AN INHOMOGENEOUS COSMOLOGICAL SOLUTION OF EINSTEIN'S EQUATIONS

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The construction of the conformally flat and inhomogeneous solution of Einstein's equations is presented. The Bondi type energy tensor has been used as a source.

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1. Introduction

There are many observational and theoretical reasons that have motivated studies of anisotropic and inhomogeneous cosmologies. Such studies are based on a number of particular inhomogeneous models forming a subset in a space of all solutions of Einstein's equations. They include such generally known solutions as Gowdy, Szekeres, Tolman solutions and some plane symmetric solutions.

Although the most intensively discussed inhomogeneous cosmologies are the perturbated FRW models, there is still a great necessity of the search for new inhomogeneous solutions of Einstein's equations which could be used in cosmological discussions e.g. on galaxy formation or on other processes in the early universe.

The construction of a simple inhomogeneous solution of Einstein's equations will be presented here. The conformally flat and spherically symmetric form of the line element will be considered. The energy-momentum tensor is the Bondi type inhomogeneous source. The considered equations, together with the natural energy conditions, give the final form of the conformal factor and the region of applicability of the metric. The obtained space-time has no symmetries except spherical ones.

2. Solution

When looking for an inhomogeneous solution of the Einstein equations one has to assume the general structure of the metric i.e. the admissible symmetries of the space-time described by the metric. We assume that in spherical coordinates t, r, θ, φ

$$ds^2 = \Omega^2(t, r)\left[dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{1}$$

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One can easily find that the metric becomes homogeneous only if the conformal factor takes the form discussed by Landau and Lifshitz [1]:

$$\Omega^2(t, r) = F^2(t^2-r^2).$$

Next simplification consists in considering the form of the conformal factor which is assumed to be additive and separable with respect to t and r:

$$\Omega^2 = F(t)+G(r). \tag{2}$$

(Assuming for instance the quantum effects in the cosmological model described by the metric (1), the above property makes the Klein–Gordon wave equation separable.) Now, the Einstein equations take the form:

$$8\pi T_{01} = \tfrac{3}{2}\Omega^{-4}\dot{F}G'$$

$$8\pi T_{00} = \Omega^{-4}\left\{\tfrac{3}{4}[\dot{F}^2+G'^2]+\Omega^2\left[-\frac{2}{r}G'-G''\right]\right\}$$

$$8\pi T_{11} = \Omega^{-4}\left\{\tfrac{3}{4}[\dot{F}^2+G'^2]-\Omega^2\left[\ddot{F}-\frac{2}{r}G'\right]\right\}$$

$$8\pi T_{22} = \Omega^{-4}\left\{-\Omega^2\left[\ddot{F}-\frac{1}{r}G'-G''\right]+\tfrac{3}{4}\left[\dot{F}^2-G'^2\right]\right\}$$

$$T_{33} = \sin^2\theta T_{22}, \tag{3}$$

where dots and primes indicate differentiation with respect to t and r, respectively.

An attempt at obtaining a solution for the hydrodynamical form of the energy-momentum tensor seems to be the wrong way of solving the problem. In that case functions $F(t)$ and $G(r)$, formally being the solution of (3), give in general a negative energy and pressure.

In order to avoid such difficulties one should take into consideration other forms of the energy tensor. One of the most promising forms was proposed by Bondi [2] (originally for the Schwarzschild metric).

In order to give more physical significance to our discussion one introduces an observer moving relative to the local Minkowski coordinates $\bar{t}, \bar{x}, \bar{y}, \bar{z}$ defined by:

$$d\bar{t} = \Omega dt, \quad d\bar{x} = \Omega dr, \quad d\bar{y} = \Omega r d\theta, \quad d\bar{z} = \Omega r \sin\theta d\varphi \tag{4}$$

with Ω being treated as constant.

In the next step, we suppose that the physical content of the world, viewed by the observer moving with the velocity v in the radial direction ($\bar{x}$), consists of:

— an isotropic fluid of density $\hat{\varrho}$ and pressure $\hat{p}$,

— isotropic radiation of energy density $3\hat{\sigma}$,

— unpolarized radiation of energy density $\hat{\varepsilon}$ travelling in the radial direction.

When viewed by this moving observer, the covariant energy tensor in Minkowski coordinates is:

$$\begin{bmatrix} \hat{\varrho}+3\hat{\sigma}+\hat{\varepsilon} & -\hat{\varepsilon} & 0 & 0 \\ -\hat{\varepsilon} & \hat{p}+\hat{\sigma}+\hat{\varepsilon} & 0 & 0 \\ 0 & 0 & \hat{p}+\hat{\sigma} & 0 \\ 0 & 0 & 0 & \hat{p}+\hat{\sigma} \end{bmatrix}$$

