REMARKS ON THE KRÓLIKOWSKI-RZEWUSKI EQUATION FOR A DISTINGUISHED COMPONENT OF A STATE VECTOR AND ASYMPTOTIC PROPERTIES OF ITS SOLUTIONS

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We generalize the Królikowski-Rzewuski equation for a distinguished component of the state vector and give the new formulae for a quasipotential and inhomogeneity occurring there. We also study the strong limit (when time goes to infinity) of the quantum evolution operator occurring in this equation. A connection of this limit with the ergodic theorem on the time average is proved.

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1. Introduction

In quite many physical problems information on how the states of a quantum system evolve in time t should be supplemented by the asymptotic form of its state vector when $t \to \infty$ [1–8]. One can calculate the time evolution of the physical system if the Hamiltonian H and the initial state $|\psi, t_0\rangle \overset{\mathrm{df}}{=} |\psi\rangle$ of the system at $t = t_0$ are known.

Frequently we are interested in some particular properties of the system only, in these which are described by the components of the state vector from a closed subspace $\mathscr{H}_{||}$ of a Hilbert space $\mathscr{H}$ [4–8]. The knowledge of the limit, when $t \to \infty$, of these components is often useful. The time evolution of the component of a state vector is described by the Królikowski–Rzewuski equation for the distinguished component of a state vector [5], by the so-called master equations [4, 6–8] and so on.

The aim of this paper is twofold: first (Section 2), we generalize the Królikowski–Rzewuski equation [5] so that the generalized equation could simply describe transitions between any two different subspaces of a Hilbert space and using the Laplace transform formalism we give the new formulae for the quasipotential and inhomogeneity occurring there. Next (Section 3), the Laplace transforms method is used as a basis for a general

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discussion of a limit (when $t \to \infty$) of the evolution operator acting in a proper subspace $\mathcal{H}_{\|}$ of a state space $\mathcal{H}$. We show that there is a connection of this limit with von Neuman's theorem on the time average.

2. Some generalization of the Królikowski–Rzewuski equation for the distinguished component of a state vector

Let us consider a physical system described by the Hamiltonian H. We assume that H is a selfadjoint and linear operator acting in a Hilbert space $\mathcal{H}$. The states of the investigated system are represented by time dependent vectors $|\psi; t\rangle$ belonging to the space $\mathcal{H}$. These states at the initial instant t_0 of time — $|\psi; t_0\rangle \overset{\text{df}}{=} |\psi\rangle$ and at the later instant t are joined together by the evolution operator $U(t, t_0)$, which belongs to the one-parameter family of operators $\{U(t, t_0)\}_{t \geqslant t_0}$

$$|\psi; t\rangle = U(t, t_0)|\psi; t_0\rangle \tag{1}$$

$$\overset{\text{df}}{=} U(t, t_0)|\psi\rangle. \tag{1'}$$

The operators $U(t, t_0)$ are defined by the equation

$$i\frac{\partial}{\partial t}U(t, t_0)\,|\psi\rangle = HU(t, t_0)\,|\psi\rangle \tag{2}$$

$U(t_0, t_0) \equiv \mathbf{1}$ is the unit operator in $\mathcal{H}$.

This means that the one-parameter family of operators $U(t, t_0) \in \{U(t, t_0)\}_{t \geqslant t_0}$ is for $t \geqslant t_0$ the abelian unitary semigroup, and

$$U(t, t_0)\,|\psi\rangle \equiv e^{-i(t-t_0)H}|\psi\rangle \tag{3}$$

so

$$U(t, t_0)\,|\psi\rangle \equiv U(t-t_0)\,|\psi\rangle. \tag{4}$$

Using relation (1) one can write the component of the state vector we are interested in in the following way

$$P|\psi; t\rangle \equiv PU(t, t_0)\,|\psi\rangle \in P\mathcal{H} \overset{\text{df}}{=} \mathcal{H}_{\|}, \tag{5}$$

$$P = P^2 = P^+ \text{ is the projection operator in } \mathcal{H}. \tag{6}$$

Thus the projector P divides the Hilbert space $\mathcal{H}$ into two orthogonal parts: $\mathcal{H}_{\|}$ and $\mathcal{H} \ominus \mathcal{H}_{\|} \overset{\text{df}}{=} \mathcal{H}_{\perp}$. The projector $Q \overset{\text{df}}{=} \mathbf{1}-P$ is associated with the orthogonal complement to the subspace $\mathcal{H}_{\|}$

$$Q \overset{\text{df}}{=} \mathbf{1}-P,$$

$$Q\mathcal{H} \overset{\text{df}}{=} (\mathbf{1}-P)\mathcal{H} \equiv \mathcal{H} \ominus \mathcal{H}_{\|} \overset{\text{df}}{=} \mathcal{H}_{\perp}. \tag{7}$$

