantum field theory as a general theoretical framework for elementary particle physics, started in the late sixties, when earlier ideas of Yang and Mills were suddenly put in the right perspective by the introduction of a unified field theory for weak and electromagnetic interactions, independently, by Weinberg and Salam in 1967, and by the subsequent proof of the renormalizability of such theories by 't Hooft in 1971. From there to the present time the development of quantum field theory has been explosive, with gauge theories also for the strong interactions, ideas for further unification of interactions and a growing amount of support by experimental results. Quantum field theory again dominates our theoretical understanding of elementary particle physics, in the form of gauge theories of which quantum electrodynamics is the oldest, the simplest and still the most successful example.

3. *Mathematical aspects. Early history*

The mathematical basis of (non-relativistic) quantum mechanics was never much of a problem. This is due to the nature of the subject, but probably also the fact that in the twenties and thirties theoretical physicists and mathematicians still spoke the same language and were interested in new developments in each other fields. Already in 1932, J. von Neumann, one of the foremost mathematicians of his generation, explained the mathematics of quantum mechanics in his book „Grundlagen der Quantenmechanik", [3], now a classic, still worth reading. His formulation, in terms of Hilbert space and its operators is used in all modern books on quantum mechanics although of course not followed in all rigorous mathematical details.

For quantized fields the situation was completely different. From the beginning it was realized that quantization of fields, as systems with an infinite number of degrees of freedom, was fundamentally much more difficult and that the divergencies that appeared in higher order perturbation theory, were connected with deep mathematical problems. When however finally, around 1950, renormalization had established itself as a practical and effective way of getting around the divergence problems, there was among the leading physicists no longer a great desire for understanding at a more fundamental level. One reason for this was of course that renormalized quantum electrodynamics, whether a largely heuristic theory or not, was a great practical success. A second reason was probably that by that time theoretical physics and mathematics, certainly in their advanced parts, had become separate subjects. Mathematics had evolved into a more abstract direction, away from the explicit language of physics. Theoretical physics had stuck to classical mathematics, except for the adoption of group theory, and had in addition developed a special sort of mathematics of its own, much of which was unintelligible to mathematicians. Among the few people that in this period took a serious interest in the mathematical problems of quantum field theory, two pioneers must be mentioned: K. O. Friedrichs and I. E. Segal, both mathematicians. They tried to understand quantum field theory as a quantized non--linear Hamiltonian system, with an infinite number of degrees of freedom. This means the following:

Quantization of a classical system with N degrees of freedom starts from a description in classical canonical variables q_j, p_j, $j = 1, \ldots N$. The associated quantum system is then described in terms of operators $\hat{q}_j, \hat{p}_j$, satisfying the Heisenberg commutation relations

$$[\hat{q}_j, \hat{q}_l] = [\hat{p}_j, \hat{p}_l] = 0, \quad [\hat{q}_j, \hat{p}_l] = i\hbar\delta_{jl}. \tag{1}$$

For a system with a finite N, such as in ordinary quantum mechanics, it is not difficult to find an explicit operator representation of these algebraic relations, moreover such a representation is, for given N, unique up to unitary transformation, according to a theorem of Von Neumann. (The standard set of such operators consists of course in this case of multiplication and differentiation operators x_j, $-i\hbar \dfrac{\partial}{\partial x_j}$ in the Hilbert space of square integrable wave functions $\psi(x_1, \ldots, x_N)$). The dynamics of the quantum system is given by the Hamiltonian, the operator expression $\hat{H}(\hat{q}_j, \hat{p}_j)$ determined by the classical Hamiltonian function $H(q_j, p_j)$.