A Lorentz transformation shows that in the local Minkowski system of coordinates the components of energy tensor are:

$$\overline{T}_{00} = (1-v^2)^{-1}(\varrho+pv^2)+\varepsilon,$$

$$\overline{T}_{11} = -(1+v^2)^{-1}(\varrho v^2+p)-\varepsilon,$$

$$\overline{T}_{22} = \overline{T}_{33} = -p,$$

$$\overline{T}_{01} = -v(1-v^2)^{-1}(\varrho+p)-\varepsilon, \tag{5}$$

where

$$\varrho \equiv \hat{\varrho}+3\hat{\sigma},$$

$$p \equiv \hat{p}+\hat{\sigma},$$

$$\varepsilon \equiv \hat{\varepsilon}(1+v)/(1-v).$$

Finally, the field equations take (in the initial system of coordinates) the following form:

$$-\tfrac{3}{2}\dot{F}G' = A,$$

$$\tfrac{3}{4}[\dot{F}^2+G'] + \Omega^2\left[\frac{2}{r}G'-G''\right] = B,$$

$$\tfrac{3}{4}[\dot{F}^2+G'^2] - \Omega^2\left[\ddot{F}-\frac{2}{r}G'\right] = C,$$

$$-\Omega^2\left[\ddot{F}-\frac{1}{r}G'-G''\right] + \tfrac{3}{4}[\dot{F}^2-G'^2] = D, \tag{6}$$

where

$$A \equiv -8\pi\Omega^6\overline{T}_{01} = (1-v^2)^{-1}v(\bar{\varrho}+\bar{p})+\bar{\varepsilon},$$

$$B \equiv 8\pi\Omega^6\overline{T}_{00} = (1-v^2)^{-1}(\bar{\varrho}+v^2\bar{p})+\bar{\varepsilon},$$

$$C \equiv 8\pi\Omega^6\overline{T}_{11} = (1-v^2)^{-1}(\bar{p}+v^2\varrho)+\bar{\varepsilon},$$

$$D \equiv 8\pi\Omega^6\overline{T}_{22} = \bar{p},$$

and

$$\bar{\varrho} \equiv 8\pi\Omega^6\varrho, \qquad \bar{p} \equiv 8\pi\Omega^6 p, \qquad \bar{\varepsilon} \equiv 8\pi\Omega^6\varepsilon.$$

We have now a set of four equations for the six physical variables $\bar{p}$, $\bar{\varrho}$, $\bar{\varepsilon}$, v (from now on denoted by p, ϱ, ε, v) and F, G; and we have to assume some suitable additional conditions.

From the physical point of view it seems reasonable to assume:

$$\varrho > 0, \qquad p > 0, \qquad v < 1 \quad \text{(in fact } v < c\text{)},$$

$$T_{00} > 0, \qquad \Omega^2 = F+G > 0. \tag{7}$$

Taking into consideration the above set of inequalities we get:

$$D+B-C > 0,$$

$$D > 0, \qquad B > 0, \qquad \Omega^2 > 0,$$

$$\left| \frac{A+D-C}{A-B-D} \right| < 1. \tag{8}$$

Since the obtaining of the general solution of the field equations (6) satisfying simultaneously inequalities (8) presents a great difficulty, let us try to find the particular solution of the simple form:

$$\Omega^2 = at^2+br^2. \tag{9}$$

Considering the inequalities (8), where A, B, C, D are expressed by (6), one obtains the set of the algebraic conditions for the coefficients a and b and for the validity region. Finally one can choose the following function:

$$\Omega^2 = 3t^2+r \tag{10}$$

as a solution of the postulated form (9), which is valid in the region where $t^2-r^2 > 0$. One obtains also the remaining functions p, ϱ, ε, v.

3. *Discussion*

The most illustrative quantities from the purely geometrical point of view are Ricci tensor and Ricci scalar, which may be expressed in the following way:

$$R_{00} = -6\Omega^{-2}+54t^2\Omega^{-4}, \qquad R_{01} = 18\Omega^{-4}tr,$$

$$R_{11} = -2\Omega^{-2}+6\Omega^{-4}r^2, \qquad R_{22} = -2r^2\Omega^{-2},$$

$$R_{33} = \sin^2\theta R_{22}, \qquad R = 6\Omega^{-6}(9t^2-r^2). \tag{11}$$

Spherical symmetry in mathematical formulation means that there exist three independent Killing vectors $K^{\mu}_{(\alpha)}$ (only vector field K components: $K^{\theta}_{(\alpha)}$ and $K^{\varphi}_{(\alpha)}$ differ from zero) satisfying the Killing equation:

$$K_{a:b}+K_{b:a} = 0.$$

Each additional symmetry would be reflected by the existence of other Killing vectors. Instead of searching for these Killing vectors one can simply note that the eigenvalues of the Ricci tensor are invariants only if Ω^2 and $(9t^2-r^2)$ are invariants (from (11)) and hence t^2 and r^2 must be constant on the orbits of the Killing vector (for general results of which this is an example, see [3]).

Thus the solution (1) with Ω^2 given by (8) has no additional symmetries except the spherical one; it presents a new inhomogeneous solution of Einstein's equations.

The obtained solution seems to have no clear physical motivation. However, there exist problems in which such a solution could be used as a mathematical model e.g. for searching for the influence of quantum effects near the cosmological singularity.

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