Now let $|\psi; t_0\rangle \equiv |\psi\rangle$ be an element of some closed subspace $\mathscr{H}_\Pi \subset \mathscr{H}$ — so we assume that our system at the initial instant t_0 has the properties described by vectors from $\mathscr{H}_\Pi$, i.e. let

$$|\psi\rangle = \Pi|\psi\rangle, \tag{8}$$

where

$$\Pi = \Pi^2 = \Pi^+. \tag{9}$$

Using assumption (8), Eq. (2) can be converted into equations describing the behaviour in time of $PU(t, t_0)\,|\psi\rangle \equiv PU(t, t_0)\Pi\,|\psi\rangle$

$$\left(i\frac{\partial}{\partial t} - PHP\right) PU(t, t_0)\Pi|\psi\rangle = PHQQU(t, t_0)\Pi|\psi\rangle, \tag{10}$$

and

$$\left(i\frac{\partial}{\partial t} - QHQ\right) QU(t, t_0)\Pi|\psi\rangle = QHPPU(t, t_0)\Pi|\psi\rangle, \tag{11}$$

where

$$PU(t_0, t_0)\Pi \equiv P\Pi, \tag{12}$$

$$QU(t_0, t_0)\Pi \equiv Q\Pi. \tag{13}$$

From this it follows that

$$i\frac{\partial}{\partial t} PU(t, t_0)\Pi|\psi\rangle\Big|_{t=t_0} = PH\Pi|\psi\rangle, \tag{14}$$

$$i\frac{\partial}{\partial t} QU(t, t_0)\Pi|\psi\rangle\Big|_{t=t_0} = QH\Pi|\psi\rangle. \tag{15}$$

Without loss of generality, we can limit our considerations to Π such that

$$P\Pi = P \tag{16}$$

or

$$P\Pi = 0. \tag{17}$$

So, let (16) hold, then we easily obtain the following properties:

$$\Pi P = (\Pi P)^2 = (\Pi P)^+ \equiv P\Pi \tag{18}$$

and

$$Q\Pi \equiv \Pi - P \tag{19}$$

$$Q\Pi = (Q\Pi)^2 = (Q\Pi)^+ \tag{20}$$

so

$$Q\Pi\mathscr{H} \equiv \Pi\mathscr{H} \ominus P\mathscr{H} \subset Q\mathscr{H} \equiv \mathscr{H}_{\perp}. \tag{21}$$

On the other hand, if (17) takes place, then

$$Q\Pi \equiv \Pi. \tag{22}$$

The operators

$$T_\Pi(t, t_0) \stackrel{\mathrm{df}}{=} PU(t, t_0)\Pi \equiv T_\Pi(t-t_0) \tag{23}$$

are not unitary and map the subspace $\mathscr{H}_\Pi \equiv \Pi\mathscr{H}$ onto subspace $\mathscr{H}_{\|} \equiv P\mathscr{H}$

$$T_\Pi(t, t_0): |\psi\rangle \in \mathscr{H}_\Pi \to T_\Pi(t, t_0)\,|\psi\rangle \in \mathscr{H}_{\|}. \tag{24}$$

Here (if (16) holds)

$$T_\Pi(t_0, t_0) \equiv P \equiv \mathbf{1}_{\mathscr{H}_{\|}} \text{ — is the unit operator in } \mathscr{H}_{\|}, \tag{25}$$

or (if (17) holds)

$$T_\Pi(t_0, t_0) \equiv 0. \tag{26}$$

In particular, if

$$\Pi \equiv \mathbf{1} \tag{27}$$

then (for $t \geqslant t_0$)

$$T_1(t, t_0) \stackrel{\mathrm{df}}{=} T(t, t_0) \equiv T(t-t_0) \tag{28}$$

is a partial isometry

$$T^+(t, t_0)T(t, t_0) \equiv P(t, t_0), \tag{29}$$

where

$$P(t, t_0) \stackrel{\mathrm{df}}{=} U^+(t, t_0)PU(t, t_0) \doteq P^2(t, t_0) = P^+(t, t_0) \tag{30}$$

as well as $T^+(t, t_0)$

$$T(t, t_0)T^+(t, t_0) = P \equiv T(t_0, t_0). \tag{31}$$

The operator $T(t, t_0)$ maps the whole space of states $\mathscr{H}$ onto $\mathscr{H}_{\|}$

$$T(t, t_0): |\psi\rangle \in \mathscr{H} \to T(t, t_0)\,|\psi\rangle \in \mathscr{H}_{\|}. \tag{32}$$

