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STATIONARY STATES IN PROPER TIME QUANTUM MECHANICS

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The stationary states in proper time quantum mechanics are identified on the basis of physically essential stationarity property. The correspondence with the usual results is established.

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1. *Introduction*

In the present article we shall give a definition of a stationary state in the relativistic quantum mechanics of a particle in an external field, in which the dynamical evolution is controlled by an additional parameter τ and the physical time is an observable. To be more specific we shall discuss the model described in [1] (see also [2]) of an electron in an external electromagnetic field. We shall show that the level structure of the stationary states in the case of nearly sharply defined mass corresponds to the usual one. At the same time, let us recall [1], the proper time quantum mechanics remains relativistically invariant in contradistinction to the usual first quantized formalism.

The dynamical evolution is determined in the proper time quantum mechanics of [1] and [2] by a Schrödinger-type equation

$$-\frac{i\hbar}{c^2}\frac{\partial}{\partial\tau}\Phi(\tau) = \mathscr{M}\Phi(\tau), \tag{1}$$

where the generator of evolution $\mathscr{M}$ is the mass operator. A stationary state cannot, however, be defined in a way naively analogous to the usual one, i.e. as an eigenstate of the generator of evolution $\mathscr{M}$. Even more, the appearance of such eigenstates in the theory should be automatically excluded in general, as in such states all averages of physical quantities would be constant, including physical time. In our case the mass eigenstates are absent in accordance with this requirement. Indeed, $\mathscr{M}$ in this case is the Dirac operator

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(see [1]), so that the eigenvalue equation is the Dirac equation:

$$\gamma^\mu \pi_\mu \Phi = m\Phi \tag{2}$$

from which the continuity equation

$$\partial_0(\Phi^\dagger \Phi) = -\vec{\nabla}\cdot(\Phi^\dagger \gamma^0 \vec{\gamma}\Phi) \tag{3}$$

follows (all denotations as in [1]). The Hilbert space of this model is $H = \boldsymbol{C}^4 \otimes L^2(\boldsymbol{R}^4, d^4x)$, therefore $\Phi(x^0, \vec{x}) \in \boldsymbol{C}^4 \otimes L^2(\boldsymbol{R}^3, d^3x)$ for all x^0. The integration of (3) over the whole $\vec{x}$-space gives

$$\frac{d}{dx^0}\int \Phi^\dagger \Phi d^3x = 0$$

or

$$\int \Phi^\dagger \Phi d^3x = \text{const.} > 0$$

which contradicts $\Phi \in H$, as $\|\Phi\|_H^2 = \int \Phi^\dagger \Phi d^3x dx^0 = \infty$.

The stationary states are not the eigenfunctions of the energy operator $cP^0 = i\hbar c\partial^0$ either (these lie outside the Hilbert space as well).

2. *The definition*

What is physically essential for a state to be stationary is the following property: there exists a frame of reference in which the sole effect of the dynamical evolution of the state is the flow of the physical time. The following two consequences are easily deduced:

1) The external field A has to be time-independent in that frame:

$$[P^0, A^\mu] = 0 \tag{4}$$

as otherwise its average would evolve with τ. Using (4) and the definition of $\mathcal{M}$ we have

$$[P^0, \mathcal{M}] = 0. \tag{5}$$

2) The evolution of the state of electron in that frame is equivalent to some translation along the physical time axis, i.e.

$$e^{-\frac{ic}{\hbar}f(s)P^0}\, e^{\frac{ic^2}{\hbar}s\mathcal{M}}\,\Phi = \Phi \qquad \text{for all } s,$$

where Φ is Schrödinger- or Heisenberg-picture state and $f(s)$ is some real function. In view of (5) the last equality can be rewritten as the following condition

$$e^{\frac{ic}{\hbar}(s\mathcal{M}c - f(s)P^0)}\,\Phi = \Phi \qquad \text{for all } s,$$

which can only be satisfied if $f(s) = \dfrac{s}{\alpha}$ (α is some real constant) and in this case is equivalent to

$$P^0\Phi = \alpha\mathcal{M}c\Phi. \tag{6}$$

On introducing a time-like unit vector n_μ ($n^2 = 1$) we can now state the definition of a stationary state in arbitrary frame:

