LETTERS TO THE EDITOR

AN APPROXIMATE SOLUTION TO THE EQUATION $\varepsilon XY\frac{dY}{dX} = X-Y(Y+1)$*

By R. E. Mickens

Department of Physics, Atlanta University, Atlanta, Georgia 30314, USA

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We extend the calculations of a previous paper to include the boundary layer behavior of the solution to the equation $\varepsilon XYY' = X-Y(Y+1)$. An iteration technique is used to calculate higher-order corrections to the solution.

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Equation (1) below arises in the study of stellar structure (Jeans 1925, Chandrasekhar 1958). In a recent paper (Mickens 1978), we obtained an exact regular perturbation solution to the equation

$$\varepsilon XY\frac{dY}{dX} = X-Y(Y+1), \quad 0<\varepsilon \ll 1, \quad X>0, \tag{1}$$

to all orders in the parameter ε. However, since Eq. (1) has boundary layer behavior near $X = 0$, the solution given by Mickens (1978) is only the outer part of the solution to Eq. (1) (Nayfeh 1981). The inner solution, which exhibits the boundary layer behavior, has to be calculated using the techniques of singular perturbation theory. The purpose of this paper is to calculate a uniformly valid first-order approximation to the solution of Eq. (1). We then show that higher-order approximates to the solution may be easily obtained by use of an iteration scheme.

The curve defined by the equation

$$X-Y(Y+1) = 0 \tag{2}$$

is an attractor curve; by this, we mean the following: For ε small and starting from an

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initial point (X_0, $Y(X_0, \varepsilon)$), the corresponding solution curve drops or rises rapidly to a neighborhood of the curve given by equation (2) and stays near it for further increase in X values. This behavior is shown in Fig. 1. (An excellent theoretical discussion of this type of behavior for first-order differential equations is given in Elsgolts (1970).) The

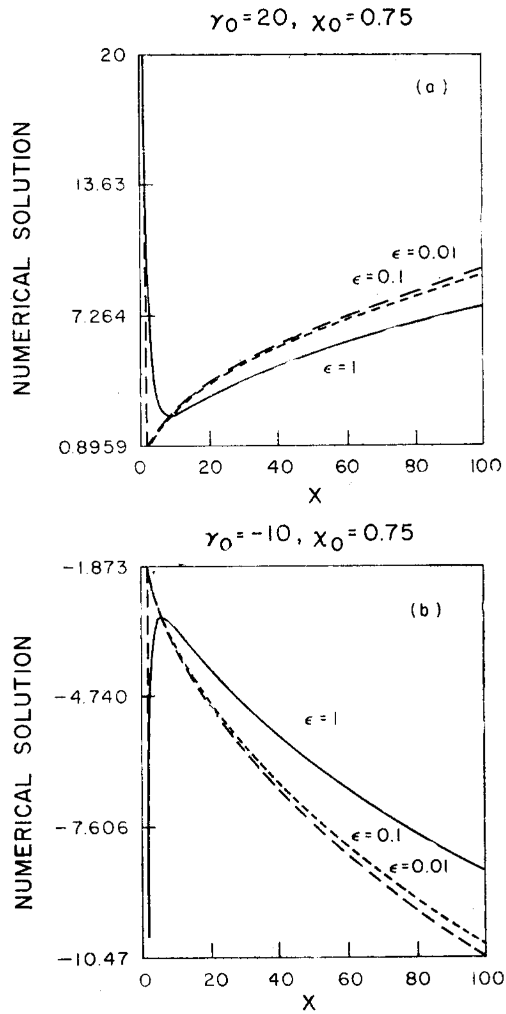

Fig. 1. Numerical solution of equation (1) for ε values of (1, 0.1, 0.01); the initial X value is 0.75 and $Y_0 = Y(0.75, \varepsilon)$

initial, very rapid behavior of $Y(X, \varepsilon)$ is a consequence of Eq. (1) having a boundary layer near $X = 0$.

To proceed with the calculation, recognition has to be made of the fact that two cases have to be considered, namely, when $Y(X_0, \varepsilon) > 0$ and $Y(X_0, \varepsilon) < 0$. (This result is a consequence of equation (1) not being defined when $Y = 0$.) A first-order approximate solution to equation (1) can be written as follows (Carrier and Pearson 1968)

$$Y(X_0, \varepsilon) > 0: \quad Y(X, \varepsilon) = -(1+\sqrt{1+4X})/2+f_1(Z), \tag{3}$$

$$Y(X_0, \varepsilon) < 0: \quad Y(X, \varepsilon) = -(1-\sqrt{1+4X})/2+f_2(Z) \tag{4}$$

where the first terms on the right-sides of equations (3) and (4) are the corresponding lowest-order contributions to the outer solution (Mickens 1978), and the functions, $f_1(Z)$ and $f_2(Z)$, are the corresponding lowest-order contributions to the inner solution. Following the work of Carrier and Pearson (1968), the variable Z takes the form