On the other hand, if

$$\Pi \equiv P \tag{33}$$

then

$$T_P(t, t_0) \stackrel{\mathrm{df}}{\equiv} T(t, t_0)P \equiv PU(t, t_0)P \stackrel{\mathrm{df}}{=} U_{\|}(t, t_0) \equiv U_{\|}(t-t_0). \tag{34}$$

The operator $U_{\|}(t, t_0)$ is not unitary

$$U_{\|}^{+}(t, t_0)U_{\|}(t, t_0) \equiv PP(t, t_0)P, \tag{35}$$

$$U_{\|}(t_0, t_0) = P \equiv \mathbf{1}_{\mathscr{H}_{\|}}. \tag{36}$$

It maps a subspace $\mathscr{H}_{\|} \equiv P\mathscr{H}$ on itself

$$U_{\|}(t, t_0): |\psi\rangle \in \mathscr{H}_{\|} \to U_{\|}(t, t_0)\,|\psi\rangle \in \mathscr{H}_{\|}. \tag{37}$$

Let us note that formula (22) can also be written as follows:

$$T_{\Pi}(t, t_0) \equiv PU(t, t_0)P\Pi + PU(t, t_0)Q\Pi \tag{38}$$

$$\stackrel{\text{df}}{=} U_{\|}(t, t_0)\Pi + J(t, t_0)\Pi, \tag{39}$$

where

$$J(t, t_0)\Pi \stackrel{\text{df}}{=} J_{\Pi}(t, t_0) \stackrel{\text{df}}{=} PU(t, t_0)Q\Pi \equiv J_{\Pi}(t-t_0) \tag{40}$$

so

$$J(t_0, t_0) \equiv J_{\Pi}(t_0, t_0) \equiv 0. \tag{41}$$

These relations are a consequence of (7) and (34).

From (38) it follows that if $P\Pi = 0$ (i.e. if (17) holds), then

$$T_{\Pi}(t, t_0) \equiv PU(t, t_0)Q\Pi \equiv J_{\Pi}(t, t_0). \tag{42}$$

In relation to states in $\mathscr{H}_{\|}$, the operator $J_{\Pi}(t, t_0)$ can be treated as an external "source". It maps subspace $\Pi\mathscr{H} \ominus P\mathscr{H}$ (or $Q\mathscr{H} \equiv \mathscr{H}_{\perp}$ — if $\Pi = \mathbf{1}$) onto subspace $\mathscr{H}_{\|} \equiv P\mathscr{H}$

$$J_{\Pi}(t, t_0): |\psi\rangle \in \Pi\mathscr{H} \ominus P\mathscr{H} \to J_{\Pi}(t, t_0)\,|\psi\rangle \in \mathscr{H}_{\|} \tag{43}$$

and, if we think in terms of particles, it describes a regeneration or creation of objects, whose state vectors belong to $\mathscr{H}_{\|}$ from the states (or particles) included in $\Pi\mathscr{H} \ominus \mathscr{H}_{\|}$.

Now, the solution of Eq. (11) with the initial conditions (13) and (15) has the form

$$QU(t, t_0)\Pi|\psi\rangle = e^{-i(t-t_0)QHQ}Q\Pi|\psi\rangle + G_Q * QHPT_{\Pi}(t, t_0)\,|\psi\rangle \tag{44}$$

where the operator product $G_Q * QHPT_{\Pi}(t, t_0)$ is defined as the convolution

$$f * g(t, t_0) \stackrel{\text{df}}{=} \int_{t_0}^{\infty} f(t-\sigma)g(\sigma)d\sigma \tag{45}$$

and $G_Q(t)$ is the retarded operator solution of the equation

$$\left(i\frac{\partial}{\partial t} - QHQ\right)G_Q(t) = Q\delta(t)$$

$$G_Q(t < 0) = 0 \tag{46}$$

that is

$$G_Q(t) = -i\theta(t)e^{-itQHQ}Q. \tag{47}$$

So, from Eq. (10) it follows that the time evolution of the distinguished component $P|\psi; t\rangle \equiv T_\Pi(t, t_0)|\psi\rangle$ (5) of the state vector $|\psi; t\rangle$ is determined, for $t \geqslant t_0$, by the equation

$$\left(i\frac{\partial}{\partial t} - PHP\right)T_\Pi(t, t_0)|\psi\rangle = \chi_\Pi(t, t_0)|\psi\rangle$$

$$-i\int_{t_0}^{\infty} K(t-\sigma)T_\Pi(\sigma)|\psi\rangle d\sigma \tag{48}$$

with the initial conditions (25) or (26).

