STATIC SOLUTIONS OF THE YANG-MILLS SYSTEM COUPLED TO A SCALAR FIELD

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It is proven that the linear Schrödinger (Klein-Gordon) field coupled to Yang-Mills fields has no non-zero static, finite energy, solutions. The case of nonlinear scalar fields is discussed, and necessary criteria for existence of nonzero solutions are given.

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1. *Introduction*

The interest in classical Yang-Mills theory stems from the hope that classical Yang-Mills fields, after semiclassical quantization, can elucidate the fundamental problem of hadronic physics, i.e., the quark confinement. As is well known, the quantum chromodynamics is asymptotically free and, morever, the asymptotic freedom alone seems to explain the scaling phenomena and the tendency of quarks to behave like pointlike and free particles in deep inelastic scattering. The asymptotic freedom has been demonstrated within the framework of the perturbative QCD, but the perturbation theory fails to incorporate the quark confinement. There is a common belief that this problem can be overcome by nonperturbative approach, i.e., by quantizing around a nonzero background [1]. Note at this point that fully consistent approach should take into account the gauge and the matter fields simultaneously. Because of that the Yang-Mills fields coupled to scalar (Schrödinger or Klein-Gordon) fields will be considered in this paper. We will study the existence of static solitons in Minkowski space, i.e., static finite energy solutions of Yang-Mills equations with sources introduced dynamically via the coupling with a scalar Schrödinger (Klein-Gordon) fields. As we have indicated above, this problem can be important from the quantum point of view. If there exist such solutions, they should be used as a background for quantum fluctuations, thus allowing for a nonperturbative investigation of the theory.

As is well known, contrary to the 4-dimensional Euclidean case, the sourceless Yang-Mills theory in Minkowski space does not possess solitons [2]. We will show that for

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the Yang-Mills-Schrödinger (YMS) and for the Yang-Mills-Klein-Gordon (YMKG) theories there are no solitons either, provided the scalar fields are linear. So if one believes in success of semiclassical quantization, then one has better use a nonlinear scalar fields.

The organization of the paper is as follows. In Section 2 we describe the model and prove our main results concerning the YMS model. In Section 3 is discussed a quantum-mechanical version of the YMS theory. In Section 4 we show that if a scalar self-interaction is included, the occurence of solitons may not be suppressed. In Section 5 the YMKG theory is considered. The last Section comprises a summary.

2. *Description of the models. YMS theory*

Let a Schrödinger (Klein-Gordon) field be an n-component SU(n) isospinor denoted by ψ. Yang-Mills potentials are denoted by A^a_μ, where the upper isospin indices change from 1 to n^2-1, while the lower space-time indices range from 0 to 3. The covariant derivative D_μ acts on ψ as follows

$$D_\mu \psi = \partial_\mu \psi - igA_\mu \psi. \tag{1}$$

A potential A is a one-form with values in a fundamental algebra of the SU(n) group.

A Yang-Mills strength field tensor is defined as usual

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{2}$$

where $A_\mu = A^a_\mu T^a$; T^a are the hermitian generators of SU(n) algebra.

A Yang-Mills potential A^a_μ, its strength field tensor and ψ transform as below under a gauge rotation h (h is in a fundamental representation of SU(n)):

$$(A_\mu)^h = hA_\mu h^{-1} - ig(\partial_\mu h)h^{-1}, \tag{3a}$$

$$(F_{\mu\nu})^h = hF_{\mu\nu}h^{-1}, \tag{3b}$$

$$(\psi)^h = h\psi. \tag{3c}$$

The Yang-Mills equations are

$$D_i F^{i0} = j^0, \tag{4a}$$

$$D_\mu F^{\mu j} = j^j, \tag{4b}$$

where the Schrödinger current (the Klein-Gordon current is given in Section 5) is

$$j^a_0 = g\psi^+ T^a \psi, \quad j^a_i = \frac{ig}{2m}(\psi^+ T^a D_i \psi - (D_i \psi)^+ T^a \psi), \tag{5}$$

and the Schrödinger equations are

$$\frac{-1}{2m} D_i D_i \psi + \frac{1}{i} D_0 \psi = 0. \tag{6}$$

The signature we used above is $(-, +, +, +)$.

