A CONVENTIONAL INTERPRETATION OF KÄHLER EQUATION

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The Kähler equation for differential forms is interpreted in flat space-time as describing a system of two Dirac particles, one of which being infinitely heavy.

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Recently, some interest was aroused among physicists around the equation introduced in 1960 by the mathematician Erich Kähler [1]. It is an equation for differential forms (or sets of antisymmetric tensor fields), following from a mathematical construction closely reminding the famous Dirac squared-root procedure leading to the equation for bispinors. The Kähler equation was lately pointed out as a natural tool to describe the lattice fermions [2, 3] and also as a possible basis to understand the phenomenon of fermionic generations [3]. On the other hand, it was suggested that this equation might be more fundamental than the Dirac equation when passing to the general relativity [4]. In this note we interpret the Kähler equation in a more cenventional way as describing in flat space-time a system of two Dirac particles, one of which is infinitely heavy.

In order to write down the Kähler equation in flat space-time let us consider the inhomogeneous external differential form

$$\omega = a(x)+\sum_{p=1}^{4}\frac{1}{p!}\,a_{\mu_1\ldots\mu_p}(x)dx_{\mu_1}\wedge\ldots\wedge dx_{\mu_p}, \tag{1}$$

where $x=(x_\mu)=(\vec{x}, it)$ $(\mu = 1, 2, 3, 4)$, $dx_\mu \wedge dx_\nu = -dx_\nu \wedge dx_\mu$, $a(x)$ is a scalar field and $a_{\mu_1\ldots\mu_p}(x)$ $(p = 1, 2, 3, 4)$ are antisymmetric tensor fields. Representing the form (1) by its components we can write

$$\omega = [a(x), a_{\mu_1\ldots\mu_p}(x)\,(p = 1, 2, 3, 4)]. \tag{2}$$

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Performing the differentiation of the form (1) we obtain

$$d\omega = \partial_\nu a(x)dx_\nu + \sum_{p=1}^{4} \frac{1}{p!} \partial_\nu a_{\mu_1 \ldots \mu_p}(x) dx_\nu \wedge dx_{\mu_1} \wedge \ldots \wedge dx_{\mu_p}$$

$$= \sum_{p=1}^{4} \frac{1}{(p-1)!} \frac{1}{p!} \sum_{\text{perm.}} (-1)^\pi \partial_{\mu_{\pi_1}} a_{\mu_{\pi_2} \ldots \mu_{\pi_p}}(x) dx_{\mu_{\pi_1}} \wedge \ldots \wedge dx_{\mu_{\pi_p}}, \tag{3}$$

where $\partial_\nu = \partial/\partial x_\nu$, or in terms of components

$$d\omega = \left[0, \frac{1}{(p-1)!} \sum_{\text{perm.}} (-1)^\pi \partial_{\mu_{\pi_1}} a_{\mu_{\pi_2} \ldots \mu_{\pi_p}}(x) \, (p = 1, 2, 3, 4)\right]. \tag{4}$$

Then making use of the operation d^+ being Hermitian conjugate of d with respect to the scalar product [3]

$$(\omega_a, \omega_b) = \frac{1}{i} \int d^4x \left[a(x)b(x) + \sum_{p=1}^{4} \frac{1}{p!} a_{\mu_1 \ldots \mu_p}(x) b_{\mu_1 \ldots \mu_p}(x)\right] \tag{5}$$

we get

$$-d^+\omega = \partial_{\mu_1} a_{\mu_1}(x) + \sum_{p=2}^{4} \frac{1}{(p-1)!} \partial_{\mu_1} a_{\mu_1 \mu_2 \ldots \mu_p}(x) dx_{\mu_2} \wedge \ldots \wedge dx_{\mu_p} \tag{6}$$

or in terms of components

$$-d^+\omega = [\partial_{\mu_1} a_{\mu_1 \mu_2 \ldots \mu_p}(x) \, (p = 1, 2, 3, 4), 0]. \tag{7}$$

It follows from Eqs. (4) and (7) that $d^2 = 0 = d^{+2}$ and

$$(dd^+ + d^+d)\omega = -\Box\omega, \tag{8}$$

where $\Box = \Delta - (\partial/\partial t)^2$. Hence, the operation $D = d - d^+$ is a square root of the d'Alambertian,

$$D^2\omega = \Box\omega, \tag{9}$$

playing, therefore, a similar role for differential forms as the Dirac operation $\gamma_\mu \partial_\mu$ for bispinors. So the Kähler equation [1]

$$(D+m)\omega = 0 \tag{10}$$

is a close analogue of the Dirac equation

$$(\gamma_\mu \partial_\mu + m)\psi = 0, \tag{11}$$

where $(\gamma_\mu) = (-i\beta\vec{\alpha}, \beta)$ and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. (The Kähler equation in curved space-time has also the form (10) [4]).

