Vol. B14 (1983) *ACTA PHYSICA POLONICA* No 7

LINEAR POTENTIAL AND SPONTANEOUS BREAKDOWN OF CHIRAL SYMMETRY*

BY O. PÈNE

Laboratoire de Physique et Hautes Energies, Orsay**

(Received September 14, 1982)

After a brief pedagogical reminder of the standard ideas about spontaneous chiral symmetry breaking, we show, after Nambu, how the ideas of B.C.S. theory of superconductivity can be applied to explain dynamically spontaneous chiral symmetry breaking. More specifically, the Bogoliubov-Valatin variational method is applied to a model of massless quarks interacting via a chiral invariant attractive color linear potential. It is shown analytically that due to the quark's negative self energy, chiral symmetry is spontaneously broken in this model for any value of the parameters.

PACS numbers: 12.40.-y

This paper describes a work which has been done in collaboration with A. Amer, A. Le Yaouanc, L. Oliver and J.-C. Raynal [1]. The pion's mass is very low compared to standard hadronic masses. The pion field has the same quantum numbers as the divergence of the isovector axial current. These two facts led physicists to elaborate more than twenty years ago the ideas of PCAC (Partial Conservation of Axial Current) and of SCSB (Spontaneous Chiral Symmetry Breaking). Since a good part of the audience was apparently extremely young at those times, it might not be completely useless to give a crude reminder of these ideas.

One assumes that the following description is a good approximation of the physical world: The Lagrangian of the strong interactions, whatever it may be, is invariant for the group $SU(2) \times SU(2)$ of chiral transformations, and the pion is the massless "Goldstone boson" for this broken dynamical symmetry.

The notion of "Goldstone boson" comes from Goldstone's theorem. The latter theorem states that if the Hamiltonian of a physical system is invariant for some continuous symmetry, then, either the ground state is invariant for that symmetry implying that all the

* Presented at the XXII Cracow School of Theoretical Physics, Zakopane, May 30 — June 12, 1982.

** Address: Laboratoire associé au centre National de la Recherche Scientifique, Bâtiment 211, Université Paris XI, 91405 Orsay, France.

transforms of a given physical state by that symmetry group are degenerate in mass, or the ground state is not invariant, implying the existence of a given number of massless bosons. The latter case is known as "spontaneous breaking of the symmetry" because the spectrum is not symmetrical although the Lagrangian remains totally symmetrical. Specifically, if the symmetry group is the SU(2) × SU(2) — chiral group, generated by the three isovector charges Q_0, $Q_{\mp}$ and the three isovector axial charges Q_0^5, $Q_{\mp}^5$, and if the vacuum is invariant only for the three isovector charges, then Goldstone's theorem implies the existence of an isotriplet of massless pseudoscalar bosons, or so to say three "massless pions". Exactly what we need. On the contrary, if the above mentioned chiral symmetry was not spontaneously broken, it would imply the existence of $J^P = 1/2^-$ baryons degenerate in mass with the nucleons. This is far from reality.

This idea has been more precisely stated by the PCAC hypothesis which includes some account of explicit non spontaneous breaking of chiral symmetry since it considers a small non-zero mass of the pion. From PCAC many "soft pions" theorems were deduced, all compatible with experiment but a small discrepancy (less than 15% of error). These successes are so striking that the validity of this scheme is now well established.

Massless quarks and chiral symmetry

In science, any answer raises new questions, namely: why is the Lagrangian of strong interactions chiral invariant? And why is that symmetry spontaneously broken? The answer to the first question is rather simple because chiral symmetry is not a far-fetched property once we admit that fermions are basic constituents of matter. Indeed, a sufficient condition to have chiral invariance is the following. A Lagrangian of, say, n_f massless fermions which are coupled to other fields only via SU(n_f)-singlet vector or axial currents, is invariant for the U(n_f) × U(n_f) chiral group defined by the following transformations:

$$\psi \rightarrow e^{i\lambda_k \alpha_k}\psi, \qquad \psi \rightarrow e^{i\lambda_k \beta_k \gamma_5}\psi,$$

$$\psi \rightarrow e^{i\alpha_0}\psi, \qquad \psi \rightarrow e^{i\beta_0 \gamma_5}\psi,$$

where λ_k are the $(n_f^2 - 1)$ Gell-Mann matrices of SU(n_f) and α_i, β_i are real parameters. ψ are quark fields and γ_5 is the Dirac matrix.

