Vol. B14 (1983) ACTA PHYSICA POLONICA No 9

LOCALLY KÄHLER GRAVITATIONAL INSTANTONS

BY M. PRZANOWSKI

Institute of Physics, Technical University, Łódź*

AND B. BRODA

Institute of Physics, Department of Theoretical Physics, Łódź University**

(Received April 27, 1982; revised version received January 11, 1983)

The paper discusses certain properties of locally Hermitian and locally Kählerian manifolds. It is shown that a manifold is locally Hermitian iff some of its spinorial connection coefficients vanish. Conditions of a similar type are obtained for locally Kählerian manifolds. Subsequently, locally Kähler-Einstein gravitational instantons are investigated. Their vacuum Einstein equations with the cosmological term have been reduced locally to a single non-linear equation of the second order for one real function. This, in turn, has enabled the authors to show that any locally Kähler gravitational instanton with $R = -4\Lambda =$ const. admits locally a real, source-free Maxwell field which for a Kähler gravitational instanton appears to be global. In the latter case, the Einstein-Maxwell equations have been reduced locally to a single differential equation of the fourth order for one real function.

PACS numbers: 04.20.-q

1. Introduction

The notion of a gravitational instanton was introduced in 1976/77 (see [1, 2, 39, 40]). By analogy to a Yang-Mills instanton [3], we define a gravitational instanton to be a solution of the classical field equations with positive definite metric which is complete and which is either compact or its curvature tensor dies away at large distances (see [2, 10]).

Recently gravitational instantons have attracted a great deal of interest [4–14]; they seem to play a distinguished role in quantum gravity.

Hacyan [12] has proved that every self-dual gravitational instanton with $R_{ab} = 0$ is a locally Kählerian manifold and he has shown that in this case the Euclidean Einstein vacuum equations can be reduced to a single second order non-linear equation for one function (the Kähler function).

Consequently the natural programme arises: to study the general, locally Kählerian manifolds. The gravitational instanton with $R_{ab} \neq 0$ will be called a *non-vacuum gravita-*

* Address: Instytut Fizyki, Politechnika Łódzka, Wólczańska 219, 93-005 Łódź, Poland.

** Address: Instytut Fizyki, Uniwersytet Łódzki, Nowotki 149/153, 90-236 Łódź, Poland.

tional instanton, otherwise, i.e., if $R_{ab} = 0$, the gravitational instanton is called a *vacuum gravitational instanton*.

In our opinion the non-vacuum gravitational instantons play a significant role when the interaction between the gravitational and other physical fields are considered. We also hope that the study of non-vacuum gravitational instantons as well as of vacuum ones will be useful for finding some procedure to generate the physical space-times from the gravitational instantons.

The main aim of the present paper is to study the geometry of a locally Kähler gravitational instanton.

In Sec. 2 we consider locally Hermitian manifolds. First, the spinorial formalism as given in [17, 20, 41] is presented and then a theorem (theorem (2.1)) concerning the necessary and sufficient conditions under which a manifold is locally Hermitian is proved. The theorem (2.2) gives the sufficient condition for a manifold being Hermitian. Then, using the results of theorems (2.3), (2.4) (analogs of the Goldberg-Sachs theorems) we find some connections between the local Hermiticity and the algebraic speciality of the Weyl tensor (propositions (2.1), (2.2)).

Sec. 3 is dedicated to the locally Kählerian manifolds. The analogs of the theorems (2.1), (2.2) are given for this case (see the theorems (3.1), (3.2)). Then the local expressions for the connection forms, the Weyl tensor field, the traceless Ricci tensor field and the curvature scalar for the locally Kählerian manifolds are given in the appropriate extended spinor frames.

In Sec. 4 locally Kähler-Einstein gravitational instantons are investigated. Their Euclidean Einstein equations have been reduced locally to a single second order non-linear differential equation (4.7) for one real function (the Kähler functon). An interesting proposition (the proposition (4.2)) presents the sufficient condition under which a locally Hermit-Einstein gravitational instanton is a locally Kähler-Einstein one.