Moreover, defining the 4×4 matrix

$$\psi^{(\omega)} = a(x) + \sum_{p=1}^{4} \frac{1}{p!}\gamma_{\mu_1}\dots\gamma_{\mu_p}a_{\mu_1\dots\mu_p}(x) \tag{12}$$

we can show [3] that

$$(\gamma_\mu\partial_\mu + m)\psi^{(\omega)} = \psi^{[(D+m)\omega]} = 0. \tag{13}$$

Thus the matrix $\psi^{(\omega)}$ satisfies the Dirac equation with respect to its first index if the corresponding form ω satisfies the Kähler equation, and vice versa. Since under Lorentz transformations we have[1]

$$\psi^{(\omega)\prime}(x') = a'(x') + \sum_{p=1}^{4} \frac{1}{p!}\gamma_{\mu_1}\dots\gamma_{\mu_p}a'_{\mu_1\dots\mu_p}(x'), \tag{14}$$

where

$$a'(x') = a(x), \quad a'_{\mu_1\dots\mu_p}(x') = \Lambda_{\mu_1\nu_1}\dots\Lambda_{\mu_p\nu_p}a_{\nu_1\dots\nu_p}(x), \tag{15}$$

we can conclude that

$$\psi^{(\omega)\prime}(x') = S\psi^{(\omega)}(x)S^{-1}, \quad S^{-1}\gamma_\mu S = \Lambda_{\mu\nu}\gamma_\nu. \tag{16}$$

Then the Dirac–Kähler equation (13) is Lorentz covariant (for the Lorentz covariance of Eq. (13) alone it would be sufficient to have $\psi^{(\omega)\prime}(x') = S\psi^{(\omega)}(x)$). So both indices of the matrix $\psi^{(\omega)}$ behave under Lorentz transformations as Dirac bispinor indices, though the second index is kinematically passive in Eq. (13).

Looking at the Dirac–Kähler equation (13) alone, one may make attractive conjectures that the second four-valued index of $\psi^{(\omega)}$, not being kinematically active, numerates some fermionic internal states e.g. four fermionic generations [3]. However, if this equation is derived from the Kähler equation (10) via the definition (12), both indices of $\psi^{(\omega)}$ are related to the usual space-time[1]. It suggests that $\psi^{(\omega)}$ satisfying Eq. (13) describes in fact a system of two Dirac particles, one of which being infinitely heavy and so kinematically passive. Thus $\psi^{(\omega)}$ is effectively a wave function of one Dirac particle, the only remnant of the second Dirac particle being the second index at $\psi^{(\omega)}$. It does not mean, however, that the Kähler equation is logically worse than the Dirac equation in approximate description of the hydrogen atom. On the contrary, one must introduce the spinor index for the proton if one wants to discuss hyperfine-structure effects in hydrogen.

At this point we should like to remark that if the Dirac equation

$$(\gamma_\mu\partial_\mu + m)\psi = 0 \tag{17}$$

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[1] if $a_{\mu_1\dots\mu_p}(x)$ are Lorentz tensor fields (and $a(x)$ is a Lorentz scalar field).

for a 4×4 matrix ψ is based not on the Kähler equation but rather on the Clifford algebra defined by the anticommutation relations

$$\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2\delta_{\mu\nu}, \quad \{\hat{\gamma}_\mu, \hat{\eta}_\nu\} = 0, \quad \{\hat{\eta}_\mu, \hat{\eta}_\nu\} = 2\delta_{\mu\nu}, \tag{18}$$

then the second index of ψ can describe some fermionic internal states e.g. four fermionic generations [5]. This follows from the fact that relations (18) can be minimally represented by $(4\times 4)\times(4\times 4)$ (or equivalently 16×16) matrices including the extended Dirac matrices $\hat{\gamma}_\mu = \gamma_\mu\otimes\mathbf{1}$, where γ_μ and $\mathbf{1}$ are the usual 4×4 Dirac matrices acting on the first and second index of ψ, respectively (when forming the direct product $\gamma_\mu\otimes\mathbf{1}$). Then $\hat{\gamma}_\mu\psi = \gamma_\mu\psi$.

It is interesting to write down the system of equations for components of the form ω, following from the Kähler equation (10) or the Dirac–Kähler equation (13). So, introducing the notation

$$a = S, \quad a_\mu = V_\mu, \quad a_{\mu\nu} = T_{\mu\nu}, \quad a_{\mu\nu\varrho} = \varepsilon_{\mu\nu\varrho\sigma}A_\sigma, \quad a_{\mu\nu\varrho\sigma} = -\varepsilon_{\mu\nu\varrho\sigma}P, \tag{19}$$

where $\varepsilon_{1234} = i$, we obtain via definition (1) or (12)

$$\begin{gathered}\partial_\mu V_\mu + mS = 0, \quad \partial_\nu T_{\nu\mu} + \partial_\mu S + mV_\mu = 0,\\ \partial_\mu V_\nu - \partial_\nu V_\mu + \varepsilon_{\mu\nu\varrho\sigma}\partial_\varrho A_\sigma + mT_{\mu\nu} = 0, \quad \partial_{[\mu}T_{\nu\varrho]} + \varepsilon_{\mu\nu\varrho\sigma}(\partial_\sigma P + mA_\sigma) = 0,\\ \partial_\mu A_\mu + mP = 0.\end{gathered} \tag{20}$$

Here, $x_4 = it$ and $\partial_4 = -i\partial/\partial t$. The tensor equations (20) imply the Klein–Gordon equations with the mass m for any of the components S, V_μ, $T_{\mu\nu}$, A_μ and P. It follows from the construction of D or $\gamma_\mu\partial_\mu$.