As a most important example, Quantum Chromodynamics (QCD) with n_f massless quarks is U(n_f) × U(n_f) chiral invariant. In particular, $n_f = 2$ (u and d massless quarks) implies the existence of a massless pion isovector. As a matter of fact, simple-minded application of Goldstone's theorem predicts an additional massless boson, an isosinglet one. This is due to the fact that the group U(n_f) × U(n_f) is equal to U(1) × U(1) × SU(n_f) × SU(n_f). The breaking of the additional U(1) axial symmetry induces the isosinglet Goldstone boson. However, a very light isoscalar pseudoscalar meson does not exist in nature! This paradox is known as the "U(1) problem". It has been at least partly solved in relation with what is known as the "triangle anomaly". We shall here completely avoid this delicate topic and treat, in fact, U(1) axial symmetry on the same footing as the SU(n_f) one.

A small explicit breaking of chiral symmetry, and hence a small pion mass is easily introduced by small Lagrangian quark masses m_u and m_d. But since we are interested in the mechanism of spontaneous symmetry breaking we shall totally neglect explicit breaking and assume exactly massless quarks in the Hamiltonian.

Nambu's model of dynamical symmetry breaking

Now we come to the main problem: given a chiral invariant Lagrangian, what causes spontaneous breaking of the symmetry, and under what conditions does it happen? We shall derive a formalism inspired by BCS theory of superconductivity and then solve the problem in the case of linear potential. But before that, let us sketch what is usually understood as "Dynamical Spontaneous Breaking of Chiral Symmetry". These ideas are mainly taken from the pioneering work of Nambu and Jona-Lasinio [2]. Let us consider a Lagrangian of free massless fermions. The Lagrangian is obviously chiral symmetric. Furthermore the ground state (the vacuum) is clearly chiral symmetric. It is simply defined as the absence of any massless fermion or anti-fermion (we shall henceforth call these fermions "quarks" although the scheme is more general):

$$b_s^{(0)}(p)\,|\Omega_0\rangle = d_s^{(0)}(p)\,|\Omega_0\rangle = 0,$$

where $b_s^{(0)}$ ($d_s^{(0)}$) are annihilators of quarks (antiquarks) and $|\Omega_0\rangle$ is the chiral invariant vacuum.

Let us now switch on a weak attractive interaction between quarks and antiquarks. The vacuum remains the chiral invariant one defined by the absence of massless quarks and antiquarks. Let us then increase slowly the attraction. At some point the attraction becomes strong enough to generate a bound state. But a bound state of massless particles has a "negative energy". In plain words this means that the former vacuum is no more the lowest energy state. The new vacuum is now, grossly speaking, filled up with quark--antiquark bound pairs. This is more precisely expressed by the fact that $\overline{\psi(0)}\psi(0)$ has a non zero vacuum expectation value:

$$\langle\bar{\psi}(0)\psi(0)\rangle_0.$$

These bound pairs have the total spin zero. This means that the spins of the constituents, projected along the direction of the momenta, are opposite, and the momenta being opposite, the helicities are equal adding up to a total helicity ± 1. For massless particles helicity and chirality are equal. As a consequence *the vacuum*, filled with chirally charged states, *is not chiral invariant*. This means spontaneous breaking of chiral symmetry. Let us now look for a more rigorous and quantitative description of dynamical symmetry breaking in a model where the interaction between quarks is given by an instantaneous potential.

The Bogolioubov–Valatin transformation and the gap equation

Let us assume a chiral invariant Hamiltonian

$$\mathscr{H}_1 = \sum_{\vec{x}} \psi^+(\vec{x})\,(-i\vec{\alpha}\cdot\vec{\nabla})\psi(\vec{x})$$

$$+\tfrac{1}{2}\sum_{\vec{x},\vec{y},\alpha} V(\vec{x}-\vec{y})\left(\psi^+(\vec{x})\frac{\lambda^\alpha}{2}\psi(\vec{x})\right)\left(\psi^+(\vec{y})\frac{\lambda^\alpha}{2}\psi(\vec{y})\right). \tag{1}$$