Let us suppose that A^a_μ and ψ are static and vanish on the boundary, which may be taken at spatial infinity. In the last case we demand in addition a sufficiently fast decrease of the fields as the radius r tends to infinity:

$$A^a_\mu \sim 1/r^{3/4+\varepsilon}, \quad \frac{\partial}{\partial r} A^a_\mu \sim 1/r^{3/2+\varepsilon}, \quad |\psi| \sim 1/r^{3/2+\varepsilon}. \tag{7}$$

It can be easily seen that these requirements are sufficient and, morever, necessary foı the finiteness of energy (although the condition on ψ can be weaker)

$$\varepsilon = \int dV \left[\tfrac{1}{2} (\tfrac{1}{2} F^a_{ij} F^{aij} + F^a_{0i} F^a_{0i}) + \frac{1}{2m} (D_i\psi)^+ D_i\psi \right].$$

Remark 1. In Yang-Mills-Higgs model (with or without scalar selfinteraction) one usually imposes boundary condition on the scalar field, $\psi(\partial\Omega) = \text{const}$ [3]. This is unlike above. Our option is closely connected to the following intention: to test basic assumptions of the bag model [4], according to which matter and gauge fields are confined within a bounded volume of space.

Let us multiply a static version of Eqs. (4a) and (6) by A^a_0 and ψ^+ respectively and integrate over all volume bounded by a boundary $\partial\Omega$. Then, integrating by part and dropping out boundary terms (which vanish because of the boundary conditions assumed above), one obtains:

$$\frac{1}{2m} \int_N (D_i\psi)^+ D_i \psi dv - g \int_N \psi^+ A_0 \psi dv = 0, \tag{8a}$$

$$-\int_N F^a_{i0} F^a_{i0} Jv = g \int_N \psi^+ A_0 \psi dv. \tag{8b}$$

We will analyze these equations. At first note the weak inequality

$$g \int_N \psi^+ A_0 \psi dv \leqslant 0, \tag{9}$$

which follows immediately from Eq. (8b), because its left hand side is nonpositive. But, if (9) holds, then all terms in (8a) are weakly positive and each of them must vanish separately, since their sum vanishes.

From (8a) it follows that

$$D_i \psi = 0 \tag{10}$$

i.e.,

$$\partial_i \psi = igA_i \psi. \tag{11}$$

The integrability condition for (11) is

$$0 = (\partial_l \partial_k - \partial_k \partial_l)\psi = igF_{kl}\psi. \tag{12}$$

From (12) it follows that $\psi = 0$ (if one chooses $F_{ik} = 0$, which satisfies (12) also, then the potential A_i^a can be gauged away to zero and from (10) we obtain $\psi = \text{const}$; since the boundary condition is $\psi(\partial\Omega) = 0$, it forces $\psi = 0$, as above).

Turning back to Eq. (8b) we see that

$$F_{0i} = 0, \tag{13}$$

i.e., the electric field is a pure gauge and the potential A_0^a can be gauged away. Inserting $\psi = 0$ and $A_0^a = 0$, we may write Eq. (4b) as

$$D_i F_{ij} = 0, \tag{14}$$

from which (after multiplying by A_j, integrating by parts and omitting boundary terms) we get

$$\int_N F_{ij}^a F^{aij} dv = 0. \tag{15}$$

Hence $F_{ij} = 0$, and the corresponding potential A_i^a could be gauged away. Thus we prove the following

Theorem 1. The Yang-Mills-Schrödinger system does not possess non-zero static finite energy solutions satisfying homogeneous boundary conditions, provided the scalar field is linear.

Remark 2. In the above calculations we have used positivity of the Killing form of SU(n). It is well known that the Killing form is positive for all classical (A(n), B(n), C(n), D(n)) and exceptional (E_6, E_7, E_8, F_4, G_2) real algebras [5]. This allows, as can be shown, for an extension of the above results for all semisimple compact Lie groups — classical and exceptional.

3. *Further discussion*

The boundary condition (7) on ψ allows for its normalization

$$\int_N \psi^+ \psi dv = 1 \tag{16}$$

and, consequently, for its quantum mechanical interpretation. Eq. (16) implies in Eq. (6) a term proportional to the Lagrange multiplier λ. Eq. (6) now reads

$$\frac{1}{2m} D_i D_i \psi - g A_0 \psi = \lambda \psi. \tag{17}$$

The multiplier λ is usually interpreted as energy of the system described by a field ψ.

