Vol. B14 (1983) ACTA PHYSICA POLONICA No 6

YANG-MILLS POTENTIALS IN THE FORM OF INFINITE $1/g$ POWER SERIES

By A. Górski

Institute of Physics, Jagellonian University, Cracow*

(Received December 3, 1982)

Solutions to the Yang-Mills equations being infinite $1/g$ power series are investigated. An explicit example of non-Abelian solution of such a type is given. In the strong coupling limit it tends to the coupling constant independent solution which have been found recently.

PACS numbers: 11.10.-z, 11.10.Np

1. Introduction

Quantum Chromodynamics (QCD) is viewed as the best candidate for a theory describing the strong interactions. However, due to its high complexity, QCD is little understood, especially in the non-perturbative strong coupling (low energy) region. This is the main reason why the classical theory (CCD) is popular. It seems to be the most interesting in the case of non-zero external c-number sources [1]. The source may provide an approximate description of a system to which the Yang-Mills (YM) fields are coupled, and whose dynamics may be ignored. Nontrivial topology and other unexpected properties of CCD suggest that it can throw some light on the full quantum theory.

Recently $1/g$-expansion for CCD was proposed to investigate the strong coupling region [2]. Here, within this approach a class of YM potentials being infinite $1/g$ power series is investigated and special non-Abelian solutions are constructed as an example. In the strong coupling limit the new solutions tend to those found recently [3].

2. Coupling constant dependence of potentials

In CCD we have no clear indications what form of the external current should be chosen. Sources with different coupling constant dependence are used in the literature (see e.g. [1, 3–5]). Here we focus our attention on this problem.

In general YM potentials and sources can be divided in the four disjunctive classes listed below

* Address: Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

Class I. To this class belong YM potentials which are finite power series (polynomials) in $1/g$. Hence they are of the form

$$A_\mu(x, g) = \sum_{n=0}^{N} \overset{n}{A}_\mu(x) \frac{1}{g^n}, \tag{1}$$

where $\overset{n}{A}_\mu$ (and $\overset{n}{\mathbf{j}}_\mu$ for the source, respectively) are the expansion coefficients.

Instantons, merons, the Sikivie-Weiss magnetic dipole solution [5] etc. are of the form (1). It is obvious that the class I solution has finite strong coupling limit

$$\lim_{g \to \infty} A_\mu(x, g) < \infty. \tag{2}$$

Class II. It contains potentials which are infinite power series in $1/g$, i.e. they are of the form

$$A_\mu(x, g) = \sum_{n=0}^{\infty} \overset{n}{A}_\mu(x) \frac{1}{g^n}. \tag{3}$$

There are no examples of such solutions in the literature. A simple example will be given in the following Section. The potentials (3) also fulfil condition (2).

Class III. To this class belong potentials with the finite strong coupling limit which are not members of the classes I and II. In fact they are non-analytic in g at $g = \infty$ but they obey condition (2).

Many well-known solutions are in class III: 't Hooft-Polyakov monopole, dyon [4] or vortex [6]. It is interesting to notice that all these potentials are accompanied by extra Higgs fields in the Lagrangian. Their coupling constant dependence is of the form $\frac{1}{g} e^{-g}$.

Class IV. This class consists of potentials with infinite or indefinite strong coupling limit:

$$\lim_{g \to \infty} A_\mu(x, g) = \infty. \tag{4}$$

The best known potential of the class IV is the Abelian YM potential, which is proportional to the coupling constant: $A_\mu \sim g$. However, the Abelian potentials, because of the well-known instabilities, seem to be irrelevant for the strong coupling region of QCD [7].

Now we are going to concentrate on the class II potentials. Expanding YM potentials in the $1/g$ power series we have in the n-th order in $1/g$ the following equations [2]

$$\mathscr{M}_\mu^{abv} \overset{n}{A}{}_v^b = J_\mu^a, \tag{5}$$

where[1]

$$\mathscr{M}_\mu^{abv} \equiv \varepsilon^{abc}\varepsilon^{cde} \overset{0}{A}{}_\mu^e \overset{0}{A}{}^{dv} + \varepsilon^{adc}\varepsilon^{ceb} \overset{0}{A}{}_\lambda^d \overset{0}{A}{}^{e\lambda}\delta_\mu^v + \varepsilon^{ade}\varepsilon^{cbe} \overset{0}{A}{}_\mu^e \overset{0}{A}{}^{dv} \tag{6}$$

---

[1] Here we limit our discussion to the SU_2 gauge group. Then ε^{abc} is the totally antisymmetric symbol.

