Vol. B14 (1983) ACTA PHYSICA POLONICA No 7

ON THE GENERAL VACUUM SOLUTION WITH A COSMOLOGICAL CONSTANT FOR BIANCHI TYPE-VI_0

By D. Lorenz

Fakultät für Physik, Universität Konstanz*

(*Received January 14, 1983*)

We investigate the field equations for Bianchi type-VI_0 space-times with a nonvanishing cosmological constant. Exact solutions are given for the vacuum case of Bertotti-Robinson-type models. In addition a reduction of the field equations to a second order differential equation is given in the general case. The general vacuum case is discussed in various equivalent ways and a transcendental solution is derived.

PACS numbers: 04.20.Jb, 98.80.Dr

1. *Introduction*

Exact vacuum solutions of Einstein's field equations with a cosmological constant Λ are known for the Bianchi-types I, II, III, V, VIII and IX (see e.g. Kramer et al. 1980). However, no general vacuum solutions with $\Lambda \neq 0$ are known for types IV, VII_h and VI_h, with the exceptional group type $h = -1/9$ (Siklos 1981).

A special situation arises in case of Bianchi type-VI_0 (with $n^\beta_\beta = 0$ in the notation of Ellis and MacCallum (1969)) which will be discussed in this paper. The metric of this class of space-times is given by

$$ds^2 = -dt^2 + R_1^2(t)dx^2 + R_2^2(t)\exp(-2qx)dy^2 + R_2^2(t)\exp(2qx)dz^2 \tag{1.1}$$

where R_i are the cosmic-scale functions and $q = \text{const}$. The field-equations to be considered are

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_1}{R_1}\frac{\dot{R}_2}{R_2} - \left(\frac{q}{R_1}\right)^2 = \varepsilon(1-\gamma) + \Lambda - \frac{e^2}{R_2^4} \tag{1.2}$$

$$2\frac{\ddot{R}_2}{R_2} + \left(\frac{\dot{R}_2}{R_2}\right)^2 + \left(\frac{q}{R_1}\right)^2 = \varepsilon(1-\gamma) + \Lambda + \frac{e^2}{R_2^4} \tag{1.3}$$

$$\left(\frac{\dot{R}_2}{R_2}\right)^2 + 2\frac{\dot{R}_1}{R_1}\frac{\dot{R}_2}{R_2} - \left(\frac{q}{R_1}\right)^2 = \varepsilon + \Lambda + \frac{e^2}{R_2^4}, \tag{1.4}$$

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* Address: Fakultät für Physik, Universität Konstanz, D-7750 Konstanz, Fed. Rep. Germany.

where e^2/R_2^4 are the components of the electromagnetic field (e^2 = const.) and the perfect fluid matter is characterized by the equation of state $p = (\gamma-1)\varepsilon$, $1 \leqslant \gamma \leqslant 2$, where ε and p are, respectively, the density and the pressure of the fluid (a dot denotes differentiation with respect to t).

The nonvanishing of the terms of Eqs. (1) involving the constant q makes the Bianchi type-VI_0 equations quite complicated. The only known general solutions are the vacuum solutions ($\varepsilon = e = \Lambda = 0$) due to Ellis and MacCallum (1969) and the stiff matter case ($\gamma = 2$) also obtained by these authors. For $\gamma = 1$ only special solutions are known (Ellis and MacCallum 1969, Evans 1978). For $\gamma \neq 1, 2$ special solutions have been found by Collins (1972) and Ruban (1978). A class of perfect fluid space-times including an electromagnetic field ($e \neq 0$) was found by Dunn and Tupper (1976) and Lorenz (1982a, b).

2. *Exact solutions*

We first consider the case $\dot{R}_1 = 0$, $\Lambda \neq 0$. It can be readily shown that no solution with $(e, \varepsilon) \neq 0$ can exist. However, for $(e, \varepsilon) = 0$, we obtain the solution

$$R_2 = (C_1 \cosh \omega t + C_2 \sinh \omega t)^{2/3} \tag{2.1}$$

with $C_1^2 = C_2^2$, $\omega^2 = -9q^2/4c$, $\Lambda = -2q^2/c$, $c = R_1^2$. If $c < 0$, this solution is real. For $\varepsilon \neq 0$ we seek solutions with the aid of the "Ansatz" (in general for $\dot{R}_1 \neq 0$)

$$\dot{R}_2^2 = \frac{g}{R_2} - q^2 + \frac{\Lambda}{3} R_2^2 - \frac{e^2}{R_2^2}, \qquad R_1 = \dot{R}_2 h, \tag{2.2}$$

where g and h are functions of R_2. We obtain

$$g' = \varepsilon(1-\gamma)R_2^2 + q^2\left(1 - \frac{R_2^2}{R_1^2}\right), \qquad (\ )' = \frac{d}{dR_2} \tag{2.3}$$

$$h' - q^2 h^3 \frac{R_2}{R_1^4} - \frac{\varepsilon}{2}\gamma h^3 \frac{R_2}{R_1^3} = 0. \tag{2.4}$$

The field equations (2) can then be decoupled to a single very complicated second order differential equation for g in case of $(e, \varepsilon, \Lambda) \neq 0$. After solving this equation the most general solution for Bianchi type VI_0 would arise. Here we present only the general vacuum equation with $(\Lambda, e) \neq 0$

$$R_2\left(g - q^2 R_2 + \frac{\Lambda}{3} R_2^3 - \frac{e^2}{R_2}\right) g'' - 6gg' + 2R_2 g'^2$$

$$- \left[(2q^2-6)R_2 - 8\frac{e^2}{R_2}\right] g' + 6q^2 g - 2q^2(2q^2+1)R_2 - 8q^2 \frac{e^2}{R_2} = 0 \tag{2.5}$$

as an illustration. However, until now no explicit solutions have been found.