It is an easy exercise to prove that for $\lambda < 0$ Theorem 1 still holds, and the case of negative energy is physically reasonable, since then one can hope that gauge and scalar fields are concentrated within a small volume of space. If, however, we set $\lambda > 0$, the fields are long-range. Indeed, if a potential A_μ^a is short-ranged, Eq. (17) becomes asymptotically

$$\frac{-1}{2m}\Delta\psi = \lambda\psi. \tag{18}$$

Its solutions behave as $1/r$ for $r \to \infty$ and λ positive, or as $\exp(-(-\lambda)^{1/2} r)/r$, for $r \to \infty$ and λ negative. Hence a scalar field is long-ranged when the energy is positive.

4. *The nonlinear Yang-Mills-Schrödinger system*

Now let us consider a modification of the Schrödinger equation, which consists in inclusion a nonlinear term $V(\psi)$ describing self-interaction of the scalar field ψ. The static Schrödinger equation is now

$$\frac{-1}{2m}D_iD_i\psi - gA_0\psi + V(\psi) = 0, \tag{19}$$

while Yang-Mills equations are unchanged, hence given by (4a, b). The question that arises here is this: *under what circumstances* Eq. (19) *coupled to* Eqs. (4a, b) *has nonzero static solutions?* An incomplete answer is given by the following:

Theorem 2. The Yang-Mills nonlinear Schrödinger (YMnS) system can possess nonzero finite energy solutions only if

$$\int_{\mathbf{R}^3} \psi^+ V(\psi)dv < 0. \tag{20}$$

The condition (20) is necessary but not sufficient.

Proof. Let us multiply (19) by ψ^+ and integrate over all space $\mathbf{R}^3$. After integration by parts and omission of boundary terms one arrives at the equality

$$\frac{1}{2m}\int_{\mathbf{R}^3}(D_i\psi)^+ D_i\psi dv - g\int_{\mathbf{R}^3}\psi^+ A_0\psi dv + \int_{\mathbf{R}^3}\psi^+ V)\psi)dv = 0. \tag{21}$$

The first and the second terms are positive if there exist nonzero solutions ψ, A^a_μ (the second term is positive because of (9)), hence the last term must be negative.

$$\int_{\mathbf{R}^3} \psi^+ V(\psi)dv < 0.$$

Thus the inequality (20) is necessary, but of course not sufficient condition.

Here is an example of $V(\psi)$ satisfying (20):

$$V(\psi) = F(|\psi|)\psi, \tag{22}$$

where $F(|\psi|)$ is any positive function that tends to a finite limit as $|\psi|$ tends to zero. This condition on F is imposed in order to guarantee that the integral

$$\int_{\mathbf{R}^3} \psi^+ V(\psi)dv = \int_{\mathbf{R}^3} |\psi|^2 F(|\psi|)dv$$

is finite (see (7) for the behaviour of ψ at infinity).

Remark 3. Theorem 2 can be extended to all semisimple compact gauge groups (see *Remark 2*).

Remark 4. There is an important class of scalar field self-interactions, $V(\psi) = \lambda(|\psi|^2 - a^2)\psi$, for which our criterion (20) does not work effectively; in practice it is not possible to determine the sign of $\lambda \int |\psi|^2(|\psi|^2 - a^2)dv$ without knowing a solution. In spite of this, the qualitative analysis still can be done without too much effort. One can proceed certain techniques taken from bifurcation theory (some elementary facts about it and related mathematical references can be found in [6]). Without going into details, we recall that the bifurcation theory is a powerful machinery for studying whether near a given exact solution there exists another. Presently we will describe it using above mentioned example.