Sec. 5 is devoted to the general non-vacuum locally Kähler gravitational instantons. We find that for any point of a locally Kähler gravitational instanton, the traceless Ricci tensor is of types ((1, 1), (1, 1)) or ((1, 1, 1, 1)) (the proposition (5.1)). Then we show that any locally Kähler gravitational instanton admits (locally) a real, source-free Maxwell field iff (locally) $R = -4\Lambda = \text{const.}$ If M is a Kähler gravitational instanton and $R = -4\Lambda = \text{const.}$ on M, then M admits a global, real, source-free Maxwell field. If $R = 0$ on M then the Kähler gravitational instanton is self-dual in the natural orientation of M. It is shown that the Einstein-Maxwell equations can be reduced (locally) to a single fourth order non-linear differential equation for one function (see (5.34)). The conclusions and the list of references close the article.

2. *Locally Hermitian manifolds*

Let M be an oriented four-dimensional connected Riemannian manifold with positive definite metric ds^2. For each point $p \in M$ there exist a neighbourhood U of p and four independent complex-valued 1-forms on U, $\{e^a\}$ $a = 1, 2, 3, 4$, such that

$$ds^2 = g_{ab} e^a \otimes e^b \qquad \text{on } U, \tag{2.1}$$

where

$$(g_{ab}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{2.2}$$

and (see [12],

$$\overline{e^1} = e^2; \quad e^3 = e^4. \tag{2.3}$$

If for any point $q \in U$ the orthonormal frame $\{E_a\}$ defined as follows

$$E^a(E_b) = \delta^a{}_b, \tag{2.4}$$

where

$$E^1 = \frac{1}{\sqrt{2}}(e^1+e^2), \quad E^2 = \frac{1}{i\sqrt{2}}(e^1-e^2)$$

$$E^3 = \frac{1}{\sqrt{2}}(e^3+e^4), \quad E^4 = \frac{1}{i\sqrt{2}}(e^3-e^4) \tag{2.5}$$

possesses the positive orientation then four 1-forms $\{e^a\}$ are called a *null tetrad on U*. (We shall omit the words "local", "on U" etc., of course, only if these abbreviations do not lead to any misunderstandings). Let us define (see e.g., [16, 17, 19])

$$(g^{AB}) := \sqrt{2}\begin{pmatrix} e^4 & e^2 \\ e^1 & -e^3 \end{pmatrix}, \tag{2.6}$$

$$A = 1, 2, \quad \dot{B} = \dot{1}, \dot{2}.$$

Then relations (2.3) lead to

$$\overline{g^{A\dot{B}}} = -g_{A\dot{B}}, \tag{2.7}$$

where the "spinorial" indices $A, B, \dot{A}, \dot{B}$ etc., are to be manipulated according to the formulae

$$\psi_A = \varepsilon_{AB}\psi^B, \quad \psi^A = \psi_B\varepsilon^{BA}, \tag{2.8}$$

$$\psi_{\dot{A}} = \varepsilon_{\dot{A}\dot{B}}\psi^{\dot{B}}, \quad \psi^{\dot{A}} = \psi_{\dot{B}}\varepsilon^{\dot{B}\dot{A}}, \tag{2.9}$$

with

$$(\varepsilon_{AB}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = :(\varepsilon^{AB}), \tag{2.10}$$

$$(\varepsilon_{\dot{A}\dot{B}}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = :(\varepsilon^{\dot{A}\dot{B}}). \tag{2.11}$$

Using the 1-forms $\{g^{A\dot{B}}\}$ one has

$$ds^2 = -\tfrac{1}{2}\, g_{A\dot{B}} \underset{s}{\otimes} g^{A\dot{B}} = g^{1\dot{2}} \underset{s}{\otimes} g^{2\dot{1}} - g^{1\dot{1}} \underset{s}{\otimes} g^{2\dot{2}} \tag{2.12}$$

where $\underset{s}{\otimes}$ denotes the symmetrized tensor product, $\Phi \underset{s}{\otimes} \Psi := \tfrac{1}{2}(\Phi \otimes \Psi - \Psi \otimes \Phi)$. Let now $(l^{A''}{}_A), (l^{\dot{A}''}{}_{\dot{A}}) \in \mathrm{SU}(2)$, and define

$$g^{A''\dot{B}''} := l^{A''}{}_A l^{\dot{B}''}{}_{\dot{B}} g^{A\dot{B}}. \tag{2.13}$$