Here

$$\chi_\Pi(t, t_0) \stackrel{\text{df}}{=} PHQe^{-i(t-t_0)QHQ}Q\Pi \equiv \chi_\Pi(t-t_0), \tag{49}$$

$$K(t) \stackrel{\text{df}}{=} \theta(t)PHQe^{-itQHQ}QHP. \tag{50}$$

This inhomogeneous equation is closely connected with the Królikowski–Rzewuski equation for the distinguished component of a state vector [5]. To obtain the Królikowski–Rzewuski equation one must put $\Pi = \mathbf{1}$ in Eq. (48).

For $\Pi = P$ we obtain the so-called homogeneous equation corresponding to (48), i.e.

$$\left(i\frac{\partial}{\partial t} - PHP\right)U_\parallel(t, t_0)|\psi\rangle = -i\int_{t_0}^{\infty} K(t-\sigma)U_\parallel(\sigma)|\psi\rangle d\sigma \tag{51}$$

$$\equiv -iK * U_\parallel(t, t_0)|\psi\rangle \tag{51'}$$

$$U_\parallel(t_0, t_0) = P. \tag{52}$$

If $U_\parallel^{-1}(t, t_0)$ exists, then Eq. (51) can be rewritten as

$$\left\{i\frac{\partial}{\partial t} - PHP - [-iK * U_\parallel(t, t_0)]U_\parallel^{-1}(t, t_0)\right\}U_\parallel(t, t_0)|\psi\rangle = 0. \tag{53}$$

So, if $U_\parallel^{-1}(t, t_0)$ exists, then we can replace the integro-differential equation by the equivalent differential one with a time-dependent "quasipotential"

$$V_\parallel(t, t_0) \stackrel{\text{df}}{=} -i[K * U_\parallel(t, t_0)]U_\parallel^{-1}(t, t_0). \tag{54}$$

One can obtain a more convenient form of this quasipotential in the Laplace transform language [11]. We introduce the Laplace transform as follows [13]

$$\mathscr{L}[f(t)](z, \tau) \equiv \tilde{f}(z, \tau) \stackrel{\text{df}}{=} \int_{\tau}^{\infty} f(t)e^{-zt}dt, \tag{55}$$

$$\operatorname{Re} z \geqslant \sigma > 0$$

and the inverse transform according to

$$f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{f}(z, \tau) e^{zt} dz. \tag{56}$$

Thus, taking the Laplace transform of Eq. (51) and then calculating $\tilde{U}_{\|}(z, t_0)$ we can write formula (54) as

$$V_{\|}(t, t_0) = -i\{ \int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{K}(z, 0)(z+iPHP+\tilde{K}(z, 0)^{-1})Pe^{z(t-t_0)}dz\}$$

$$\times \{ \int_{\sigma-i\infty}^{\sigma+i\infty} (z+iPHP+\tilde{K}(z, 0))^{-1} Pe^{z(t-t_0)}dz\}^{-1} \equiv V_{\|}(t-t_0), \tag{57}$$

where

$$\tilde{K}(z, 0) \equiv \tilde{K}(z, \tau)|_{\tau=0} = PHQ(z+iQHQ)^{-1}QHP, \tag{58}$$

and

$$(z+iPHP+\tilde{K}(z, 0))^{-1}Pe^{-zt_0} \equiv iPR(iz, H)Pe^{-zt_0} = P\tilde{U}(z, t_0)P \equiv \tilde{U}_{\|}(z, t_0) \tag{59}$$

$$R(s, H) \overset{\text{df}}{=} (s-H)^{-1} \text{ is the resolvent of the Hamiltonian } H. \tag{60}$$

Let us note that an operator $U_{\|}(t, t_0)$ being a solution of Eq. (51) also fulfils the following equation:

$$\left\{ i\frac{\partial}{\partial t} - H_{\|}(t, t_0) \right\} U_{\|}(t, t_0)\,|\psi\rangle = 0, \tag{61}$$

$$U_{\|}(t_0, t_0) = P,$$

where

$$H_{\|}(t, t_0) \overset{\text{df}}{=} i\left[\frac{\partial}{\partial t} U_{\|}(t, t_0) \right] U_{\|}^{-1}(t, t_0). \tag{62}$$

And now, taking into account (2), (6), (7) and (34) we can write that

$$H_{\|}(t, t_0) \equiv PHP + PHQQU(t-t_0)P[PU(t-t_0)P]^{-1} \tag{63}$$

$$\equiv PHP + V_{\|}(t-t_0) \tag{64}$$

that is

$$V_{\|}(t-t_0) = PHQ\{ \int_{\sigma-i\infty}^{\sigma+i\infty} QR(iz, H)Pe^{z(t-t_0)}dz\}$$

$$\times \{ \int_{\sigma-i\infty}^{\sigma+i\infty} PR(iz, H)Pe^{z(t-t_0)}dz\}^{-1}. \tag{65}$$