As can be easily seen, the YMnS system has one exact solution $A_\mu^a = 0$, $\psi = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$; linearization near this solution gives

$$\frac{-1}{2m} \Delta\delta\psi - \lambda a^2 \delta\psi = 0, \tag{23a}$$

$$\Delta\delta A_\mu^a = 0, \tag{23b}$$

where variations δA_μ^a, $\delta\psi$ satisfy homogeneous conditions and the Coulomb gauge condition $\partial_i A_i^a = 0$. It is easily seen that for certain values of λa^2 (assuming that the boundary $\partial\Omega$ is finite; in the unbounded, noncompact case the spectrum of Δ is continuous and a theoretic--bifurcation discussion is harder — we omit that case) the linear system (23a) has nonzero solutions $\delta\psi$. This means (according to bifurcation theory [6]) that there exists a nonzero solution of the full YMnS system if and only if certain "second order" condition is satisfied.

The term $V(\psi) = \lambda(|\psi|^2 - a^2)\psi$ is typical for the Yang-Mills-Higgs theories, in which a large number of monopole-like solutions was discovered (see e.g., [3]). However, they satisfy nonhomogeneous boundary conditions: at infinity $|\psi(\infty)| = \text{const} \neq 0$.

5. *The Yang-Mills-Klein-Gordon system*

The YMKG system of equations is (assuming a linear scalar field);

$$(D_i D_i - D_0 D_0)\psi = 0 \tag{24a}$$

$$D_\mu F^{\mu\nu} = j^\nu. \tag{24b}$$

Here the signature is $(-, +, +, +)$, the covariant derivative is as in (1) and the current j_ν^a is given by

$$j^{a,\nu} = ig[\psi^+ T^a D^\nu \psi - (D^\nu \psi)^+ T^a \psi]. \tag{25}$$

The static equations are as follows

$$D_k D_k \psi + g^2 (A_0)^2 \psi = 0, \tag{26a}$$

$$(D_k F_{0k})^a = ig[\psi^+ T^a(-igA_0)\psi - (-igA_0\psi)^+ T^a\psi], \tag{26b}$$

$$(D_\mu F^{k\mu})^a = ig[\psi^+ T^a(\partial^k - igA^k)\psi - ((\partial^k - igA^k)\psi)^+ T^a\psi]. \tag{26c}$$

Multiplying Eq. (26a) by ψ^+, (26b) by A_0^a, integrating by parts and omitting boundary terms (as for the Schrödinger field, we impose homogeneous boundary conditions) one gets the equations:

$$\int (D_i\psi)^+ D_i\psi dv = 2g^2 \int \psi^+ A_0^2 \psi dv, \tag{27a}$$

$$\int F_{i0}^a F_{i0}^a dv = -g^2 \int \psi^+ A_0^2 \psi dv. \tag{27b}$$

From (27b) it follows that $A_0^a = 0$ or $\psi = 0$. After computations similar to the ones of Section 2, one obtains

$$A_\mu^a = 0, \qquad \psi = 0. \tag{28}$$

Hence we prove

Theorem 3. YMKG system does not possess non-zero static finite energy solutions such that vanish on the boundary (possibly at infinity), provided the scalar field is linear.

Theorem 3 holds for all semisimple compact groups (compare the Remark 2).

Remark 5. Nonzero solutions are not excluded, provided that the scalar field is nonlinear. In this case one can repeat all considerations from Section 4 and arrive at identical conclusions.

6. *Summary*

We studied the problem of existence of static solutions of the Yang-Mills-Schrödinger (Klein-Gordon) equations. The scalar fields were in the fundamental representation of the algebra of a gauge group. We proved that in these models there are no nonzero static solutions provided that they vanish sufficiently fast at infinity and that the scalar Schrödinger (Klein-Gordon) field is linear. The case of a self-interaction of a scalar field was also investigated, using global as well as local (bifurcation theory) techniques. Nonzero solutions may exist whenever the nonlinearities are of type $V(\psi) = F(|\psi|)\psi$, where F is some positive function that tends to a finite limit as $|\psi| \to 0$. This holds for both Schrödinger and Yang-Mills fields coupled to the Yang-Mills fields and for all semisimple and compact gauge groups. The assumption that the scalar fields are in fundamental representation, is in fact irrelevant. As one can show, all previous results are true, provided that they carry an adjoint representation of any compact semisimple gauge group.

At the end let us stress that above results indicate that phenomenological assumptions of the bag model [4] cannot be obtained within the framework of matter-gauge theory, provided the matter field is linear. The existing lagrangian models overcome this difficulty by introducing new unobservable fields [7] — but this strips the whole charm from the theory which is attractive because of its conceptual simplicity.

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