One can easily check that there exists a null tetrad $\{e^{a''}\}$ (only one of course) such that

$$(g^{A''\dot{B}''}) = \sqrt{2}\begin{pmatrix} e^{4''} & e^{2''} \\ e^{1''} & -e^{3''} \end{pmatrix}. \tag{2.14}$$

Conversely, for each null tetrad $\{e^{a''}\}$ there exist 2×2 complex matrices $(l^{A''}{}_A), (l^{\dot{A}''}{}_{\dot{A}}) \in \mathrm{SU}(2)$ such that (2.14) holds These matrices are defined by (2.14) uniquely up to the simultaneous changing of the signs in both matrices. Keeping in mind the connections between the null tetrads and rightly oriented orthonormal frames (see (2.4), (2.5)) we conclude that our construction leads to the cencrete realization of the group isomorphism

$$\mathrm{SO}(4, \boldsymbol{R}) \cong \mathrm{SU}(2) \times \dot{\mathrm{S}}\mathrm{U}(2)/\mathrm{Z}_2, \tag{2.15}$$

here Z_2 is the cyclic group $\{1, -1\}$.

Then one can construct two complex spinor bundles over U, the first one denoted by S (the bundle of undotted spinors) is connected with the leftmost factor SU(2) in (2.15) and the second one $\dot{S}$ (the bundle of dotted spinors) is connected with the rightmost factor SU(2) [9, 17]. For the future we shall conventionally distinguish the leftmost factor SU(2) and the rightmost one by means of a dot, as follows

$$\mathrm{SO}(4, \boldsymbol{R}) = \mathrm{SU}(2) \times \dot{\mathrm{S}}\mathrm{U}(2)/\mathrm{Z}_2. \tag{2.16}$$

Having all that, we can treat $\{g^{A\dot{B}}\}$ as components of the bundle section

$$g^{A\dot{B}} e_A \otimes e_{\dot{B}} = g^{A\dot{B}}{}_a e_A \otimes e_{\dot{B}} \otimes e^a \in \Gamma(S \otimes \dot{S} \otimes T^{*c}) \tag{2.17}$$

where T^{*c} is the complexified cotangent bundle and $\Gamma(\ldots)$ symbolizes the set of bundle sections. Therefore, we say that $\{g^{A\dot{B}}\}$ defines a spinor-valued 1-form, or that $g^{A\dot{B}}$ is a spinor--valued 1-form. Similarly ε_{AB}, ε^{AB}, $\varepsilon_{\dot{A}\dot{B}}$, $\varepsilon^{\dot{A}\dot{B}}$ are spinor-valued 0-forms, or simply, spinor fields.

Let ω be p-form

$$\Gamma(\overset{p}{\Lambda} T^{*c}) \ni \omega = \frac{1}{p!}\,\omega_{a_1\ldots a_p} e^{a_1} \wedge \ldots \wedge e^{a_p}; \tag{2.18}$$

then we define Hodge's star operation

$$* : \Gamma(\overset{p}{\Lambda} T^{*c}) \to \Gamma(\overset{4-p}{\Lambda} T^{*c}) \tag{2.19}$$

according to the formula [16]

$$* \omega := -\frac{1}{p!(4-p)!} \exp\left[\frac{i\pi}{2} p(4-p)\right] \varepsilon_{a_1 \dots a_p b_1 \dots b_{4-p}} \omega^{a_1 \dots a_p} e^{b_1} \wedge \dots \wedge e^{b_{4-p}}. \tag{2.20}$$

One can easily find that this definition of the star operation assures that $** = 1$ It is evident how to extend Hodge's star operation onto arbitrary spinor-valued forms.