If Φ is a state of a system for which

$$[n_\mu P^\mu, A^\nu] = 0, \tag{7}$$

then Φ is stationary if

$$n_\mu P^\mu \Phi = \alpha \mathcal{M} c \Phi. \tag{8}$$

Φ is here Schrödinger- or Heisenberg-picture state (the corresponding two versions of the definition are equivalent). From now on we shall employ the Schrödinger picture (in which states satisfy the evolution equation (1)). The equations (7) and (8) are the covariant analogues of (4) and (6) respectively.

In all further considerations we shall use this particular frame in which the external field is time-independent (in this frame $n_\mu = \delta_{\mu 0}$).

Combining the stationarity condition (6) with the evolution equation (1) we obtain the following differential equation:

$$\left(\frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial \tau}\right) \Phi(\tau, x) = 0 \tag{9}$$

which is solved by the substitution

$$\Phi(\tau, x) = \Psi(\tau - \alpha t, \vec{x}). \tag{10}$$

It is now clear that if Φ lies in the Hilbert space H for arbitrarily chosen $\tau = \tau_0$ (hence $\Psi(s, \vec{x}) \in \boldsymbol{C}^4 \otimes L^2(\boldsymbol{R}^4)$), then it remains there for all τ and moreover is a regular distribution on $\boldsymbol{C}^4 \otimes S(\boldsymbol{R}^5)$, so that the whole postulate IV of [1], section 4, is automatically fulfilled[1].

The equation (6) is a generalization of the usual definition of a stationary state, the main novelty being that the energy levels have non-zero width, which is proportional to the spread in mass. The constant α plays the role analogous to that of the energy eigenvalue of the usual theory.

3. The correspondence

To compare the results of the present model with the usual theory let us take the Fourier transformation from $\Phi(\tau, x)$ and $\Psi(s, \vec{x})$ over τ and s respectively:

$$\hat{\Phi}(m, x) = \frac{1}{\sqrt{2\pi \frac{\hbar}{c^2}}} \int e^{-\frac{imc^2}{\hbar}\tau} \Phi(\tau, x) d\tau,$$

$$\hat{\Psi}(m, \vec{x}) = \frac{1}{\sqrt{2\pi \frac{\hbar}{c^2}}} \int e^{-\frac{imc^2}{\hbar}s} \Psi(s, \vec{x}) ds.$$

[1] This means that every solution of the evolution equation which satisfies the stationarity condition is a state as defined by postulates of [1].

Let B denote arbitrary operator which does not depend on time variable t (hence B acts on $C^4 \otimes L^2(R^3, d^3x)$ for every fixed x^0). Then we have the equalities:

$$\int \Phi^\dagger(\tau, x)B\Phi(\tau, x)d^4x = c \int \Psi^\dagger(\tau-\alpha t, \vec{x})B\Psi(\tau-\alpha t, \vec{x})d^3x dt$$

$$= \frac{c}{\alpha}\int \Psi^\dagger(s, \vec{x})B\Psi(s, \vec{x})d^3x ds = \frac{c}{\alpha}\int \hat{\Psi}^\dagger(m, \vec{x})B\hat{\Psi}(m, \vec{x})d^3x dm.$$

Therefore according to the postulate III of [1], Section 4, the physical averages can be computed from

$$\langle B\rangle_\Phi = \frac{\int \hat{\Psi}^\dagger(m, \vec{x})\gamma^0 B\hat{\Psi}(m, \vec{x})d^3x dm}{\int \hat{\Psi}^\dagger(m, \vec{x})\gamma^0\hat{\Psi}(m, \vec{x})d^3x dm} \tag{11}$$

(B is any observable independent of t). The energy average is

$$\langle cP^0\rangle_\Phi = \alpha\langle \mathcal{M}\rangle_\Phi c^2. \tag{12}$$