In the case, when

$$\left[\frac{\partial}{\partial t} U_{\|}(t, t_0), U_{\|}(t, t_0)\right]_- = 0 \tag{66}$$

we have

$$H_{\|}(t, t_0) \equiv i \frac{\partial}{\partial t} \ln U_{\|}(t, t_0). \tag{67}$$

There are cases when (66) takes place, e.g. when $\mathscr{H}_{\|}$ is one-dimensional [9], in the Lee model [10] and so on.

We can express the solutions $T_\Pi(t, t_0)$ of the inhomogeneous equation (48) by the solutions $U_{\|}(t, t_0)$ of the homogeneous one (51). According to a general theory of differential equations [17] for a solution $T_\Pi(t, t_0)$ of Eq. (48) we obtain

$$T_\Pi(t, t_0) = U_{\|}(t, t_0)\Pi + U_{\|} * \chi_\Pi(t, t_0). \tag{68}$$

Comparing (68) with expressions (39) and (40) we see that

$$J_\Pi(t, t_0) \equiv PU(t, t_0)Q\Pi = U_{\|} * \chi_\Pi(t, t_0). \tag{69}$$

Formula (68) enables us to rewrite the inhomogeneous integro-differential Eq. (48) as a differential one. This has already been done for the homogeneous equation (51) — see (53), (54) and (61), (62). The homogeneous integro-differential equation (51) corresponds to the inhomogeneous one (48). Thus the differential equation (61) equivalent to Eq. (51) must also correspond to Eq. (48). So, the differential equation obtained instead of the integro-differential (48) should contain the homogeneous differential one (61) with the effective Hamiltonian $H_{\|}(t, t_0)$ (62), (64) and, additionally, some timedependent inhomogeneity, which we will denote as $j_\Pi(t, t_0)$. In other words, the inhomogeneous differential equation for $T_\Pi(t, t_0)$ should have the following form

$$\left\{i \frac{\partial}{\partial t} - H_{\|}(t, t_0)\right\} T_\Pi(t, t_0) \,|\psi\rangle = j_\Pi(t, t_0)\,|\psi\rangle. \tag{70}$$

(Here the initial conditions should be the same as for Eq. (48)). Now, putting the expression (68) for $T_\Pi(t, t_0)$ into Eq. (70), and taking into account that $U_{\|}(t, t_0)$ fulfils (61), one obtains that

$$j_\Pi(t, t_0) \equiv \left\{i \frac{\partial}{\partial t} - H_{\|}(t, t_0)\right\} J_\Pi(t, t_0) \tag{71}$$

where $J_\Pi(t, t_0)$ — see (69), and in terms of the Laplace transforms

$$J_\Pi(t, t_0) \equiv \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} (z + iPHP + \tilde{K}(z, 0))^{-1}$$

$$\times PHQ(z+iQHQ)^{-1}Q\Pi e^{z(t-t_0)}dz \tag{72}$$

$$\equiv \frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} PR(iz, H)Q\Pi e^{z(t-t_0)}dz \tag{73}$$

and for the definition of $H_{\|}(t, t_0)$ see (62), (64), (56); for $\tilde{K}(z, 0)$ see (58).

It is easy to verify that Eqs. (48) and (70) are equivalent if an inverse to the operator $U_{\|}(t, t_0)$ exists, because then the operator $H_{\|}(t, t_0)$ exists.

Formulae (57) and (72) can be treated as a starting point for the approximate calculations of a quasipotential $V_{\|}(t, t_0)$ and inhomogeneity $j_{\Pi}(t, t_0)$ by different methods (depending on particular needs). Formulae (23), (34), (38), (55) will be useful for the calculations of the asymptotic form of $T_{\Pi}(t, t_0)$ and so on.