Now, let us define the spinor-valued 2-form $S^{AB} \in \Gamma(S \underset{s}{\otimes} S \otimes \overset{2}{\Lambda} T^{*c})$ as follows [16, 17, 41]

$$S^{AB} := \tfrac{1}{2} \varepsilon_{\dot{R}\dot{S}} g^{A\dot{R}} \wedge g^{B\dot{S}}. \tag{2.21}$$

One finds that

$$* S^{AB} = S^{AB}. \tag{2.22}$$

Analogously, the spinor-valued 2-form $S^{\dot{A}\dot{B}} \in \Gamma(\dot{S} \underset{s}{\otimes} \dot{S} \otimes \overset{2}{\Lambda} T^{*c})$ defined by the formula

$$S^{\dot{A}\dot{B}} := \tfrac{1}{2} \varepsilon_{RS} g^{R\dot{A}} \wedge g^{S\dot{B}} \tag{2.23}$$

fulfils the relation

$$* S^{\dot{A}\dot{B}} = - S^{\dot{A}\dot{B}}. \tag{2.24}$$

By (2.6) the Riemannian connection on M induces in a natural manner connections on S and $\dot{S}$. These connections are uniquely defined by the formula (the first Cartan structure equation) [16, 17, 19, 41]

$$D g^{A\dot{B}} = d g^{A\dot{B}} + \Gamma^{A}{}_{C} \wedge g^{C\dot{B}} + \Gamma^{\dot{B}}{}_{\dot{C}} \wedge g^{A\dot{C}} = 0, \tag{2.25}$$

where $\Gamma := (\Gamma^{A}{}_{B})$ and $\dot{\Gamma} := (\Gamma^{\dot{A}}{}_{\dot{B}})$ are the connection matrices on S and $\dot{S}$ respectively; they are traceless

$$\Gamma^{A}{}_{A} = 0 = \Gamma^{\dot{A}}{}_{\dot{A}}. \tag{2.26}$$

Now the curvatures on S and $\dot{S}$ are defined as follows

$$R^{A}{}_{B} := d\Gamma^{A}{}_{B} + \Gamma^{A}{}_{C} \wedge \Gamma^{C}{}_{B} =: D\Gamma^{A}{}_{B}; \quad R^{\dot{A}}{}_{\dot{B}} := d\Gamma^{\dot{A}}{}_{\dot{B}} + \Gamma^{\dot{A}}{}_{\dot{C}} \wedge \Gamma^{\dot{C}}{}_{\dot{B}} =: D\Gamma^{\dot{A}}{}_{\dot{B}}. \tag{2.27}$$

Then the decomposition of these curvatures into the irreducible components [9, 16, 17, 20, 41] is of the form

$$R_{AB} = -\tfrac{1}{2} C_{ABCD} S^{CD} + \frac{R}{24} S_{AB} + \tfrac{1}{2} C_{AB\dot{C}\dot{D}} S^{\dot{C}\dot{D}},$$

$$R_{\dot{A}\dot{B}} = -\tfrac{1}{2} C_{\dot{A}\dot{B}\dot{C}\dot{D}} S^{\dot{C}\dot{D}} + \frac{R}{24} S_{\dot{A}\dot{B}} + \tfrac{1}{2} C_{CD\dot{A}\dot{B}} S^{CD}, \tag{2.28}$$

where

$$C_{ABCD} = C_{(ABCD)} := \tfrac{1}{16}\, S_{AB}{}^{ab} C_{abcd} S_{CD}{}^{cd},$$

$$C_{\dot{A}\dot{B}\dot{C}\dot{D}} = C_{(\dot{A}\dot{B}\dot{C}\dot{D})} := \tfrac{1}{16}\, S_{\dot{A}\dot{B}}{}^{ab} C_{abcd} S_{\dot{C}\dot{D}}{}^{cd},$$

$$C_{AB\dot{C}\dot{D}} = C_{(AB)\dot{C}\dot{D}} = C_{AB(\dot{C}\dot{D})} := \tfrac{1}{4}\, g_{A\dot{C}a} g_{B\dot{D}b} C^{ab} \tag{2.29}$$

and C_{abcd} is the conformal curvature tensor field (the Weyl tensor field), C_{ab} is the traceless Ricci tensor field and R is the scalar curvature; moreover,

$$S^{AB} =: \tfrac{1}{2}\, S^{AB}{}_{ab} e^a \wedge e^b, \qquad S^{\dot{A}\dot{B}} =: \tfrac{1}{2}\, S^{\dot{A}\dot{B}}{}_{ab} e^a \wedge e^b. \tag{2.30}$$