For $N = \infty$ the situation is much less simple. Consider for the quantization of fields the simple example of a real scalar field $\phi(x)$, nonlinearly self coupled according to the equation $(\Box + m^2)\phi(x) = \lambda\phi(x)^3$, $(\hbar = c = 1)$. As a classical field it is described by canonical variables $\phi(\vec{x}, t), \Pi(\vec{x}, t) = \frac{\partial}{\partial t}\phi(\vec{x}, t)$, an infinite set, parametrized by the variable $\vec{x}$. Canonical quantization requires finding a set of operators $\hat{\phi}(\vec{x}, t_0)$, $\hat{\Pi}(\vec{x}, t_0)$, for some fixed time t_0, in a Hilbert space $\mathscr{H}$, satisfying the Heisenberg relations

$$[\hat{\phi}(\vec{x}, t_0), \hat{\Pi}(\vec{y}, t_0)] = i\delta(\vec{x}-\vec{y}),$$

$$[\hat{\phi}(\vec{x}, t_0), \hat{\phi}(\hat{y}, t_0)] = [\hat{\Pi}(\vec{x}, t_0), \hat{\Pi}(y, t_0)] = 0. \tag{2}$$

Having this, the full quantum field, i.e. the operator $\hat{\phi}(x) = \hat{\phi}(\hat{x}, t)$ for all t, is in principle determined by the usual relation $\hat{\phi}(\vec{x}, t) = e^{i\hat{H}(t-t_0)}\hat{\phi}(\vec{x}, t_0)e^{-i\hat{H}(t-t_0)}$, with the Hamiltonian $\hat{H}$ given by the expressions

$$\hat{H} = \hat{H}_0 + \lambda\hat{H}_1,$$

$$\hat{H}_0 = \tfrac{1}{2}\int\left(\hat{\Pi}(\vec{x}, t_0)^2 + \sum_{j=1}^{3}\left(\frac{\partial\hat{\phi}(\vec{x}, t_0)}{\partial x_j}\right)^2 + m^2\phi(\vec{x}, t_0)^2\right)d\vec{x},$$

$$\hat{H}_1 = \tfrac{1}{4}\int\hat{\phi}(\vec{x}, t)\, d\vec{x}. \tag{3}$$

Friedrichs, and somewhat later Segal, studied systems of canonical operators, satisfying (2). For such an infinite set, this turned out to be a highly non-trivial mathematical problem. They discovered that there exist, contrary to the case with N finite, more than one solution, in fact an uncountably infinite number of essentially different, ,,inequivalent" representations of (2). The system obtained by a standard free field ,,second quantization" procedure, using creation and annihilation operators, is just one rather special example, which is, as it turns out, of only limited applicability in a rigorous theory of interacting quantum fields. (The fact that there is a δ-function in (2) is at this stage not yet an essential difficulty; there is an obvious rephrasing of (2), using testfunctions, like in rigorous distribution theory.)

Friedrichs's main idea was to realize (2) by trying to find generalizations of the standard set of canonical operators $x_j, -i\hbar\frac{\partial}{\partial x_j}$ of ordinary quantum mechanics, using Hilbert spaces of functionals, i.e. functions of variables that are themselves functions, and appropriate notions of functional multiplication and differentiation. He was greatly hampered by the lack, at that time, of a rigorous theory of measure and integration in function spaces. His most important papers on the mathematical problems of quantum fields were collected in a book, which appeared in 1953 [4].

Segal's work on quantum fields over many years has been characterized by the application of a broad spectrum of widely different ideas. His two main contributions to the problem of representations of (2) have been a reformulation of (2) in terms of so called Weyl operator systems and the use of abstract algebraic ideas. Both notions deserve a brief explanation here.