ψ are quark fields (a sum over n_f quark fields may be understood), λ^α are color Gell-Mann matrices and $V(\vec{x})$ is a potential which we assume for simplicity to be spherically symmetrical[1]. Chiral invariance is ensured by the fact that $\psi^+\psi$ is a chiral conserving current. One might as well add interactions via $\psi^+\vec{\alpha}\psi$ ($\vec{\alpha} = \gamma_0\vec{\gamma}$) currents but currents with an odd number of γ-matrices between ψ^+ and ψ would not be chiral invariant. The goal is to find the vacuum, that is to say the state in Hilbert space which minimizes the energy density. But Hilbert space is a very complicated space in a field theory, even in a simplified field theory like this one. The way out is to restrict ourselves to a manageable subspace of Hilbert space large enough to give a decent approximation of reality. The most common technique is of course perturbation, starting from the free quark Hamiltonian. But this is totally irrelevant in our case for the following reason: starting from a chiral invariant unperturbed vacuum, no perturbation is able to generate a non chiral invariant one. We need then to find some orbit in Hilbert space which goes, in some sense, beyond perturbation. This is given by Bogoliubov–Valatin transformations.

Let us write the quark field at $t = 0$:

$$\psi(\vec{x}) = \frac{1}{n^{3/2}}\sum_{\vec{k},s}[u_s^{(0)}(\vec{k})b_s^{(0)}(\vec{k})+v_s^{(0)}(\vec{k})d_s^{(0)+}(-\vec{k})]e^{i\vec{k}\cdot\vec{x}}, \tag{2}$$

where $u_s^{(0)}$, $v_s^{(0)}$ are free massless Dirac spinors. We may as well write the same field as

$$\psi(\vec{x}) = \frac{1}{n^{3/2}}\sum_{\vec{k},s}[u_s(\vec{k})b_s(\vec{k})+v_s(\vec{k})d_s^+(-\vec{k})]e^{i\vec{k}\cdot\vec{x}}, \tag{3}$$

where u_s, v_s are now *any* Dirac spinors verifying the normalization conditions

$$u_{s'}^+(\vec{k})u_s(\vec{k}) = v_{s'}^+(\vec{k})v_s(\vec{k}) = \delta_{ss'},$$
$$u_{s'}^+(\vec{k})v_s(\vec{k}) = v_{s'}^+(\vec{k})u_s(\vec{k}) = 0. \tag{4}$$

u_s and v_s might be for instance free massive Dirac spinors, but we do not need to restrict ourselves to solutions of Dirac equations. Once given $u_s(\vec{k})$ and $v_s(\vec{k})$, equality of (2) and

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[1] We use a discrete space for simplicity and the continuum limit is quite trivial. a is the lattice spacing and n^3 the number of lattice points.

(3) define b_s, d_s and hermitian conjugates as linear combinations of $b_s^{(0)}$, $d_s^{(0)}$ and h.c. These are the Bogoliubov–Valatin transformations. Let us consider the class of states defined by

$$b_s(\vec{k})\,|\Omega\rangle = d_s(\vec{k})\,|\Omega\rangle = 0. \tag{5}$$

This is only an implicit definition of $|\Omega\rangle$. One can happily get sure that $|\Omega\rangle$ exists by building it up explicitly via a unitary transformation $|\Omega\rangle = U|\Omega_0\rangle$. Please, take this result for granted. Our task is now to minimize the energy density inside that class of states. Before going further we must clarify how we understand the Hamiltonian $\mathscr{H}_1$. We can take it as a simple product of fields, or as

$$\mathscr{H}_2 = N^0(\mathscr{H}_1), \tag{6}$$

where N^0 means normal order of the fields expressed in terms of the $b_s^{(0)}$, $d_s^{(0)}$ and h.c. In literature one usually uses $\mathscr{H}_2$ [3, 4]. We postpone the discussion of $\mathscr{H}_1$ and $\mathscr{H}_2$ to the end, after gathering more information. We will now follow the calculation for the two cases.