Observe that with (2.6), (2.21), (2.23) and (2.25), using also (2.30), one finds

$$\Gamma_{AB} = -\tfrac{1}{4}\, \Gamma_{ab} S_{AB}{}^{ab}, \qquad \Gamma_{\dot{A}\dot{B}} = -\tfrac{1}{4}\, \Gamma_{ab} S_{\dot{A}\dot{B}}{}^{ab} \tag{2.31}$$

where $(\Gamma^a{}_b)$ is the Riemannian connection matrix on the complexified tangent bundle T^c. Then we have:

$$\Gamma(\overset{2}{\Lambda} T^{*c} \otimes \overset{2}{\Lambda} T^{*c}) \ni (\mathscr{C}_{ab} + *\mathscr{C}_{ab}) = \tfrac{1}{2}\, S^{AB} C_{ABCD} S^{CD}, \tag{2.32}$$

$$\Gamma(\overset{2}{\Lambda} T^{*c} \otimes \overset{2}{\Lambda} T^{*c}) \ni (\mathscr{C}_{ab} - *\mathscr{C}_{ab}) = \tfrac{1}{2}\, S^{\dot{A}\dot{B}}{}_{ab} C_{\dot{A}\dot{B}\dot{D}\dot{C}} S^{\dot{C}\dot{D}}, \tag{2.33}$$

where

$$\mathscr{C}_{ab} \pm *\mathscr{C}_{ab} := \tfrac{1}{2}\, C_{abcd} e^c \wedge e^d \pm *\tfrac{1}{2}\, C_{abcd} e^c \wedge e^d. \tag{2.34}$$

Therefore, the spinor field C_{ABCD} defines uniquely and is uniquely defined by the self-dual part of the Weyl tensor field; similarly, $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ represents the anti-self-dual part of the Weyl tensor field.

Now, using the relation (2.7) and the definitions of our spinorial objects one finds the following relations (see [17, 20, 41],

$$\overline{\Gamma^{AB}} = \Gamma_{AB}, \quad \overline{\Gamma^{\dot{A}\dot{B}}} = \Gamma_{\dot{A}\dot{B}}, \quad \overline{S^{AB}} = S_{AB}, \quad \overline{S^{\dot{A}\dot{B}}} = S_{\dot{A}\dot{B}}, \tag{2.35}$$

$$\overline{C^{ABCD}} = C_{ABCD}, \quad \overline{C^{\dot{A}\dot{B}\dot{C}\dot{D}}} = C_{\dot{A}\dot{B}\dot{C}\dot{D}}, \quad \overline{C^{AB\dot{C}\dot{D}}} = C_{AB\dot{C}\dot{D}}, \quad \bar{R} = R. \tag{2.36}$$

Then using relations $\overline{C^{ABCD}} = C_{ABCD}$, $\overline{C^{\dot{A}\dot{B}\dot{C}\dot{D}}} = C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ we can give the Petrov-Penrose classification of the Weyl tensor analogously as it has been done in complex relativity (see [16, 12, 20]),

$$\begin{array}{ccc} C_{ABCD} & & C_{\dot{A}\dot{B}\dot{C}\dot{D}} \\ [1-1-1-1] & & [\dot{1}-\dot{1}-\dot{1}-\dot{1}] \\ \downarrow & & \downarrow \\ [2-2] & \times & [\dot{2}-\dot{2}] \\ \downarrow & & \downarrow \\ [-] & & [\dot{-}] \end{array}$$

Fig. 2.1. The Penrose-Plebański diagram

Now we would like to introduce the notion of a *locally Hermitian manifold.* A locally Hermitian manifold is a generalisation of a Hermitian manifold ([18], [21] vol. II, [24]) and such a generalisation is necessary for our purposes. Let $N \subset M$ be some open submanifold of M with a metric induced by the metric ds^2 on M. Manifold N is said to be a *locally Hermitian manifold* if for each point $p \in N$ there exist an open neighbourhood $U \subset N$ of p and a complex structure J on U such that the metric ds^2 is Hermitian on U with respect to J

$$ds^2(JX, JY) = ds^2(X, Y) \tag{2.37}$$

for any vector fields X, Y on U; in other words U is a Hermitian manifold with respect to J. We prove now the following important theorem

Theorem 2.1.