Discussions of systems of canonical operators $\hat{q}_j, \hat{p}_j$ are unnecessarily complicated by the fact that such operators are unbounded, i.e. not defined on all vectors in Hilbert space. One avoids this by using Weyl operators, unitary operators defined as

$$W(\alpha, \beta) = e^{i\sum_{j=1}^{N} \alpha_j \hat{p}_j} e^{i\sum_{j=1}^{N} \beta_j \hat{q}_j} e^{-\frac{i}{2}\sum_{j=1}^{N} \alpha_j \beta_j} \tag{4}$$

for every set of $2N$ real numbers $\alpha = \alpha_1, \ldots, \alpha_N, \beta = \beta_1, \ldots, \beta_N$ (again with $\hbar = 1$). These Weyl operators satisfy the Heisenberg commutation relations in an exponentiated form

$$W(\alpha, \beta)W(\alpha', \beta') = W(\alpha+\alpha', \beta+\beta')e^{\frac{i}{2}\sum_{j=1}^{N}(\alpha_j\beta_j' - \alpha_j'\beta_j)}. \tag{5}$$

The introduction of unitary operators in this way has the additional advantage that it leads to a general formulation of the problem of finding representations of the canonical commutation relations that include in an obvious way the case of an infinite number of degrees of freedom, such as boson quantum fields. In this formulation one considers as given a real vector space $\mathcal{M}$, possibly of infinite dimension, together with an anti-symmetric bilinear form b, which is non-degenerate, i.e. if $b(u, v) = 0$, for some u and all v in $\mathcal{M}$, then u must be the zero vector. A Weyl system associated with $(\mathcal{M}, b)$ is then a system of unitary operators $\{W(u)\}$, in a Hilbert space $\mathcal{H}$ such that

$$W(u)W(v) = W(u+v)e^{-ib(u,v)} \tag{6}$$

for all u, v in $\mathcal{M}$, together with a certain continuity property in the dependence of $W(u)$ on u, to ensure that canonical operators in the usual sense, can be obtained by differentiation of these unitary operators. The study of representations of canonical commutation relations becomes in this way the study of general Weyl systems. A finite dimensional $\mathcal{M}$ has only one (irreducible) Weyl system, (Von Neumann's uniqueness theorem again), for an infinite dimensional $\mathcal{M}$ there are many different ones.

Segal undertook the systematic investigation of Weyl systems in the context of an algebraic formulation of quantum theory, that he had proposed already in 1947 [5]. In standard quantum theory a state is a unit vector ψ in a Hilbert space $\mathcal{H}$ and physical observables are described by self adjoint operators A in $\mathcal{H}$. Experimental predictions are made in terms of expectation values $E(A) = (\psi, A\psi)$. The values of this expectation functional E, on all A in $\mathcal{H}$, determines in fact the vector ψ uniquely, up to an irrelevant phase factor. In Segal's point of view quantum theory should start from the specification of an abstract algebra $\mathcal{A}$ of observables, in fact a so called C^*-algebra. States are then linear functionals E, of a certain type, defined on all elements of $\mathcal{A}$. There is a central representation theorem, now known as the GNS theorem, after Gelfand, Naimark and Segal. This asserts that one can construct, for each given E, a unique representation of the algebra $\mathcal{A}$ by operators in a Hilbert space, with a well-defined unit vector ψ_E, such that the functional E becomes again an expectation functional in the usual sense: $E(A) = (\psi_E, A\psi_E)$, for all A in $\mathcal{A}$. One may think that with this one is back in the situation of the standard formulation of quantum theory. There is however a catch: Two different state functionals E_1 and E_2 can in this way be represented as expectation functionals with respect to state vectors ψ_{E_1} and ψ_{E_2}. The corresponding representations of $\mathscr{A}$ may however turn out to be inequivalent, the vectors are then in different Hilbert spaces $\mathscr{H}_{E_1}$ and $\mathscr{H}_{E_2}$, that can in no way be identified. Segal studied canonical commutation relations such as (1) and (2) in terms of representations of algebras of observables, generated in an abstract way by the canonical operators in their Weyl system form. His ideas on algebras and their functionals have been very influential, even outside the subject of quantum field theory, e.g. in statistical mechanics of infinite systems. The main features of Segal's early work are discussed in his 1963 book [6]. For an introduction to his ideas on Weyl systems and C^*-algebras, see also [7], and for the explicit applications to simple field theoretic situations, connected with the relations (2), in their proper formulation, without δ-functions, see [8].