If $\dot{R}_1 = 0$ we obtain a solution of (2) in case of $(e, \varepsilon) = 0$

$$g = q^2 k_2 - \frac{q^2}{3c} R_2^3 + a, \qquad a = \text{const.} \tag{2.6}$$

By setting $f = \dot{R}_2^2$, $u = R_2$ and $a = 0$ we find the solution

$$f = -\frac{q^2}{c} u^2, \qquad \Lambda = -\frac{2q^2}{c}, \tag{2.7}$$

which corresponds to the Robinson-Bertotti type solution given by Siklos (1981) and turns out to be identical with the explicit solution (2.1). For $a \neq 0$ exact solutions can be obtained in terms of elliptic functions. Introducing the new time variable τ defined by $d\tau = R_2^{-1/2} dt$ we obtain three different kind of solutions

(*i*) $c > 0$, $a > 0$

$$R_2 = (ac/q^2)^{1/3} \frac{1 + 3^{1/2} + (1 + 3^{1/2}) cn(\tau\omega_1, k)}{1 + cn(\tau\omega_1, k)} \tag{2.8}$$

(*ii*) $c < 0$, $a > 0$

$$R_2 = (-ac/q^2)^{1/3} \frac{3^{1/2} - 1 + (3^{1/2} + 1) cn(\tau\omega_2, k')}{1 - cn(\tau\omega_2, k')} \tag{2.9}$$

(*iii*) $c < 0$, $a < 0$

$$R_2 = (ac/q^2)^{1/3} \frac{3^{1/2} + 1 - (3^{1/2} - 1) cn(\tau\omega_3, k')}{1 + cn(\tau\omega_3, k')}, \tag{2.10}$$

where

$$\omega_1 = 3^{1/4}(aq^4/c^2)^{1/6}, \qquad \omega_2 = -3^{1/4}(aq^4/c^2)^{1/6},$$

$$\omega_3 = 3^{1/4}(-aq^4/c^2)^{1/6}, \qquad k^2 = (2 + 3^{1/2})/4 \quad \text{and} \quad k'^2 = 1 - k^2.$$

3. *The general vacuum case*

In this section we derive various equivalent differential equations which determine the general vacuum solution with $\Lambda \neq 0$ for Bianchi type VI_0. We start our journey through the regime of differential equations by introducing the new time variable η by $d\eta = R_1 R_2^2 dt$. From (1) we obtain in case of $(e, \varepsilon) = 0$

$$R_1^2 \{R_2^4 R_1^2 [\ln (R_1 R_2)]^{\cdot}\}^{\cdot} = 2q^2 + 2\Lambda R_1^2 \tag{3.1}$$

$$\{R_2^4 R_1^2 [\ln R_2^2]^{\cdot}\}^{\cdot} = 2\Lambda \tag{3.2}$$

$$R_2^4 R_1^4 (\ln R_2)^{\cdot} [\ln (R_2 R_1^2)]^{\cdot} = q^2 + \Lambda R_1^2, \qquad (\ )^{\cdot} = \frac{d}{d\eta} \tag{3.3}$$

Eq. (3.2) gives

$$R_2^4 R_1^2 [\ln R_2^2]^{\cdot} = 2\Lambda(\eta - \eta_0), \tag{3.4}$$

where η_0 is an integration constant which we set equal to zero for convenience. Substituting (3.4) into (3.3) we obtain

$$R_2 \ddot{R}_2 \eta^2 + \dot{R}_2^2 [2\eta^2 + a^2 R_2^4] = 0, \tag{3.5}$$

where $a^2 = q^2/\Lambda^2$. One easily verifies that

$$R_2^4 = -a^2\eta^2, \quad R_1^2 = -2a^2 \tag{3.6}$$

is a special solution, which is in fact identical with (2.1) and (2.7). However, no other polynomial solutions in terms of η can be found.