Let us note that the Duffin–Kemmer–Petiau equation describing one spin-0 or spin-1 particle [6] can be written in the form

$$\tfrac{1}{2}[\gamma_\mu, \partial_\mu\psi^{(\omega)}] + m\psi^{(\omega)} = 0 \tag{21}$$

implying via the definitions (12) and (19) the system of equations

$$\begin{gathered}mS = 0, \quad \partial_\nu T_{\nu\mu} + mV_\mu = 0,\\ \partial_\mu V_\nu - \partial_\nu V_\mu + mT_{\mu\nu} = 0, \quad \partial_\mu P + mA_\mu = 0, \quad \partial_\mu A_\mu + mP = 0\end{gathered} \tag{22}$$

which reduces to the Proca and Klein–Gordon equations, viz.

$$(\Box - m^2)V_\mu = 0, \quad \partial_\mu V_\mu = 0 \tag{23}$$

and

$$(\Box - m^2)P = 0. \tag{24}$$

In Eq. (21) *both* Dirac indices of $\psi^{(\omega)}$ are kinematically active. Notice that the familiar Duffin–Kemmer–Petiau matrices β_μ are represented in Eq. (21) by

$$\beta_\mu = \tfrac{1}{2}(\gamma_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \gamma_\mu^c), \tag{25}$$

where $\gamma_\mu^c = c^{-1}\gamma_\mu c = -\gamma^T$ with γ_μ and $\mathbf{1}$ being the usual 4×4 Dirac matrices. In fact, $\psi^{(\omega)}\gamma_\mu = (\mathbf{1}\otimes\gamma_\mu^T)\psi^{(\omega)} = -(\mathbf{1}\otimes\gamma_\mu^c)\psi^{(\omega)}$, so that

$$\beta_\mu\partial_\mu\psi^{(\omega)} = \tfrac{1}{2}[\gamma_\mu, \partial_\mu\psi^{(\omega)}]. \tag{26}$$

The counterpart in form space of the Duffin–Kemmer–Petiau equation (21) involves, beside d, the differentiation $\tilde{d}$ defined by the formula

$$\tilde{d}\omega = \partial_\nu a dx_\nu + \sum_{p=1}^{4} dx_{\mu_1} \wedge \ldots \wedge dx_{\mu_p} \wedge \frac{1}{p!}\partial_\nu a_{\mu_1\ldots\mu_p}(x)dx_\nu \tag{27}$$

which leads to

$$\tilde{d}\omega = \left[0, \frac{(-1)^{p-1}}{(p-1)!}\sum_{perm}(-1)^\pi \partial_{\mu_{\pi_1}} a_{\mu_{\pi_2}\ldots\mu_{\pi_p}}(x)\;(p = 1, 2, 3, 4)\right] \tag{28}$$

and

$$-\tilde{d}^+\omega = [(-1)^{p-1}\partial_{\mu_1}a_{\mu_1\ldots\mu_p}(x)\;(p = 1, 2, 3, 4), 0]. \tag{29}$$

Then $\tilde{d}^2 = 0 = \tilde{d}^{+2}$ and $\tilde{D}^2\omega = \Box\omega$ with $\tilde{D} = \tilde{d}-\tilde{d}^+$. The antisymmetrized equation in form space

$$[\tfrac{1}{2}(D-\tilde{D})+m]\omega = 0 \tag{30}$$

implies the Duffin–Kemmer–Petiau equation (21), and vice versa, leading to the same system of tensor equations (22).

In conclusion, our interpretation of the Kähler equation is that it describes a system of two Dirac particles in the limiting case when one of these particles can be considered as infinitely heavy. If it is so, the spatial coordinates $\vec{x}$ can be taken as relative coordinates referring the Dirac particle of mass m to the other Dirac particle of mass M with $M/m \to \infty$ ($m > 0$). As long as $M/m = \infty$ there is no difference between these relative coordinates and some non-relative coordinates since even in the interaction case the origine of an inertial frame of reference can be located at the material point of mass $M = \infty$ (which is then at rest in this frame). One can wonder, however, what may happen with this picture if one supposes that the mass of a point particle cannot exceed some maximal value, say, the Planck mass $M_{\mathrm{PL}} \sim 10^{19}$ GeV/c^2. Should in this case the Dirac–Kähler equation (13) be simply replaced by the two-body Breit equation with the masses $m \ll M \to M_{\mathrm{PL}}$ or, rather, might something extraordinary happen?

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