To calculate the energy density $\mathscr{E}_1$ of $\mathscr{H}_1$ in the state $|\Omega\rangle$ (5) we simply apply Wick's theorem to the fields b_s, d_s and h.c. The result is

$$v\mathscr{E}_1 = 3n_f \sum_{\vec{k}} \mathrm{Tr}\,[\vec{\alpha}\cdot\vec{k}\Lambda_-(\vec{k})] + \frac{4}{v}\,\frac{n_f}{2} \sum_{\vec{k},\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')\,\mathrm{Tr}\,[\Lambda_+(\vec{k})\Lambda_-(\vec{k}')], \tag{7}$$

where $v = (an)^3$ is the volume of space, $\tilde{V}(\vec{k}-\vec{k}')$ the Fourier transform of $V(\vec{x})$ and

$$\Lambda_+(\vec{k}) = \sum_s u_s(\vec{k})u_s^+(\vec{k}); \qquad \Lambda_-(\vec{k}) = \sum_s v_s(\vec{k})v_s^+(\vec{k}) = 1-\Lambda_+(\vec{k}). \tag{8}$$

Note that the energy density only depends on the projectors $\Lambda_-(\vec{k})$ (or $\Lambda_+(\vec{k})$), independently of the choice of a spin basis in the subspaces defined by Λ_+ and Λ_-. We will therefore simply parametrize the class of states (5) by the $\Lambda_-(\vec{k})$ with the condition that it remains a projector (of rank 2): $\Lambda_-^2 = \Lambda_-$.

Starting form $\mathscr{H}_2$ (6) gives the same equation for $v\mathscr{E}_2$ as (7) with $\Lambda_\pm(\vec{k})$ substituted by

$$\Lambda_\pm^{(d)}(\vec{k}) = \Lambda_\pm(\vec{k}) - \Lambda_\pm^{(0)}(\vec{k}), \tag{9a}$$

where $\Lambda_\pm^{(0)}(\vec{k})$ are the free massless projectors

$$\Lambda_\pm^{(0)}(\vec{k}) = \tfrac{1}{2}\,(1 \pm \vec{\alpha}\cdot\vec{k}). \tag{9b}$$

$\mathscr{E}_1$ and $\mathscr{E}_2$ are functionals of the real matrix functions $\Lambda_-(\vec{k})$ and our aim is to find the function $\Lambda_-(\vec{k})$ which minimizes $\mathscr{E}_1(\mathscr{E}_2)$. To simplify the problem we take for granted that the rotational invariance as well as P, C, T are not spontaneously broken. We can then parametrize $\Lambda_-(\vec{k})$ as

$$\Lambda_-(k) = \tfrac{1}{2}\left[1 - \frac{A(\vec{k})}{E(\vec{k})}\beta - \frac{B(\vec{k})}{E(\vec{k})}\vec{\alpha}\cdot\hat{k}\right] \tag{10}$$

with $E(\vec{k}) = [(A(\vec{k}))^2 + (B(\vec{k}))^2]^{1/2}$.

Comparing with the free massive projector

$$\Lambda_-^{(m)}(\vec{k}) = \tfrac{1}{2}\left(1 - \frac{m}{E}\beta - \frac{\vec{\alpha}\cdot\vec{k}}{E}\right)$$

the physical meaning of the real functions $A(\vec{k})$ and $B(\vec{k})$ appears clearly. $A(\vec{k})$ is an effective $\vec{k}$-dependent "mass" and $B(\vec{k})$ an effective renormalisation of the kinetic energy. Note that A and B are defined by (10) only up to a multiplicative constant.

A necessary, but not sufficient condition for $\Lambda_-(\vec{k})$ to minimize $\mathscr{E}_1$ is that $\mathscr{E}_1$ is an extremum:

$$\begin{aligned}\delta\mathscr{E}_1 &= 3n_f \sum_{\vec{k}} \left\{\delta\Lambda_-(\vec{k})\left[\vec{\alpha}\cdot\vec{k} + \frac{4}{3v}\tfrac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')(1-2\Lambda_-(\vec{k}'))\right]\right\} \\ &= 3n_f\,\mathrm{Tr}\,\{\delta\Lambda^-(\vec{k})H(\vec{k})\} = 0\end{aligned} \tag{11}$$

with

$$H(\vec{k}) = \vec{\alpha}\cdot\vec{k} + \frac{4}{3v}\tfrac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')(1-2\Lambda_-(\vec{k}')) \tag{12}$$

for any $\delta\Lambda_-$ such that $(\Lambda_- + \delta\Lambda_-)^2 = (\Lambda_- + \delta\Lambda_-)$ i.e. to first order in $\delta\Lambda_-$

$$\Lambda_-\delta\Lambda_- + \delta\Lambda_-\Lambda_- = \delta\Lambda_-. \tag{13}$$