$$Z = X^{\delta}, \quad \delta = \delta(\varepsilon) \tag{5}$$

where $\delta(\varepsilon)$ is, for the moment, an unknown function of the parameter ε. The differential equations satisfied by $f_1(Z)$ and $f_2(Z)$ may be obtained by substituting Eqs. (3) and (4) into Eq. (1) and letting $X \to 0$ at fixed Z. Doing this gives

$$Z\frac{df_1}{dZ} = -(1+f_1), \quad \varepsilon\delta(\varepsilon) = 1, \tag{6}$$

$$Z\frac{df_2}{dZ} = -f_2, \quad \varepsilon\delta(\varepsilon) = 1. \tag{7}$$

These equations may be easily solved to give

$$f_1 = -1+A_1/X^{1/\varepsilon}, \tag{8}$$

$$f_2 = A_2/X^{1/\varepsilon}, \tag{9}$$

where A_1 and A_2 are arbitrary constants. Therefore, the first approximation to the solution of Eq. (1) is

$$Y(X_0, \varepsilon) > 0: \quad Y_1(X, \varepsilon) = -(3-\sqrt{1+4X})/2+A_1/X^{1/\varepsilon}, \tag{10}$$

$$Y(X_0, \varepsilon) < 0: \quad Y_1(X, \varepsilon) = -(1+\sqrt{1+4X})/2+A_2/X^{1/\varepsilon}. \tag{11}$$

The following features of Eqs. (10) and (11) should be pointed out:

(*i*) The second terms on the right-sides of Eqs. (10) and (11) give rise to the boundary layer. As ε decreases in value, the boundary layer behavior becomes more prominent.

(*ii*) The first terms on the right-sides of Eqs. (10) and (11) give the "attractor" curves to which the solution approaches for increase in X. For sufficiently small ε and any given set of initial conditions, $(X_0, Y(X_0, \varepsilon))$, the solution curve is essentially identical with the "attractor" curve except for a small interval of X values in the neighborhood of X_0; see Fig. 1.

Higher-order approximations to the solution of Eq. (1) may be gotten by using the techniques of singular perturbation theory (Bender and Orszag, 1978). However, for the particular equation studied in this paper, we have discovered an iteration procedure which converges very fast. The iteration scheme is obtained by noting that Eq. (1) is a quadratic function of Y. If we solve for Y and denote the n-th iterate as Y_n, then our iteration scheme is defined as follows

$$Y_n(X, \varepsilon) = -(\tfrac{1}{2})\{(1+\varepsilon X Y'_{n-1}) \pm \sqrt{(1+\varepsilon X Y'_{n-1})^2+4X}\} \tag{12}$$

where the $(+)$ or $(-)$ sign is associated, respectively, with the solution which has the initial condition $Y(X_0, \varepsilon)$ greater than or less than zero and $Y_1(X, \varepsilon)$ is correspondingly taken to be either Eq. (10) or (11). For example, for $Y(X_0, \varepsilon) > 0$, then

$$\begin{aligned} Y_2(X, \varepsilon) = {} & -(\tfrac{1}{2}) + A/2Z - \varepsilon X/2\sqrt{1+4X} + (\tfrac{1}{2})\{1+4X+\varepsilon^2X^2/(1+4X) \\ & + 2\varepsilon X/\sqrt{1+4X} - 2A/Z + A^2/Z^2 - 2A\varepsilon X/Z\sqrt{1+4X}\}^{1/2} \end{aligned} \tag{13}$$

where $Z = X^{1/\varepsilon}$, the initial conditions are $(X_0, Y(X_0, \varepsilon))$

$$A = \frac{C_4^2 - C_2}{2C_1(C_4 - C_3 - 1)},$$

$$C_1 = 1/X_0^{1/\varepsilon}, \quad C_3 = \varepsilon X_0/\sqrt{1+4X_0},$$

$$C_2 = 1+4X_0+\varepsilon^2X_0^2/(1+4X_0)+2\varepsilon X_0/\sqrt{1+4X_0},$$

$$C_4 = 2Y(X_0, \varepsilon)+1+\varepsilon X_0/\sqrt{1+4X_0}.$$

Comparison of the solution given by the iteration scheme of Eq. (12) with the numerical solution obtained by using an efficient ODE integrator shows that the iteration procedure converges very fast; in fact

$$\frac{Y_c - Y_n}{Y_c} = O(\varepsilon^n) \tag{14}$$

where Y_c is the numerical solution and Y_n is calculated using equations (10), (11) and (12). Figure 1 gives plots of $Y_2(X, \varepsilon)$ for several sets of initial values: (0.75, 20.0) and (0.75, −10.0).

In summary, we have presented a technique for obtaining approximate analytical solutions to Eq. (1). The technique consists of using singular perturbation theory to calculate a first-approximate to the solution; higher-order approximates are gotten by use of the iteration scheme given in Eq. (12).

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