An open submanifold $N \subset M$ of M is a locally Hermitian manifold iff for each point $p \in N$ there exist an open neighbourhood $V \subset N$ of p and spinor frames on V such that

$$\Gamma_{\dot{1}\dot{1}1\dot{1}} = \Gamma_{\dot{1}\dot{1}2\dot{1}} = 0 \tag{2.38}$$

or

$$\Gamma_{111\dot{1}} = \Gamma_{111\dot{2}} = 0 \tag{2.39}$$

on V, where

$$\Gamma_{\dot{1}\dot{1}A\dot{B}}g^{A\dot{B}} := \Gamma_{\dot{1}\dot{1}}, \quad \Gamma_{11A\dot{B}}g^{A\dot{B}} := \Gamma_{11}. \tag{2.40}$$

Proof:

First we prove the "if part" of the theorem. Let p be any point of N and let $V \subset N$ be an open neighbourhood of p such that there exist spinor frames on V for which

$$\Gamma_{\dot{1}\dot{1}1\dot{1}} = \Gamma_{\dot{1}\dot{1}2\dot{1}} = 0.$$

Then with the above assumption, from the first Cartan structure equation (2.25) one finds that

$$g^{1\dot{2}} \wedge g^{2\dot{2}} \wedge dg^{A\dot{2}} = 0 \tag{2.41}$$

on V. Using the Frobenius theorem [22, 23], the relations (2.7) and then the independence of the four 1-forms $\{g^{A\dot{B}}\}$

$$g^{1\dot{1}} \wedge g^{1\dot{2}} \wedge g^{2\dot{1}} \wedge g^{2\dot{2}} = 4E^1 \wedge E^2 \wedge E^3 \wedge E^4 \neq 0 \tag{2.42}$$

we find that there exists a local complex chart $\{U, \{z^\alpha\}\}$ of V ($\alpha = 1, 2$; $p \in U$) such that

$$g^{1\dot{2}} = f^1{}_\alpha dz^\alpha, \quad g^{2\dot{2}} = f^2{}_\alpha dz^\alpha$$

$$g^{2\dot{1}} = \overline{g^{1\dot{2}}} = \overline{f^1{}_\alpha dz^\alpha}, \quad g^{1\dot{1}} = -\overline{g^{2\dot{2}}} = -\overline{f^2{}_\alpha dz^\alpha}, \tag{2.43}$$

where $\{f^{\alpha}{}_{\beta}\}$ are four functions on U and for each point of U

$$\det(f^{\alpha}{}_{\beta}) \neq 0 \tag{2.44}$$

Using (2.12) one has

$$ds^2 = g_{\alpha\bar{\beta}}dz^{\alpha} \otimes dz^{\bar{\beta}} + g_{\bar{\beta}\alpha}dz^{\bar{\beta}} \otimes dz^{\alpha} \tag{2.45}$$

on U, where, $\bar{\beta} = \bar{1}, \bar{2}$; $dz^{\bar{\beta}} := \overline{dz^{\beta}}$ and

$$g_{\alpha\bar{\beta}} := \tfrac{1}{2}(f^{1}{}_{\alpha}\overline{f^{1}{}_{\beta}} + f^{2}{}_{\alpha}\overline{f^{2}{}_{\beta}}) =: g_{\bar{\beta}\alpha}. \tag{2.46}$$

From (2.46) it follows that the matrix $(g_{\alpha\bar{\beta}})$ is Hermitian

$$(g_{\alpha\bar{\beta}})^{+} = (g_{\alpha\bar{\beta}}). \tag{2.47}$$

Let us define the complex structure J on U

$$J\left(\frac{\partial}{\partial z^{\alpha}}\right) = i\frac{\partial}{\partial z^{\alpha}}, \quad J\left(\frac{\partial}{\partial z^{\bar{\alpha}}}\right) = -i\frac{\partial}{\partial z^{\bar{\alpha}}}. \tag{2.48}$$

Hence

$$ds^2(JX, JY) = ds^2(X, Y) \tag{2.49}$$

for any vector fields X, Y on U.

Therefore, the metric ds^2 is Hermitian on U with respect to the complex structure J on U. Now as $p \in N$ is an arbitrary point of N then, by the definition, N is a locally Hermitian manifold. If $\Gamma_{111\dot{1}} = \Gamma_{111\dot{2}} = 0$ on V, the proof proceeds similarly (interchange $g^{1\dot{2}} \leftrightarrow g^{2\dot{1}}$ in (2.43)).