and

$$J^a_\nu \equiv \overset{n-1}{j}{}^a_\nu - \Delta \overset{n-2}{A}{}^a_\nu + \partial_\nu \partial^\mu \overset{n-2}{A}{}^a_\mu - \sum_{i=0}^{n-1} [\varepsilon^{abc}\partial^\mu(\overset{i}{A}{}^b_\mu \overset{n-i-1}{A}{}^c_\nu)$$
$$+\varepsilon^{abc}\overset{i}{A}{}^{b\mu}(\partial_\mu \overset{n-i-1}{A}{}^c_\nu - \partial_\nu \overset{n-i-1}{A}{}^c_\mu)] - \sum_{\substack{i,j=0\\ i+j\leqslant n}}^{n-1} \varepsilon^{abc}\varepsilon^{cde}\overset{i}{A}{}^b_\mu \overset{j}{A}{}^{d\mu} \overset{n-i-j}{A}{}^c_\nu. \tag{7}$$

More detailed discussion of equation (5) was given in [2] and it will not be repeated here.

In [3] a non-Abelian solution to (5) was found in the $g \to \infty$ limit. There it was assumed that the $\overset{0}{A}_\mu$ term has the generalized Maxwell-type form [8]:

$$\overset{0}{A}{}^a_\mu(x) = A^a(x)\alpha_\mu(x), \tag{8'}$$

where $A^a(x)$, $\alpha_\mu(x)$ are arbitrary regular functions. This solution was shown to have interesting group-topological properties. We would like to find class II solutions with the same strong coupling limit. Their existence will support correctness of the limiting procedure proposed in [2, 3].

To solve the infinite chain of equations (5) for $n = 0, 1, 2, \dots$ the simplifying Ansatz (8') is assumed for an arbitrary coefficient $\overset{n}{A}{}^a_\mu(x)$

$$\overset{n}{A}{}^a_\mu(x) = \overset{n}{A}{}^a(x)\overset{n}{\alpha}_\mu(x) \tag{8}$$

and, similarly as in [3], we also assume that

$$\overset{n}{\alpha}_\mu(x) \equiv \delta_{\mu 0}\overset{n}{\alpha}(x), \tag{9}$$

where $\overset{n}{\alpha}(x)$ are arbitrary functions.

Assumptions (8), (9) allow us to write the YM equations explicitly in the simple form

$$\delta_{\mu 0}\Delta \overset{n-2}{A} + \sum_{i=0}^{n-1} \overset{n-i-1}{A} \times \partial_\mu \overset{i}{A}$$
$$-\delta_{\mu 0} \sum_{\substack{i,j=0\\ i+j\leqslant n}}^{n} \overset{i}{A}\times(\overset{j}{A}\times \overset{n-i-j}{A}) = \overset{n-1}{j}_\mu, \tag{10}$$

where we limit our discussion to the static YM potentials and the metric tensor signature is chosen to be $(+++-)$.

However, even equations (10) form a set of nonlinear partial differential equations which seem to be too complicated to be solved generally. For that reason we make an additional Ansatz assuming that all coefficients in the expansion (3) are parallel in the group ("color") space:

$$\overset{n}{A}(x) = \overset{0}{A}(x) \equiv A(x), \quad n = 1, 2, 3, \dots \tag{11}$$

and

$$\overset{0}{\alpha}(x) \equiv 1. \tag{11'}$$

The assumptions (11) imply the following simple form of the YM equations (10)

$$\Delta(\overset{n-2}{\alpha} A) = \overset{n-1}{j}_0, \tag{12a}$$

$$A \times \partial_k A \sum_{i=0}^{n-1} \overset{i}{\alpha} \overset{n-i-1}{\alpha} = \overset{n-1}{j}_k \tag{12b}$$

and an additional restriction for the external current

$$\overset{0}{j}_0 \equiv 0. \tag{13}$$

The condition (13) means that only space-components of the external current can be nonzero in the $g \to \infty$ limit, as was in [3].

Equation (12b) implies the following consistency conditions for the source terms

$$\overset{n}{j}_k = \overset{0}{j}_k \sum_{i=0}^{n} \overset{i}{\alpha} \overset{n-i}{\alpha}. \tag{14}$$

The full potential can be written as

$$A_\mu = \delta_{\mu 0} A(x) \sum_{n=0}^{\infty} \overset{n}{\alpha}(x) \frac{1}{g^n} \tag{15}$$

and the space-components of the external current have to be

$$j_k = \overset{0}{j}_k \sum_{n=0}^{\infty} \sum_{i=0}^{n} \overset{i}{\alpha} \overset{n-i}{\alpha} \frac{1}{g^n}. \tag{16}$$

Hence $j^a_\mu(x)$ satisfies the following gauge invariant condition

$$\text{rang}\, \|j^a_\mu\| = 1, \tag{17}$$

where j^a_μ is treated as a 3×4 matrix.