Summarizing one may say that Friedrichs and Segal discovered many interesting properties of the phenomenon of inequivalent representations of the canonical commutation relations for infinite systems. Segal in particular put the theory of free fields on a sound mathematical basis and inspired rigorous work on related linear field theories. Both were however unable, at that time, to formulate, let alone solve, the fundamental dynamical problem of non-linear quantum field theory in a rigorous way.

In its essence the problem is that of formulating properly heuristic formulae such as (3). It can be narrowed down to the question how and in what space the expressions for $\hat{H}$, $\hat{H}_0$ and $\hat{H}_1$ are defined as operators in an exact mathematical sense. As an example, an easy calculation with creation and annihilation operators shows that the action of the operator $\hat{H}_1$, as given by the expression in (3), on the vacuum vector, and in fact on any vector, gives a "vector" with infinite length, i.e. not in the Hilbert space. This has nothing to do with the heuristic use of plane waves states, common in quantum mechanics, but is an essential difficulty caused by the δ-function distribution character of the operator fields, and which presents here contrary to (2) a very hard problem, which cannot simply be overcome by the insertion of test functions. In principle this and similar phenomena are explained by the occurrence of inequivalent representations. The free and interacting fields act in different spaces. It is however extremely difficult to give this general idea an explicit and workable form. In any case it is not surprising, because of this, that conventional perturbation theory, pretending to expand quantities of the interacting theory in terms of the free theory, leads in first instance to divergent results.

4. The Wightman and Haag-Kastler axioms

In the late fifties and early sixties, in the period of stagnation in quantum field theory, physicists felt a renewed need for reflection on its foundations.

A. S. Wightman started his work on field theory in 1956 with a paper on the general properties of vacuum expectation values of product of field operators [9]. His results developed in the following years into a systematic and mathematically rigorous account of basic concepts in quantum field theory. In a certain way his approach was more modest than the earlier attempts sketched in the preceding section. It avoided a direct attack on the problem of non-linear dynamics and concentrated instead on the formulation, as precisely and unambiguously as possible, of those general properties that any relativistic quantum field theory must be expected to possess. These were grouped together in a system of axioms from which further interesting properties could be derived. In these axioms the standard concepts of quantum theory appeared in specialized forms appropriate to field theory, with an emphasis on fields as basic operators in the Heisenberg picture, the importance of the vacuum state as ground state, the rôle of the requirements of special relativity, and with great attention to mathematical details as an essential ingredient.

For the example of a real scalar field a loosely formulated set of Wightman axioms runs as follows:

1. There is a Hilbert space $\mathscr{H}$ of state vectors.
2. In this space there are hermitian field operators $\phi(x)$, for every space-time point x.
3. The space can be generated by repeated application of these operators on a single special state vector, the vacuum state Ω.
4. Relativistic covariance is ensured by the existence of unitary operators $U(a, \Lambda)$, for each space-time translation a and Lorentz transformation Λ, transforming the field according to $U(a, \Lambda)\phi(x)U(a, \Lambda)^{-1} = \phi(\Lambda^{-1}(x-a))$ and leaving the vacuum state invariant.
5. The theory is local in the sense that $(x-y)^2 < 0$ implies $[\phi(x), \phi(y)] = 0$.
6. The energy operator H, generator of pure time translations $U(a, 1) = e^{-iHt}$, with $a = (t, 0, 0, 0)$, has a positive spectrum with a single discrete eigenvalue at 0, non-degenerate and with Ω as eigenvector.

Mathematical refinements are necessary to make these partially heuristic statements into rigorous axioms. The main one is connected with the δ-function-like character of the field operators: The impossibility of having field operators at sharp space-time points. One has to consider field operators depending on "smearing functions", or test functions, in the language of distribution theory, in accordance with the symbolic formula $\phi(f) = \int \phi(x)f(x)d^4x$. Assumptions on the class of test functions to be used and on the precise dependence of $\phi(f)$ on such f are necessary. The smeared field operators $\phi(f)$ are still unbounded operators, this leads to further technical assumptions on the existence of a common invariant domain D of vectors in $\mathscr{H}$ on which the field operators are defined, for all admissable test functions f.