The solution (10) and the evolution equation (1) become under Fourier transformation

$$\hat{\Phi}(m, x) = e^{-\frac{i}{\hbar}\alpha mc^2 t}\hat{\Psi}(m, \vec{x}) \tag{13}$$

and

$$(\mathcal{M}^{(\alpha mc^2)} - m)\hat{\Psi}(m, \vec{x}) = 0 \tag{14}$$

respectively, where $\mathcal{M}^{(E)}$ is the Dirac operator in which cP^0 is replaced by E. As $\hat{\Psi}(m, \vec{x}) \in L^2$ the state has a non-zero mass spread Δm which satisfies

$$\Delta m \cdot \Delta\tau \geqslant \frac{\hbar}{c^2} \tag{15}$$

where $\Delta\tau$ is the spread of $\Psi(s, \vec{x})$ in s. It is now easily seen that a strictly stationary state can have arbitrarily small mass spread. Indeed, in this case $\Delta\tau$ can take arbitrarily large value, as the system does not undergo any change, which would introduce some characteristic time. For such state the equation (6) tends to the eigenvalue equation for P^0 and the width of energy levels tends to zero. Moreover, the equations (11)–(14) can be approximated by

$$\langle B\rangle_\Phi = \frac{\int \hat{\Psi}^\dagger(\vec{x})\gamma^0 B\hat{\Psi}(\vec{x})d^3x}{\int \hat{\Psi}^\dagger(\vec{x})\gamma^0\hat{\Psi}(\vec{x})d^3x}, \tag{16}$$

$$E = \alpha m_0 c^2, \tag{17}$$

$$\hat{\Phi}(x) = e^{-\frac{i}{\hbar}Et}\hat{\Psi}(\vec{x}), \tag{18}$$

$$(\mathcal{M}^{(E)} - m_0)\hat{\Psi}(\vec{x}) = 0, \tag{19}$$

where $\hat{\Phi}(x) \equiv \hat{\Phi}(m_0, x)$, $\hat{\Psi}(\vec{x}) \equiv \hat{\Psi}(m_0, \vec{x})$ and other values of $\hat{\Phi}(m, x)$ and $\hat{\Psi}(m, \vec{x})$ are of no importance. Clearly the energy levels are identical as in the usual theory and are characterized by the same quantum numbers. The averages of physical quantities which do not take sharp values on $\hat{\Psi}(\vec{x})$ (i.e. $\hat{\Psi}(\vec{x})$ is not an eigenstate of these quantities) are changed according to the insertion of the metric operator in (16).

If some deviation from strict stationarity is allowed and Δt is the characteristic time of transition, then beside (15) we have $\frac{\Delta\tau}{\alpha} \leqslant \Delta t$ $\left(\frac{\Delta\tau}{\alpha}\right.$ may be regarded as the physical time spread as is suggested by (10)$\left.\right)$, hence $\alpha\Delta m \cdot \Delta t \geqslant \frac{\hbar}{c^2}$. Consequently, when a perturbation is present the energy levels are bound to have some non-zero width, which together with the characteristic time satisfies the uncertainty relation.

4. *Discussion*

It may be objected that the conjectured correspondence of the results of the present model with the well corroborated structure of energy levels of the usual theory is only to be achieved for specially chosen states (these with negligible mass spread). The question therefore arises what is the physical interpretation of other states allowed by the model. To this we can answer that this model provides a framework within which physically realizable states can be treated (in relativistically covariant way), but it does not pretend to give a criterion allowing to decide which states are in fact realized in nature. In the usual theory it suffices to insert the observable mass of real particle into the wave equation, here we also have to determine the mass distribution. The problem of determining this distribution, we must however admit, remains open. Sometimes the mass (or mass square) eigenstates are chosen (as in the works by Horwitz, Piron and Reuse cited in [1]). From our point of view this can only be justified as a limit constituted by the approximation (16)–(19). In general, as we have heuristically indicated, the mass spread remains in some connection with the time spread. It may therefore prove necessary, in order to clarify the experimental situation, to reexamine the role of physical time and the way it is measured in microphysics.

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