3. The ergodic properties of the Królikowski–Rzewuski equation

In this Section we shall investigate the limits $\lim_{t\to\infty} |\psi; t\rangle$ and $\lim_{t\to\infty} P|\psi; t\rangle$ (weak or strong — if they exist), strictly speaking, we shall study the limits (weak or strong) $\lim_{t\to\infty} U(t, t_0)$ and $\lim_{t\to\infty} PU(t, t_0)\Pi \equiv \lim_{t\to\infty} T_{\Pi}(t, t_0)$. If they exist, they can be calculated precisely by means of the Laplace transform formalism (i.e. by the use of the so-called Abel limit technique [16]). Namely, as is well known, if for a function of the parameter $t-f(t)$ the $\lim_{t\to\infty} f(t)$ and the Laplace transform $\tilde{f}(z, t_0)$ (2.55) exist, then [13, 16]

$$\lim_{t\to\infty} f(t) = \lim_{\substack{z\to 0 \\ |\arg z| < \frac{\pi}{2}-\varepsilon}} z\tilde{f}(z, t_0) \tag{1}$$

where $0 < \varepsilon$ is an arbitrarily small parameter.

In our case (see (2.2), (2.3), (2.55), (2.60))

$$\tilde{U}(z, t_0) = i(iz-H)^{-1}e^{-zt_0} \equiv iR(iz, H)e^{-zt_0} \tag{2}$$

and (see (2.23), (2.48), (2.55))

$$\tilde{T}_{\Pi}(z, t_0) \equiv \widetilde{PU(z, t_0)\Pi} = iP(iz-H)^{-1}\Pi e^{-zt_0}$$
$$\equiv iPR(iz, H)\Pi e^{-zt_0} = P\tilde{U}(z, t_0)\Pi e^{-zt_0}. \tag{3}$$

Let us define the set of states

$$\mathcal{N}(H) \overset{\text{df}}{=} \{|\varphi\rangle \in \mathcal{H}: H|\varphi\rangle = 0\}. \tag{4}$$

The set $\mathcal{N}(H)$ is the kernel of the operator H. It is a closed subspace of $\mathcal{H}$.

Let B be the projector on $\mathcal{N}(H)$

$$\mathcal{N}(H) \overset{\text{df}}{=} B\mathcal{H} \quad \text{and} \quad B = B^2 = B^+. \tag{5}$$

The subspace $\mathcal{N}(H)$ contains the null vector 0 and (possibly) other vectors $|\varphi\rangle \neq 0$, which have a simple physical interpretation — they are the vacuum states. (The vacuum may be degenerate.)

Now, because H is selfadjoint, there exists a spectral measure $E(\lambda)$ such that [12, 16]

$$\langle H\psi|\eta\rangle = \int_{-\infty}^{\infty} \lambda d\langle E(\lambda)\psi|\eta\rangle, \tag{6}$$

(where λ is a real and $|\psi\rangle \in \mathcal{D}(H)$), and for each $|\psi\rangle \in \mathcal{H}$

$$\|z\tilde{U}(z, t_0)\,|\psi\rangle\|^2 \equiv \|zR(iz, H)e^{-zt_0}|\psi\rangle\|^2$$

$$= \int_{-\infty}^{\infty} \frac{|z|^2}{|iz+\lambda|^2} e^{-2(\mathrm{Re}\, z)t_0} d\,\|E(\lambda)\,|\psi\rangle\|^2, \tag{7}$$

i.e.

$$\lim_{\substack{z\to 0\\ |\arg z| < \frac{\pi}{2} - \varepsilon}} \|zU(z, t_0)\,|\psi\rangle\|^2$$

$$\equiv \lim_{\substack{z\to 0\\ |\arg z| < \frac{\pi}{2} - \varepsilon}} \int_{-\infty}^{\infty} \frac{|z|^2}{|iz+\lambda|^2} e^{-2(\mathrm{Re}\, z)t_0} d\,\|E(\lambda)\,|\psi\rangle\|^2. \tag{8}$$

We have the restrictions

$$|\arg z| < \frac{\pi}{2} - \varepsilon. \tag{9}$$

So, if

$$z = \mathrm{Re}\, z + i\, \mathrm{Im}\, z \overset{\mathrm{df}}{=} a + ib \tag{10}$$

then

$$\left|\frac{b}{a}\right| \equiv |\mathrm{tg} \arg z| \overset{\mathrm{df}}{=} |\mathrm{tg}\, \alpha| < \infty, \tag{11}$$

and

$$\frac{|z|^2}{|iz+\lambda|^2} \equiv \frac{a^2+b^2}{a^2+(b+\lambda)^2} < 1+\mathrm{tg}^2\, \alpha \leqslant C \tag{12}$$

(here $C = \mathrm{const} < 1+\mathrm{ctg}^2\, \varepsilon$).