Now we prove the "only if part". We assume that an open submanifold $N \subset M$ of M is a locally Hermitian manifold. Hence, if p is any point of N then there exist an open neighbourhood $V \subset N$ of p and a local complex chart $\{V, \{z^{\alpha}\}\}$ of N such that the metric ds^2 on V is of the form (2.45), with (2.47). From (2.47) it follows that there exist four functions $\{f^{\alpha}{}_{\beta}\}$ on V such that the metric tensor is determined by (2.46).

Therefore there exist spinor frames on V such that (2.43) or

$$\begin{gathered} g^{2\dot{1}} = f^{1}{}_{\alpha}dz^{\alpha}, \quad g^{2\dot{2}} = f^{2}{}_{\alpha}dz^{\alpha}, \\ g^{1\dot{2}} = \overline{g^{2\dot{1}}} = \overline{f^{1}{}_{\alpha}dz^{\alpha}}, \quad g^{1\dot{1}} = -\overline{g^{2\dot{2}}} = -\overline{f^{2}{}_{\alpha}dz^{\alpha}} \end{gathered} \tag{2.50}$$

hold on V. (Let us recall that M is oriented). Then using the first Cartan structure equation (2.25) one easily finds that

if (2.43) holds then $\Gamma_{\dot{1}\dot{1}1\dot{1}} = \Gamma_{\dot{1}\dot{1}2\dot{1}} = 0$.

if (2.50) holds then $\Gamma_{111\dot{1}} = \Gamma_{111\dot{2}} = 0$.

Now, as p is an arbitrary point of N then we conclude that the "only if part" of the theorem has been proved. This completes the proof. ■

Note that from (2.7) and (2.35) it follows that

$$\Gamma_{\dot{1}1\dot{1}1\dot{1}} = -\overline{\Gamma_{\dot{2}2\dot{2}2\dot{2}}}, \tag{2.51}$$

$$\Gamma_{\dot{1}1\dot{2}\dot{1}} = \overline{\Gamma_{\dot{2}2\dot{1}\dot{2}}}, \tag{2.52}$$

$$\Gamma_{111\dot{1}} = -\overline{\Gamma_{222\dot{2}}}, \tag{2.53}$$

$$\Gamma_{111\dot{2}} = \overline{\Gamma_{222\dot{1}}}. \tag{2.54}$$

Hence

$$\Gamma_{\dot{1}1\dot{1}1\dot{1}} = \Gamma_{\dot{1}1\dot{2}\dot{1}} = 0 \Leftrightarrow \Gamma_{\dot{2}2\dot{2}2\dot{2}} = \Gamma_{\dot{2}2\dot{1}\dot{2}} = 0 \tag{2.55}$$

and

$$\Gamma_{111\dot{1}} = \Gamma_{111\dot{2}} = 0 \Leftrightarrow \Gamma_{222\dot{2}} = \Gamma_{222\dot{1}} = 0. \tag{2.56}$$

Using (2.31) and (2.6) one has

$$\Gamma_{\dot{1}1\dot{1}1\dot{1}} = -\frac{1}{\sqrt{2}}\Gamma_{414}, \quad \Gamma_{\dot{2}2\dot{2}2\dot{2}} = \frac{1}{\sqrt{2}}\Gamma_{323}, \tag{2.57}$$

$$\Gamma_{\dot{1}1\dot{2}\dot{1}} = -\frac{1}{\sqrt{2}}\Gamma_{411}, \quad \Gamma_{\dot{2}2\dot{1}\dot{2}} = -\frac{1}{\sqrt{2}}\Gamma_{322}, \tag{2.58}$$

$$\Gamma_{111\dot{1}} = -\frac{1}{\sqrt{2}}\Gamma_{424}, \quad \Gamma_{222\dot{2}} = \frac{1}{\sqrt{2}}\Gamma_{313}, \tag{2.59}$$

$$\Gamma_{111\dot{2}} = -\frac{1}{\sqrt{2}}\Gamma_{422}, \quad \Gamma_{222\dot{1}} = -\frac{1}{\sqrt{2}}\Gamma_{311}. \tag{2.60}$$

Of course if N is a Hermitian manifold then it is also locally Hermitian but the converse statement does not hold genetally (e.g., S^4, see [25]). However, if N possesses an almost complex structure J and if N is also locally Hermitian with respect to J, then N is a Hermitian manifold (see [21], vol. II, p. 321).