Take a base where Λ_- is diagonal by blocks. Then you see that (13) is equivalent to $\delta\Lambda_-$ being antidiagonal. Equation (11), that is $\delta\mathscr{E}_1 = 0$ for any antidiagonal $\delta\Lambda_-$ means that $H(\vec{k})$ must be diagonal. In other terms, independently of any basis, equations (11) and (13) are equivalent to (12) and (14):

$$[\Lambda_-(\vec{k}), H(\vec{k})] = 0. \tag{14}$$

Using (10) and (14) we get

$$H(\vec{k}) = A(\vec{k})\beta + B(\vec{k})\vec{\alpha}\cdot\hat{k}, \tag{15}$$

where the arbitrary constant in A and B has now been fixed. Equation (12) now reads

$$A(\vec{k}) = \frac{4}{3v}\cdot\tfrac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\frac{A(\vec{k}')}{\sqrt{A^2(\vec{k}')+B^2(\vec{k}')}}, \tag{16a}$$

$$B(\vec{k}) = |\vec{k}| + \frac{4}{3v}\cdot\tfrac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\frac{B(\vec{k}')}{\sqrt{A^2(\vec{k}')+B^2(\vec{k}')}}(\hat{k}\cdot\hat{k}'). \tag{16b}$$

This is known as the "gap equation" in superconductivity. Had we started from $\mathscr{H}_2$, a derivation along similar lines leads to the same equation (16a) but (16b) is now substituted by

$$B(\vec{k}) = |\vec{k}| - \frac{4}{3v} \cdot \tfrac{1}{2} \sum_{\vec{k}'} \tilde{V}(\vec{k}-\vec{k}') \left\{ 1 - \frac{B(\vec{k}')}{\sqrt{A^2(\vec{k}')+B^2(\vec{k}')}} \right\} (\hat{k} \cdot \hat{k}'). \tag{17b}$$

What have we gained? We have expressed the extremum condition as a pair of coupled non linear integral equations. Such equations are rather tough to solve, and after solving them we must still get sure that we got a local minimum, and then wonder if it is the absolute minimum. We better leave aside this ambitious program which is beyond our present reach, and turn toward a simpler problem: gather some information on what happens in the vicinity of the chiral invariant vacuum.

Instability equation

A solution of the gap equation breaks spontaneously chiral symmetry if $A(\vec{k}) \neq 0$. In particular

$$\langle \Omega | \bar{\psi}(0)\psi(0) | \Omega \rangle = \frac{3n_f}{n} \sum_{\vec{k}} \mathrm{Tr}\,(\Lambda_-(\vec{k})\beta_0) = \frac{6n_f}{n} \sum_{\vec{k}} \frac{-A(\vec{k})}{E(\vec{k})}. \tag{18}$$

We now look for chiral invariant solutions of gap equation. From (16) we find trivially (for $\mathscr{H}_1$)

$$A^{(0)}(\vec{k}) = 0,$$

$$B^{(0)}(\vec{k}) = |\vec{k}| + \frac{4}{3v} \cdot \tfrac{1}{2} \sum_{\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')(\hat{k} \cdot \hat{k}') \tag{19}$$

and from (17) (for $\mathscr{H}_2$)

$$A^{(0)}(\vec{k}) = 0, \quad B^{(0)}(\vec{k}) = |\vec{k}|. \tag{20}$$

Solution (20) corresponds to standard massless quarks. Solution (19) is more surprising. What is the meaning of the second term in $B^{(0)}(\vec{k})$? Inspection shows that it corresponds to a self-energy. One could also derive this term via first order perturbation. The normal ordering of $\mathscr{H}_2$ eliminates this self-energy term in (20) because it corresponds to a closed loop with only one $V(\vec{x})$ insertion. But if we start from $\mathscr{H}_1$ this term cannot be avoided. Indeed (20) is not even a solution of the gap equation if we start from $\mathscr{H}_1$.