There is a fundamental theorem for such an axiomatic formulation, the reconstruction theorem, proved by Wightman in his first paper. It asserts that a field theory is completely characterized by its vacuum expectation values, and that the full operator theory can be recovered in a unique way from the knowledge of all vacuum expectation values of products of field operators, as a set of (generalized) functions. It was much later demonstrated by Borchers, see [10] or [11], that there is an alternative formulation of the Wightman scheme, rather different at first sight, but completely equivalent, in which this theorem becomes just a natural modification of the GNS theorem, mentioned earlier. Its proof then becomes very simple.

Summarizing again one can say that the Wightman approach did not provide an explicit way of handling non-linear interactions, but that it gave at least a framework in which basic concepts of field theory, such as field operators, states, the vacuum, relativistic covariance, etc. could be rigorously discussed. It also led to a rigorous proof of some important general theorems, the CPT theorem, the spin-statistics theorem, analyticity theorems for Green's functions. Last but not least, it had a healthy influence on the language of elementary particle physics. In addition to Wightman, many other people contributed to its development, which by 1965 was essentially finished. A standard text book on the subject is [12].

A different but related axiomatic scheme was proposed in 1964 by Haag and Kastler, [13], and developed further by them and others in the following years. In this approach the fields themselves are no longer the basic objects. This rôle is played by so called local algebras $\mathscr{A}(O)$, assumed to exist for each bounded open set O in space-time. They are thought of as generated in some sense by the fields and should contain the same information. An important part of the formalism goes back to Segal's early ideas on algebras and quantum theory, in its general C^*-algebra form. A set of axioms in this spirit might go as follows:

1. For each bounded open set O of space-time there exists a C^*-algebra $\mathscr{A}(O)$. These algebras are consistent with each other in the sense that if $O_1 \subset O_2$ then $\mathscr{A}(O_1)$ can be identified in a natural way with a subalgebra of $\mathscr{A}(O_2)$. This makes it possible to construct as a limit in a certain sense a large C^*-algebra $\mathscr{A}$, containing the $\mathscr{A}(O)$, for all O, as subalgebras. $\mathscr{A}$ is called the algebra of quasi-local observables.

2. Locality: For two sets O_1 and O_2, spacelike separated, one has $[\mathscr{A}(O_1), \mathscr{A}(O_2)] = 0$, as subalgebras of $\mathscr{A}$.

3. Lorentz covariance: There is a representation of the inhomogeneous Lorentz group by automorphisms of the algebra $\mathscr{A}$. When a Lorentz transformation maps O_1 onto O_2, then the associated automorphisms map $\mathscr{A}(O_1)$ onto $\mathscr{A}(O_2)$ correspondingly.

4. There is given a state functional E on $\mathscr{A}$, invariant under the Lorentz automorphisms and satisfying further requirements, such as spectral properties in some sense.

Using the GNS representation theorem one obtains from these data a Hilbert space operator theory, with unitary operators for the Lorentz transformations, and an invariant vacuum vector state corresponding with the given functional E.

The local algebra approach is in some sense complementary to Wightman theory. On one hand its basic objects are simpler, both from a physical and a mathematical point of view, on the other hand it remains further removed from quantum field theory as it is used heuristically, but quite effectively, in particle physics. Its main features were well established around 1970. See for an exposition Section 6 of [12], or for a brief review [14].

5. Constructive quantum field theory

In the main the material discussed so far has been accepted as standard background for all rigorous investigations in quantum field theory for at least ten years. This section brings us into the present.