One can see that

$$\lim_{\substack{z\to 0\\ |\arg z| < \frac{\pi}{2} - \varepsilon}} \frac{|z|^2}{|iz+\lambda|^2} e^{-2(\mathrm{Re} z)t_0} \equiv \begin{cases} 0 \text{ if } \lambda \neq 0 \\ 1 \text{ if } \lambda = 0. \end{cases} \tag{13}$$

Now

(a) let $|\psi\rangle \in \mathscr{H} \ominus \mathscr{N}(H)$ then from (12), (13) and (7) we conclude (by Lebesque theorem on dominated convergence) that

$$\lim_{\substack{z\to 0\\ |\arg z|<\frac{\pi}{2}-\varepsilon}} \|z\tilde{U}(z, t_0)\,|\psi\rangle\| = 0. \tag{14}$$

(b) let $|\psi\rangle \in \mathscr{N}(H)$, then

$$(iz-H)^{-1}|\psi\rangle \equiv \frac{1}{iz}\,|\psi\rangle. \tag{15}$$

In this way, we have found that for all $|\psi\rangle \in \mathscr{N}(H)$

$$\lim_{\substack{z\to 0\\ |\arg z|<\frac{\pi}{2}-\varepsilon}} \|iz(iz-H)^{-1}e^{-zt_0}|\psi\rangle - |\psi\rangle\| = 0. \tag{16}$$

Collecting the results (a), (b) we can write

$$s-\lim_{\substack{z\to 0\\ |\arg z|<\frac{\pi}{2}-\varepsilon}} z\tilde{U}(z, t_0) \equiv s-\lim_{\substack{z\to 0\\ |\arg z|<\frac{\pi}{2}-\varepsilon}} izR(iz, H)e^{-zt_0} = B. \tag{17}$$

Hence (see (3))

$$s-\lim_{\substack{z\to 0\\ |\arg z|<\frac{\pi}{2}-\varepsilon}} z\widetilde{PU(z, t_0)\Pi} \equiv s-\lim_{\substack{z\to 0\\ |\arg z|<\frac{\pi}{2}-\varepsilon}} z\tilde{T}_\Pi(z, t_0) = PB\Pi. \tag{18}$$

One can easily prove that $s-\lim\limits_{z\to 0} z\tilde{U}(z, t_0)$ is a projector. Indeed, from the first resolvent equation (i.e. the Hilbert equation) [12, 16] we have that (see (2))

$$sze^{(s+z)t_0}\tilde{U}(s, t_0)\tilde{U}(z, t_0)\,|\psi\rangle$$

$$= \frac{s}{s-z}\, ze^{zt_0}\tilde{U}(z, t_0)\,|\psi\rangle - \frac{z}{s-z}\, se^{st_0}\tilde{U}(s, t_0)\,|\psi\rangle. \tag{19}$$

So, taking the limit $z \to 0$ $\left(\text{if } |\arg z| < \dfrac{\pi}{2} - \varepsilon\right)$ we obtain (see (17)):

$$se^{st_0}\tilde{U}(s, t_0)B|\psi\rangle = B|\psi\rangle. \tag{20}$$

From this and (17) it follows that

$$\lim_{\substack{s\to 0\\ |\arg s|<\frac{\pi}{2}-\varepsilon}} e^{st_0}s\tilde{U}(s, t_0)B|\psi\rangle = B|\psi\rangle \tag{21}$$

i.e. that for all $|\psi\rangle \in \mathscr{H}$

$$B^2|\psi\rangle = B|\psi\rangle. \tag{22}$$

Using the identity (17) and the property (1) one can notice that for the unitary quantum evolution operator $U(t, t_0)$ the strong limit $s-\lim_{t\to\infty} U(t, t_0)$ does not exist, except for the trivial case when $\mathscr{N}(H) \equiv \mathscr{H}$. Indeed, let $U_\infty \overset{\text{df}}{=} s-\lim_{t\to\infty} U(t, t_0)$ exist. This is possible if and only if $U_\infty^+ U_\infty = \mathbf{1}$ and contradicts formulae (17) and (1), because $B^+B \neq \mathbf{1}$ if $\mathscr{N}(H) \neq \mathscr{H}$, i.e. if $H \neq 0$. From this and (2.7) it follows that if $s-\lim_{t\to\infty} PU(t, t_0) \equiv s-\lim_{t\to\infty} T(t, t_0)$ exists then the $s-\lim_{t\to\infty} QU(t, t_0)$ cannot exist and so on.

So, there do not exist the $s-\lim_{t\to\infty} U(t, t_0)$ in $\mathscr{H}$ (though for each $|\psi\rangle \in \mathscr{N}(H)$, $\lim_{t\to\infty} \|U(t, t_0)\,|\psi\rangle - B|\psi\rangle\| = 0$ and $B^+B|\psi\rangle = |\psi\rangle$), nevertheless the weak limit $w-\lim_{t\to\infty} U(t, t_0)$ and for $P < \mathbf{1}$ the limit $s-\lim_{t\to\infty} T_{II}(t, t_0)$ may exist in $\mathscr{H}$.