Now the fact that (19) and (20) are solutions of gap equations does not prove that these states are stable, they may be "metastable". We indeed expect, for symmetry reasons, that when the symmetry is spontaneously broken, the chiral invariant solution is a maximum in some direction of Hilbert space. Hence we must now solve the problem of whether solutions (19) and (20) are local minima or not. The question whether a local minimum is also an absolute one is difficult and out of our scope here. Let us give go $\Lambda_-(\vec{k})$ infinitesimal variations $\delta\Lambda_-(\vec{k})$ around the solution $\Lambda_-^{(0)}(\vec{k})$ of (19) and (20). If, in some direction of Hilbert space, the corresponding $\delta^2\mathscr{E}$ is negative, then $\Lambda_-^{(0)}(\vec{k})$ is not a local minimum. $\Lambda_-^{(0)}(\vec{k})$ is given by (9b) for solution (19) as well as (20). Let us write $\Lambda_-(\vec{k})$ as (10) with

$$-\frac{A(\vec{k})}{E(\vec{k})} = 2\varphi(\vec{k}) \quad \text{infinitesimal,} \tag{21}$$

then

$$\delta\Lambda_-(\vec{k}) \simeq \varphi(\vec{k})\beta + \varphi^2(\vec{k})\vec{\alpha}\cdot\hat{k}; \tag{22}$$

from (7) we get

$$\delta^2\mathscr{E}_1 = \frac{3n_f}{v}\sum_{\vec{k}} \mathrm{Tr}\,[\vec{\alpha}\cdot\vec{k}(\Lambda_-(\vec{k})-\Lambda_-^{(0)}(\vec{k}))]$$

$$+\frac{4n_f}{v^2}\,\tfrac{1}{2}\sum_{\vec{k},\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')\,\mathrm{Tr}\,[\Lambda_-(\vec{k})\Lambda_+(\vec{k}')-\Lambda_-^{(0)}(\vec{k})\Lambda_+^{(0)}(\vec{k}')]. \tag{23}$$

Calling $H^{(0)}(\vec{k})$ the value of $H(\vec{k})$ for solution (19), we have from (9a)

$$H^{(0)}(\vec{k}) = \vec{\alpha}\cdot\vec{k} + \frac{4}{3v}\cdot\tfrac{1}{2}\sum_{\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')\,(1-2\Lambda_-^{(0)}(\vec{k}')) = B^{(0)}(\vec{k})\vec{\alpha}\cdot\hat{k}, \tag{24}$$

with $B^{(0)}$ given by (19).

$$\delta^2\mathscr{E}_1 = \frac{3n_f}{v}\sum_{\vec{k}} \mathrm{Tr}\,[H^{(0)}(\vec{k})\Lambda_-^{(d)}(\vec{k})]$$

$$+\frac{4n_f}{v^2}\cdot\tfrac{1}{2}\sum_{\vec{k},\vec{k}'} V(\vec{k}-\vec{k}')\,\mathrm{Tr}\,[\Lambda_-^{(d)}(\vec{k})\Lambda_+^{(d)}(\vec{k}')] \tag{25}$$

and from (22) we get

$$\delta^2\mathscr{E}_1 = \frac{6n_f}{v}\left\{\sum_{\vec{k}} 2B^{(0)}(\vec{k})\,(\varphi(\vec{k}))^2 - \frac{4}{3v}\sum_{\vec{k},\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')\varphi(\vec{k})\varphi(\vec{k}')\right\}. \tag{26}$$

Starting from $\mathscr{H}_2$, the energy density is zero for solution (20). Then $\delta^2\mathscr{E}_2$ is simply given by (7) with $\Lambda_\pm$ subsituted by $\delta\Lambda_\pm$ (which is the infinitesimal form of $\Lambda_\pm^{(d)}$). This leads to

(26) with $B^{(0)}(\vec{k})$ given by (20):

$$\delta^2\mathscr{E}_2 = \frac{6n_{\mathrm{f}}}{v}\left\{\sum_{\vec{k}} 2|\vec{k}|\,(\varphi(\vec{k}))^2 - \frac{4}{3v}\sum_{\vec{k},\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')\varphi(\vec{k})\varphi(\vec{k}')\right\}. \tag{27}$$

The question is now to find some real function $\varphi(\vec{k})$ for which $\delta^2\mathscr{E}_1$ ($\delta^2\mathscr{E}_2$) is negative. Since (26) and (27) are homogeneous in φ we can multiply φ by a real number without changing the sign of the expression, and we may thus restrict ourselves to normalized functions:

$$\frac{1}{v}\sum_{\vec{k}} (\varphi(\vec{k}))^2 = 1. \tag{28}$$

This does not contradict the infinitesimal character of φ since we may multiply a solution φ by any infinitesimal real constant.