Constructive quantum field theory is a program for the construction of explicit models of non-linear quantum field theory, satisfying the Wightman and Haag-Kastler axioms. It was started around 1967 by J. Glimm and A. Jaffe and developed by them and many others in subsequent years. See their book [15] for a recent review, with an extensive bibliography. The main idea in this approach was to attack the problem of non-linear dynamics in steps of increasing difficulty, first studying crudely simplified models, in lower space-time dimensions, "tamed" by various cut-offs in space and momentum variables, which after rigorous results had been obtained, where then gradually removed in such a way that the results remained valid in some sense in the limit. Around 1972 progress in the subject gained new momentum by the introduction of ideas from probability theory, providing a very fruitful link with classical statistical mechanics. Results from the later work of Segal and earlier heuristic ideas of Schwinger and Symanzik were important. The transformation of quantum field theory problems into an area of statistical mechanics is often called Euclidean quantum field theory, because it involves analytical continuation of the real time in Minkowski space to an imaginary time parameter, changing thereby the Lorentz group into the Euclidean group.

So far constructive field theory has given us complete and rigorous theories for polynomially self interacting scalar fields and Yukawa-like models in 2 dimensional space-time. There are also interesting results in 3 dimensions, but almost nothing for the 4 dimensional space-time of the real world. Constructive field theory in the proper sense seems now to have come to a standstill, however the intimately related area of statistical mechanics where field theoretical methods are effective, is still in full growth, see e.g. [16].

6. *Quantum electrodynamics*

Quantum Electrodynamics was for a long time the only successful example of a quantum field theory, the only one of which the physical validity could not be doubted. Nevertheless investigations into the mathematical problems of field theory used to treat it as a rather special case, the study of which could be better avoided, because of its additional complications. Standard text books such as [12] exclude quantum electrodynamics, and more particularly the quantized Maxwell field, even in the treatment of free field situations. In recent years this attitude has become more and more untenable. Quantum electrodynamics has moved from the position of an isolated singular case to a central rôle as the simplest of the gauge fields that now dominate particle physics.

The special problems that arise in attempts at rigorous formulation of quantum electrodynamics, and that appear, in an even more severe form in non-Abelian gauge theories, are connected with the following two groups of well-known phenomena:

1. The theory contains for the photons two field operators, the tensor field $F_{\mu\nu}(x)$ and the vector field $A_\mu(x)$. In classical electromagnetism $F_{\mu\nu}(x)$ is the physical field consisting of electric and magnetic field strength, the $A_\mu(x)$ can be seen purely as a convenient auxiliary quantity. In the quantum situation the rôle of the $A_\mu(x)$ is an essential one. Although in a certain sense the quantized $A_\mu(x)$, as well as the spinor field $\psi(x)$, are unphysical, and although physical quantities depend only on the $F_{\mu\nu}(x)$ and on bilinear expressions in $\psi(x)$, such as current densities, no acceptable formulation of quantum electrodynamics in terms of the A_μ and ψ fields alone is known or seems possible. Matters are also complicated by the fact that the meaning of the relation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and of the non-uniqueness of the A_μ is far from straightforward. At the root of all this is the concept of gauge transformations which at the classical level is perfectly clear and can be formulated in an elegant manner, especially for non-abelian theories, in terms of differential geometry, but which

as a quantum notion is very poorly understood. The similarity with ordinary symmetries is only superficial. The standard mathematical language for symmetries in quantum theory, that of representations of groups by unitary operators in Hilbert space does not apply.

2. There seems to be a general incompatibility between manifest Lorentz covariance and the use of Hilbert space (i.e. with a positive definite inner product) as the space of quantum states. Both these properties are required to hold simultaneously in the Wightman and Haag-Kastler schemes.

In view of this it is clear that the standard mathematically rigorous formulations of quantum field theory, in particular as embodied in the Wightman and Haag-Kastler approaches, are in need of much further development, and because of the second point, probably also of some drastic modifications.

