LETTERS TO THE EDITOR

LOCALLY HERMIT–EINSTEIN, SELF-DUAL GRAVITATIONAL INSTANTONS

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It is shown that for locally Hermit-Einstein, self-dual gravitational instanton, Einstein's equations can be reduced to a single, second-order, non-linear equation.

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It is well known [1, 2] that every self-dual gravitational instanton with vanishing Ricci tensor ($R_{ab} = 0$) is a locally Kählerian manifold, i.e., for each point there exist local complex coordinates $\{z^\alpha\}$, $\alpha = 1, 2$, such that

$$(i) \qquad ds^2 = g_{\alpha\bar{\beta}} dz^\alpha \otimes dz^{\bar{\beta}} + g_{\bar{\beta}\alpha} dz^{\bar{\beta}} \otimes dz^\alpha,$$

$$(ii) \qquad d(g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}) = 0, \tag{2}$$

where $\bar{\beta} = \bar{1}, \bar{2}$; $z^{\bar{\beta}} := \overline{z^\beta}$; $\overline{g_{\alpha\bar{\beta}}} = g_{\beta\bar{\alpha}}$; $g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha}$.

Moreover, for some local complex coordinates

$$g_{\alpha\bar{\beta}} = K_{,\alpha\bar{\beta}}, \tag{3}$$

where $K = K(z^\alpha, z^{\bar{\alpha}})$ is a real solution of the following second-order, non-linear equation

$$K_{,1\bar{1}} K_{,2\bar{2}} - K_{,1\bar{2}} K_{,2\bar{1}} = 1 \tag{4}$$

(", " denotes the partial derivative). On the other hand, as it has been shown in [2], the metric tensor of any locally Kählerian manifold which is also the Einstein manifold

$$R_{ab} = -\Lambda g_{ab}, \tag{5}$$

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is defined locally by (3), where $K = K(z^\alpha, z^{\bar\alpha})$ appears to be a real solution of the equation

$$K_{,1\bar 1}K_{,2\bar 2} - K_{,1\bar 2}K_{,2\bar 1} = e^{-\Lambda K}. \tag{6}$$

In the present note we study the case of a non-Kählerian, but still locally Hermit-Einstein, self-dual gravitational instanton. Therefore, we assume (1) and (5) with $\Lambda \neq 0$, and

$$d(g_{\alpha\bar\beta}dz^\alpha \wedge dz^{\bar\beta}) \neq 0. \tag{7}$$

It has been shown [3] recently that the above assumptions lead to the following results: for some local complex coordinates

$$g_{1\bar\alpha} = F_{,\bar\alpha}, \tag{8}$$

where $F = F(z^\alpha, z^{\bar\alpha})$ is the complex function which enter into the second Cartan structure equations as follows

$$g_{2[\bar 2,\bar 1]} = \frac{g}{F} \tag{9}$$

$$(\ln H)_{,1} = \frac{\Lambda}{3} F, \tag{10}$$

$$(\ln H)_{,\alpha\bar\beta} - \frac{2}{H}\delta^2{}_\alpha \delta^{\bar 2}{}_{\bar\beta} = \frac{\Lambda}{3} g_{\alpha\bar\beta}, \tag{11}$$

here, $g := \det(g_{\alpha\bar\beta})$, $H := \dfrac{F\cdot\bar F}{g} > 0.$

From (11) one obtains

$$g_{1\bar 1} = \frac{3}{\Lambda} K_{,1\bar 1}, \quad g_{1\bar 2} = \frac{3}{\Lambda} K_{,1\bar 2} = \overline{g_{2\bar 1}},$$

$$g_{2\bar 2} = \frac{3}{\Lambda}(K_{,2\bar 2} - 2e^{-K}), \tag{12}$$

where the real function $K = K(z^\alpha, z^{\bar\alpha})$ is defined by the formula

$$K := \ln H. \tag{13}$$

It is easy to verify the above with (10) and (12). Eq. (9) is equivalent to the following equation

$$g = \left(\frac{3}{\Lambda}\right)^2 K_{,1} K_{,\bar 1} e^{-K}, \tag{14}$$

or, by (12)

$$K_{,1\bar 1}K_{,2\bar 2} - K_{,1\bar 2}K_{,2\bar 1} - (2K_{,1\bar 1} + K_{,1}K_{,\bar 1})e^{-K} = 0. \tag{15}$$

Concluding our results, we see that the metric of a locally Hermit-Einstein, self-dual gravitational instanton is determined locally by the real solution of one of equations, (4), (6) or (15). (Observe that the metric defined by (12) and (15) is positive definite iff

$$\left. \frac{3}{\Lambda} K_{,1\bar{1}} > 0. \right) \tag{16}$$

Now, if conjecture 2.2 of [2], which asserts that if the Weyl tensor of a differentiable manifold M is self-dual then M is locally Hermitian, is true for Einstein manifolds at least, then, gathering results of [1, 2] and the present note, we find that the metric of a self--dual gravitational instanton satisfying (5) is defined locally by some real solution of the appropriate second-order, non-linear differential equation.

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