Looking for the minimum of the right hand side of (26) and (27) is nothing else than applying the variational method to the search of the lowest eigenvalue of the hermitian operators:

$$\mathbf{H}(\vec{k}, \vec{k}') = 2B^{(0)}(\vec{k})\delta_{\vec{k},\vec{k}'} - \frac{4}{3v}\tilde{V}(\vec{k}-\vec{k}') \tag{29}$$

or, after Fourier transforming $\tilde{V}$,

$$\mathbf{H} = 2B^{(0)}(\vec{k}) - \tfrac{4}{3}V(\vec{x}), \tag{30}$$

$B^{(0)}$ being given by (19) or (20) according to the case. $-4/3$ is simply due to color. The equation

$$\lambda\varphi = \mathbf{H}\varphi \tag{31}$$

is called "instability equation". If it has no negative λ solution, the chiral invariant state is a local minimum (at least for the class of states (5)). If (31) has a negative eigenvalue, there is spontaneous breaking of chiral symmetry. With $B^{(0)}(\vec{k}) = |\vec{k}|$ equation (31) has already been derived via another method by Casher [3]. This equation is a Schrödinger equation with the non relativistic kinetic energy replaced by an ultra-relativistic one. Somehow it means that pairs may not get negative energy for the vacuum to remain stable. But, be careful! In principle the spectrum of relativistic pairs is given by the Bethe–Salpeter equation. By an happy surprise it simplifies here to a Schrödinger equation. When we start from $\mathscr{H}_1$, $B^{(0)}$ is given by (19). Again we have a Schrödinger equation, but this time we must add the self-energy term to the kinetic energy. Let us now discuss the specific example of linear potential.

Linear potential

We start from equations (1) and (6) with $V(\vec{r}) = -\dfrac{g^2}{8\pi}r$. We need a minus sign to compensate the minus coming from color contraction and get an attractive potential.

Now the Fourier transform of r is not well defined for $\vec{k} \to 0$ due to the infra-red singular behaviour of r. We therefore regularise it:

$$r = \lim_{m\to 0} \frac{2}{m^2} \frac{e^{-mr} - 1 + mr}{r}, \tag{32}$$

$$\tilde{V}(\vec{k}) = -\frac{g^2}{8\pi} \int d\vec{r} |\vec{r}| e^{i\vec{k}\cdot\vec{r}} = \lim_{m\to 0} g^2 \left\{ \frac{1}{k^2(k^2+m^2)} - \frac{2\pi^2}{m} \delta(\vec{k}) \right\}. \tag{33}$$

We have taken the continuous limit $a \to 0$ and $n \to \infty$. To get continuous integrals we simply have to replace $a^3 \sum_{\vec{x}}$ by $\int d\vec{x}$ and $(1/v) \sum_{\vec{k}}$ by $\int d\vec{k}/(2\pi)^3$. One can see from (33) that the singularity at $k^2 \to 0$ is "compensated" by a $\delta(\vec{k})$ singularity which multiplies an infinite coefficient. We shall now apply test functions to (26) and (27). It is enough to find one test function which gives a negative value to be sure that the chiral invariant vacuum is not stable. Inspecting (27) it is clear that the operator $\mathbf{H} = 2k + \frac{g^2}{6\pi} r$ is positive definite and will never give negative $\delta^2 \mathscr{E}_2$. This result is of course more general: *It is impossible to induce spontaneous breaking with a positive definite potential and normal ordered Hamiltonian* $\mathscr{H}_2$. One may of course substract from $V(\vec{x})$ a big enough positive constant. But this seems quite unelegant and arbitrary. The model looks rather unphysical if different potentials giving to same force between quarks should give completely different results concerning the vacuum!