Among the various attempts at adapting Wightman's axiomatic field theory to the requirements of gauge theory is the most extensive and systematic so far that of Strocchi and his collaborators. Already in 1967 Strocchi drew attention to the fact that in a manifestly covariant formulation such obvious classical relations as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu = 0$ lead to contradictions when used for the operator fields [17]. In 1974 he and Wightman wrote a long paper on the super selection rules in quantum electrodynamics, in which the outlines of an improved axiomatic scheme appeared [18]. This was followed by a long series of papers, either by Strocchi alone, or in collaboration with others, e.g. Fröhlich and Morchio, in which this axiomatic scheme was developed further, refined in mathematical details and above all enlarged in scope to serve as a general framework for possibly non-abelian gauge theories. The basic philosophy of Strocchi's work is that it is possible, in practical calculations, to work exclusively in a single non-covariant gauge with a positive definite metric such as the Coulomb gauge, but that this is too narrow and therefore unacceptable from a general theoretical point of view, and that consequently indefinite metrics have to be admitted as an essential part of the formulation. This takes the form of the use of state spaces with two inner products, such as occur in the original, non-rigorous Gupta-Bleuler formalism. One inner product is Lorentz invariant, is natural in a certain sense, but it is not positive definite. A second auxiliary inner product, not invariant but positive definite, connected with the first through a so called metric operator, makes the state space into a Hilbert space. This space contains a physical subspace. There are definitions of physical operators, gauge transformations and gauge invariance. All this is still in full progress. There are many imaginative concepts but also problems and loose ends. The introduction of the second inner product keeps the situation within the range of traditional Hilbert space methods. This has to be paid for with a certain awkwardness and artificiality of the resulting formalism. Typical for that is, for instance, the fact that the operators that represent the Lorentz group are not only non-unitary, but even unbounded, and this already in the free field case! In related work Mintchev and d'Emilio have tried to make the formalism mathematically more natural by stressing the mathematical properties of the indefinite metric structure itself. See e.g. [19]. A similar but more radical step in this direction is taken in [20] where a formalism is proposed in which Hilbert space as general state space is abandoned, in favour of more general topological inner product space that emerge naturally from the mathematical properties of the Wightman n-point functions. No auxiliary Hilbert space inner products are needed and a physical Hilbert space structure appears automatically, only where this is required by physical interpretation. As an example a complete and systematic derivation of the various free field gauges, covariant and non-covariant, is given.

The second major direction in axiomatics for quantum electrodynamics is based on the ideas of the Haag-Kastler local algebra approach. It starts with a discussion of gauge transformations, physical observables and superselection sectors in three papers by Doplicher, Haag and Roberts, the first of which is [21]. This in itself is not applicable to quantum electrodynamics and a fortiori non abelian gauge theories. It forms however the background for recent work such as that of Buchholz [22]. As is usual in the local algebra framework there is directness in simple but important physical principles, but important elements from heuristic field theory, such as in this case the vector potential operator $A_\mu(x)$, are lacking from the formalism.

A different line in the application of C^*-algebra concepts on the problems of quantum electrodynamics is followed by C. A. Hurst and his school. Their work consists of a further development of Segal's Weyl systems approach. See e.g. [23].

Finally a lot of rigorous or semi-rigorous work on separate mathematical aspects of quantum electrodynamics, such as e.g. connected with the infra-red problem, could be mentioned. There is the older work of Kibble [24] and Roepstorff [25], the work of Zwanziger, e.g. [26] and more recently of Streater et al. [27], Hochstenbach [28] and many others.

Although much has been achieved, there still does not exist a single rigorous framework for quantum electrodynamics, which is really satisfactory in the sense that it is able to unify the valuable points of view of the various approaches and connect the many results obtained separately. As long as we do not have a good understanding of the mathematical structure of quantum electrodynamics, there is no hope for general gauge theories, which are much more complicated and where we are confronted with the major problem of giving a mathematical meaning at the quantum field level to the differential geometric concepts that determine the structure of gauge theories.

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