Thus we proceed further by introducing the variable $g = R_2^3$. We find

$$\eta^2 \ddot{g} + b^2 \dot{g}^2 g^{1/3} = 0, \tag{3.7}$$

where $b^2 = q^2/3\Lambda^2$. Another change of variable

$$g = \eta^{3/2}\tau(\xi), \quad \xi = \ln \eta \tag{3.8}$$

yields

$$\tau'' + 2\tau' + \tfrac{3}{4}\tau + b^2(\tau' + \tfrac{3}{2}\tau)^2 \tau^{1/3} = 0, \quad (\ )' = \frac{d}{d\xi}. \tag{3.9}$$

Changing the variables from

$$p(\tau) = \tau'(\xi), \tag{3.10}$$

to

$$u = p + \tfrac{3}{2}\tau \tag{3.11}$$

we arrive at an Abel second-order nonlinear differential equation (Kamke (1977))

$$[u + g(\tau)]u' = f_2(\tau)u^2 + f_1(\tau)u + f_0(\tau), \quad (\ )' = \frac{d}{d\tau}. \tag{3.12}$$

Analytic solutions to such an equation can be readily obtained only if there exist certain very special relationships between the functions

$$f_0(\tau) = \tfrac{3}{4}\tau, \quad f_1(\tau) = -\tfrac{1}{2}, \quad f_2(\tau) = -b^2\tau^{1/3}, \quad g(\tau) = -\tfrac{3}{2}\tau. \tag{3.13}$$

Our equation (3.12) does not satisfy these relationships.

However, we can perform another set of transformations in hope of finding the explicit solution. From

$$y(\tau) = (u - \tfrac{3}{2}\tau)E, \quad E = \exp(-\textstyle\int f_2 d\tau) \tag{3.14}$$

we obtain

$$yy' = (-2-c^2\tau^{4/3})\exp(c^2/4)\exp(4\tau/3)y-(3c^2/4)\tau^{7/3}\exp(c^2/2)\exp(8\tau/3) \quad (3.15)$$

where $c^2 = q^2/\Lambda$.

Unfortunately the integral

$$F(\tau) = \int h_1 d\tau, \quad h_1(\tau) = (-2-c^2\tau^{4/3})\exp(c^2/4)\exp(4\tau/3) \quad (3.16)$$

cannot be evaluated in closed form so that the method described by Kamke (p. 27) does not apply.

Another reduction of (3.12) can be obtained by

$$u-\tfrac{3}{2}\tau = \frac{1}{y(\tau)}, \quad (3.17)$$

and we arrive at an Abel first-order differential equation

$$y' = A_3(\tau)y^3+A_2(\tau)y^2+A_1(\tau)y, \quad (3.18)$$

where

$$A_3(\tau) = (3c^2/4)\tau^{7/3}\exp(c^2/2+8\tau/3) \quad (3.19)$$

$$A_2(\tau) = (2+c^2\tau^{4/3})\exp(c^2/4+4\tau/3) \quad (3.20)$$

$$A_1(\tau) = (c^2/3)\tau^{1/3}. \quad (3.21)$$

This equation is very similar to that considered by Shikin (1967) and Jacobs (1969) for the radiation-magnetic case for the Bianchi type-I model with $\gamma = 4/3$. As pointed out by Jacobs, "it is this equation which presently frustrates our efforts to obtain the analytic solution". Applying the method given by Jacobs (due to Camppolattaro) we obtain the following transcendental solution of (3.18). Eq. (3.18) is rewritten in the form

$$y' = A_3y[y-B_1(\tau)][y-B_2(\tau)], \quad (3.22)$$

where B_1 and B_2 are defined as the roots of the equation

$$B^2+(A_2/A_3)B+(A_1/A_3) = 0 \quad (3.23)$$

with

$$A_2/A_3 = (4c^2/3)[2+c^2\tau^{4/3}]\tau^{-7/3}\exp(-c^2/2-4\tau/3) \quad (3.24)$$

$$A_1/A_3 = 4\exp(-c^2/2)\tau^{-2}. \quad (3.25)$$

A partial fraction expansion

$$\frac{1}{y(y-B_1)(y-B_2)} = \frac{F(\tau)}{y}+\frac{G(\tau)}{y-B_1}+\frac{H(\tau)}{y-B_2} \quad (3.26)$$

leads to the relations

$$F = 1/(B_1 B_2), \quad G = (2B_1 + B_2)/(B_1 B_2(B_2 - B_1)), \tag{3.27}$$

$$H = -(2B_2 + B_1)/(B_1 B_2). \tag{3.28}$$

By defining

$$J(y, \tau) = F(\tau) \ln y + G(\tau) \ln (y - B_1) + H(\tau) \ln (y - B_2) = K = \text{const.}, \tag{3.29}$$

we obtain the expression

$$F' \ln y + G' \ln (y - B_1) + H' \ln (y - B_2) - GB_1'/(y - B_1) - HB_2'/(y - B_2) + A_3 = 0, \tag{3.30}$$

where $(\ )' = \frac{d}{d\tau}$. Multiplying (3.30) by H and (3.29) by H', subtracting, and exponating yields the transcendental solution

$$y^{(HF' - FH')}(y - B_1)^{(HG' - GH')} \exp\left[-HGB_1'/(y - B_1) - H^2 B_2'/(y - B_2)\right]$$
$$= \exp(-HA_3 - KH'). \tag{3.31}$$

Our solution would be immediately completed by the equation

$$R_1^2 = 3\Lambda \tau^{-1/3} y \tag{3.32}$$

which can be derived from (3.2) in the same manner.

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