If the external source is chosen to be the same as in [3] then the YM potential has interesting group-topological properties. In particular it is non-Abelian, with nontrivial topology and non-Abelian holonomy group. This is due to the fact that an additional factor in (15) is color-independent. Hence the properties of the potential (15) are similar to those of $A(x)$.

It should be also stressed that if space-components of the external current are given then the coefficients $\overset{i}{A}\,\overset{}{\alpha}$ as well as the temporal-components of the external current $\overset{i}{j}_0(x)$ are uniquely determined. This will be shown in the following Section, where the explicit form of the potential will be expressed in terms of the external source.

3. *Explicit solution*

Thanks to the simple form of the potential, equations (12) can be solved without specifying the external current. Let us define

$$\alpha(x, g) = \sum_{n=0}^{\infty} \overset{n}{\alpha}(x) \frac{1}{g^n}. \tag{18}$$

Then

$$\alpha(x, \infty) = 1. \tag{18'}$$

Hence, we can write the potential in the compact form

$$A_\mu(x, g) = \delta_{\mu 0} A(x) \alpha(x, g) \tag{19}$$

and the consistency condition (14) can be written as

$$j_k(x, g) = \overset{0}{j}_k(x) \alpha^2(x, g). \tag{20}$$

Because the color-vectors j_k and $\overset{0}{j}_k$ are parallel in the group (color) space we can define their quotient by

$$\lambda_k \equiv j_k^1/\overset{0}{j}{}_k^1 = j_k^2/\overset{0}{j}{}_k^2 = j_k^3/\overset{0}{j}{}_k^3. \tag{21}$$

Using the notation (21) we are able to express the function $\alpha(x, g)$ in terms of the space--components of the external current

$$\alpha(x, g) = (\lambda_k)^{1/2}. \tag{22}$$

Hence the potential (19) was determined by the space-components of the external current. The form of the color-dependent function $A(x)$ is not discussed here because it was done in [3]. It is also obvious from (18') that the solution (19) with appropriate external source tends to the solution given in [3] in the strong coupling limit.

As it was noticed previously, the temporal-components of the external source can be also determined uniquely by the space-component:

$$j_0(x, g) = \frac{1}{g} \Delta[A(x)\,(\lambda_k(x, g))^{1/2}]. \tag{23}$$

Expanding the potential in the $1/g$ power series we have from (22)

$$A_\mu(x, g) = \delta_{\mu 0} A(x) \left\{ 1 + \frac{1}{g} \tfrac{1}{2} j_k / \overset{0}{j}_k + \frac{1}{g^2} \left[\tfrac{1}{2} \overset{2}{j}_k / \overset{0}{j}_k - \tfrac{1}{8} (\overset{1}{j}_k / \overset{0}{j}_k)^2 \right] + \dots \right\}. \tag{24}$$

The potential is the infinite power series in $1/g$ and it is analytic at $g = \infty$. Arbitrary order expansion coefficients can be easily computed from (22).

5. *Conclusions*

In this paper solutions to the YM equations with sources were classified according to their coupling constant dependence. A simple example of a class II potential (being an infinite power series in $1/g$) was explicitly constructed. Such solutions were not discussed in the literature as yet.

The obtained potential tends to the one found recently [3] in the $g \to \infty$ limit. If the source term is appropriately chosen then the potential has interesting group-topological properties, the same as in the strong coupling limit. In particular it is non-Abelian and it has nontrivial topology [3].

I am indebted to Dr H. Arodź for critical reading of the manuscript.

REFERENCES

[1] See, for example, P. Pirila, P. Presnajder, *Nucl. Phys.* **B142**, 229 (1978); H. Arodź, *Phys. Lett.* **78B**, 129 (1978); R. Jackiw, L. Jacobs, C. Rebbi, *Phys. Rev.* **D20**, 474 (1979); E. A. Freedman, L. Wilets, S. D. Ellis, E. M. Henley, *Phys. Rev.* **D22**, 3128 (1980); C. H. Oh, R. Teh, W. K. Too, *Phys. Lett.* **101B**, 337 (1981).

[2] A. Górski, *J. Phys.* **A16**, 849 (1983).

[3] A. Górski, Jagellonian University preprint, TPJU 9/82 (1982), *Phys. Rev.* **D28**, (1983) in press.

[4] A. Actor, *Rev. Mod. Phys.* **51**, 461 (1979).

[5] P. Sikivie, N. Weiss, *Phys. Rev. Lett.* **40**, 1411 (1978).

[6] M. B. Nielsen, P. Olesen, *Nucl. Phys.* **B61**, 45 (1973).

[7] J. E. Mandula, *Phys. Lett.* **67B**, 175 (1977); see also, M. Magg, *Phys. Lett.* **74B**, 246 (1978); P. Sikivie, *Phys. Rev.* **D20**, 877 (1979).

[8] A. Górski, *J. Phys.* **A15**, 239 (1982).