Using the ergodic theorem on the time average (von Neuman's) [14, 15]

$$s-\lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} U(\sigma, t_0) d\sigma = B \tag{23}$$

and the formulae (1), (17) it is easy to deduce that if the weak limit $w-\lim_{t\to\infty} U(t, t_0)$ exists then it must be equal to the weak limit (when $t \to \infty$) of the time average of the operator $U(t, t_0)$, i.e. if $w-\lim_{t\to\infty} U(t, t_0)$ exists, then

$$w-\lim_{t\to\infty} U(t, t_0) \equiv w-\lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} U(\sigma, t_0) d\sigma = B \tag{24}$$

or otherwise if $w-\lim_{t\to\infty} |\psi; t\rangle$ exists, then

$$w-\lim_{t\to\infty} |\psi; t\rangle \equiv w-\lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} |\psi; \sigma\rangle d\sigma = B|\psi\rangle. \tag{24'}$$

Let us remark that relation (23) follows from the property (1) also — introducing an auxiliary quantity $h(t, t_0) \overset{\text{df}}{=} \int_0^{t-t_0} U(\sigma, t_0) d\sigma$, calculating the Laplace transform $\mathscr{L}[(t-t_0)^{-1} h(t, t_0)]\,(z, t_0)$ and taking the $s-\lim_{z\to 0} z\mathscr{L}[(t-t_0)^{-1} h(t, t_0)]\,(z, t_0)$ one obtains the same result as in the formula (17).

Formulae (1), (18) imply for $P < \mathbf{1}$ that if the strong (or weak) limit $\lim_{t\to\infty} T_{II}(t, t_0)$ exists then it is equal to the strong (or weak) limit (when $t \to \infty$) of the time average of the operator $T_{\Pi}(t, t_0) \equiv PU(t, t_0)\Pi$ (2.23), (2.48), i.e. if (in the strong or weak sense) $\lim_{t\to\infty} T_{\Pi}(t, t_0) \equiv \lim_{t\to\infty} PU(t, t_0)\Pi$ exists, then

$$\lim_{t\to\infty} T_{\Pi}(t, t_0) = \lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} T_{\Pi}(\sigma, t_0) d\sigma = PB\Pi \tag{25}$$

(in the strong or weak topology in $\mathscr{H}$ respectively), and analogously to (24′) for the vector $P|\psi; t\rangle$ (see (2.4), (2.5), (2.24)).

Conditions of the existence of these limits, some consequences and applications of the relationship found above will be discussed in future.

REFERENCES

[1] A. Messiah, *Quantum mechanics*, Amsterdam 1963.
[2] J. D. Bjorken, S. D. Drell, *Relativistic quantum fields*, McGraw-Hill 1965.
[3] A. I. Akhiezer, S. V. Peletminskii, *Metody statisticheskoi fiziki*, Moskva 1977.
[4] E. B. Davies, *Quantum theory of open systems*, London, New York, San Francisco 1976.
[5] W. Królikowski, J. Rzewuski, *Bull. Acad. Pol. Sci.* **4**, 19 (1956); *Nuovo Cimento* **25B**, 739 (1975) and references one can find therein.
[6] L. P. Horwitz, J. P. Marchand, *Rocky Mount. J. Math.* **1**, 225 (1974).
[7] R. Zwanzig, *Physica* **30**, 1109 (1964).
[8] G. V. Chester, *Rep. Progr. Phys.* **26**, 411 (1963).
[9] W. Garczyński, K. Urbanowski, *Bull. Acad. Pol. Sci., Ser. Sci. Math. Astr. Phys.* **25**, 427 (1977).
[10] W. Garczyński, K. Urbanowski, *Acta Phys. Pol.* **B8**, 199 (1977).
[11] K. Urbanowski, *Bull. Acad. Pol. Sci., Ser. Sci. Phys. Astr.* **27**, 155 (1979).
[12] W. Mlak, *An introduction to the theory of Hilbert spaces*, Warszawa 1970 (in Polish).
[13] e.g. M. A. Lavrent'ev, B. V. Schabat, *Metody teorii funktsii kompleksnogo peremennogo*, Moskva 1973.
[14] T. Kato, *Perturbation theory for linear operators*, Berlin, Heidelberg, New York 1966.
[15] N. Dunford, J. T. Schwartz, *Linear operators*, Vol. 1, New York 1958.
[16] E. Hille, R. S. Phillips, *Functional analysis and semi-groups*, Providence 1957.
[17] e.g. L. Schwartz, *Analyse mathematique*, Paris 1967.