Happily enough $\mathscr{H}_1$ does not suffer from such a disease: any arbitrary constant in $V(\vec{r})$ is cancelled by the opposite constant appearing in the self-energy. This is encouraging and we now concentrate on equation (26). We first try the simplest test function:

$$\varphi(\vec{k}) = (4\pi\alpha)^{3/4} \exp(-\alpha k^2/2). \tag{34}$$

Using (33) we have

$$-\tfrac{4}{3} \frac{1}{(2\pi)^3} \int d\vec{k} d\vec{k}' \varphi(\vec{k}) \varphi(\vec{k}') \tilde{V}(\vec{k}-\vec{k}') = \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{(2\pi)^3}{3\alpha} \tag{35}$$

and

$$\int d\vec{k} 2k \varphi^2(k) = \left(\frac{\alpha}{\pi}\right)^{3/2} (2\pi)^3 \frac{4\pi}{\alpha^2}. \tag{36}$$

It is rather lengthy but not too difficult to derive

$$\lim_{m\to 0} \int d\vec{k}' \left\{ \frac{1}{(\vec{k}-\vec{k}')^2 [(\vec{k}-\vec{k}')^2 + m^2]} - \frac{2\pi^2}{m} \delta(\vec{k}-\vec{k}') \right\} (\hat{k}\cdot\hat{k}') = -\frac{4\pi}{k}, \tag{37}$$

whence

$$\frac{4}{3(2\pi)^3} \int d\vec{k} d\vec{k}' \tilde{V}(\vec{k}-\vec{k}') (\hat{k}\cdot\hat{k}') \varphi^2(\vec{k}) = -\left(\frac{\alpha}{\pi}\right)^{3/2} (2\pi)^3 \frac{4}{3\pi\alpha} \tag{38}$$

and finally from (26) and (35)–(38)

$$\delta\mathscr{E}_1 = \frac{6n_f}{v}\left(\frac{\alpha}{\pi}\right)^{3/2}(2\pi)^3\left\{\frac{4\pi}{\alpha^2}+\frac{g^2}{3\alpha}\left(1-\frac{4}{\pi}\right)\right\}. \tag{39}$$

Since $1-4/\pi$ is negative $\delta^2\mathscr{E}_1$ becomes negative for

$$12\pi/\alpha < g^2(4/\pi-1). \tag{40}$$

As a conclusion, for any value of g^2 the smallest eigenvalue of **H** *is negative and consequently the chiral invariant vacuum is unstable.*

Conclusion

What is the physical significance of this result? It is clear that a potential model is but a crude approximation of QCD. Furthermore we neglected the $1/r \cdot \log r$ short distance potential between quarks. Our aim was indeed not to take into account all the complexity of QCD, but only to understand something about the relation between confinement and SCSB. Therefore we simply used the confining part of the potential. And if we understood something, this "something" is undoubtedly the *crucial role of self-energy in SCSB.*

Thus, a posteriori we prefer the choice of a non-normal ordered Hamiltonian $\mathscr{H}_1$ to the normal ordered one $\mathscr{H}_2$. Is this choice theoretically grounded? If we knew how to derive some approximate potential from QCD, there would be no ambiguity. But in 4 dimensions of space and time we do not even know if a "potential" has any theoretical meaning. We shall then rely on what is known: in 2 dimensions [5] QCD does lead to a Hamiltonian closely similar to $\mathscr{H}_1$. Once more the non-normal ordered Hamiltonian seems to be favored!

It is worth noticing that at this stage of research, the results obtained by variational techniques on potential models seem to agree with completely different techniques: Lattice gauge theory. Namely it seems that in lattice gauge theory, whenever the coupling constant is strong enough to generate confinement, it also generates SCSB, in agreement with our present result. But the reciproc il is not true. As shown by Finger et al. [4], a non confining potential with strong enough coupling constant may induce instability of the chiral invariant vacuum. We shall end with the optimistic conclusion that these variational techniques are not pure nonsense and that, due to their relative simplicity and to the wide range of problems they may tackle, they deserve some attention.

The author would like to warmly thank and congratulate our Polish hosts who took up the challenge of organizing this year the Zakopane school in spite of all what is happening, and who incredibly succeeded.

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