The following theorem gives the sufficient condition for an open submanifold $N \subset M$ of M being a Hermitian manifold.

Theorem 2.2.

Let $N \subset M$ be an open submanifold of M so that there exist (global) spinor frames on N such that (2.38) or (2.39) hold on N; then N is a Hermitian manifold.

Proof:

Let $\{e_A\}$, $\{e_{\dot{A}}\}$ be the spinor frames on a submanifold $N \subset M$ such that (2.38) holds on N and let $\{e^a\}$ be the null tetrad on N defined by $\{e_A\}$, $\{e_{\dot{A}}\}$, according to (2.6). Then (2.38) and (2.51), (2.52), (2.57), (2.58) imply

$$\Gamma_{414} = \Gamma_{411} = \Gamma_{323} = \Gamma_{322} = 0 \quad \text{on } N. \tag{2.61}$$

Let us define the following tensor field on N

$$\Gamma(T^c \otimes T^{*c}) \ni J := ie_3 \otimes e^3 - ie_4 \otimes e^4 + ie_2 \otimes e^2 - ie_1 \otimes e^1, \tag{2.62}$$

where T^c, T^{*c} are complexified tangent and cotangent, respectively, bundles of N and $e_a \in \Gamma(T^c)$ for any $a = 1, 2, 3, 4$; moreover, $e^a(e_b) = \delta^a{}_b$.

One can verify that J is an almost complex structure on N and then, that (2.61) is equivalent to the vanishing of the torsion of J. Therefore, by the Newlander-Nirenberg theorem ([21] vol. II) J is a complex structure on N. Moreover, one finds that the metric ds^2 on N is Hermitian with respect to J. Hence N is a Hermitian manifold.

Suppose that (2.39) holds on $N \subset M$. Consequently

$$\Gamma_{424} = \Gamma_{422} = \Gamma_{313} = \Gamma_{311} = 0 \quad \text{on } N. \tag{2.63}$$

Let us define the tensor field $'J$ on N as follows

$$\Gamma(T^c \otimes T^{*c}) \ni {'J} := ie_3 \otimes e^3 - ie_4 \otimes e^4 + ie_1 \otimes e^1 - ie_2 \otimes e^2. \tag{2.64}$$

Then one can show analogously as in the previous case that by (2.63) $'J$ appears to be a complex structure on N and the metric ds^2 is Hermitian with respect to $'J$ i.e., N is a Hermitian manifold. ■

(Compare this theorem with the one concerning integrability of the so called "modified almost Hermitian structure", Flaherty ([26] p. 199)).

Notice that the complex structure $J('J$, resp.) in the theorem 2.1 is equal to the complex structure $J('J$, resp.) as intruduced in the theorem 2.2.

Now the problem arises: what is the connection (if any) between the local Hermiticity and the algebraic speciality of the Weyl tensor? Recall that the spinor $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ (C_{ABCD}, resp.) is called algebraically special at some point $p \in M$ if it is one of the types, $[2\dot{-}2]$ or $[\dot{-}]$ ($[2-2]$ or $[-]$, resp.) at p. Moreover, $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ (C_{ABCD}, resp.) is of type $[2\dot{-}2]$ ($[2-2]$, resp.) at p iff there exists a spinor frame $\{e_{\dot{A}}\}$ ($\{e_A\}$, resp.) at p such that

$$C_{\dot{1}\dot{1}\dot{1}\dot{1}} = C_{\dot{1}\dot{1}\dot{1}\dot{2}} = 0, \qquad C_{\dot{1}\dot{1}\dot{2}\dot{2}} \neq 0, \tag{2.65}$$

$$(C_{1111} = C_{1112} = 0, \qquad C_{1122} \neq 0, \text{resp.}) \tag{2.66}$$

Notice, that by (2.36) one has

$$C_{\dot{1}\dot{1}\dot{1}\dot{1}} = 0 \Leftrightarrow C_{\dot{2}\dot{2}\dot{2}\dot{2}} = 0; \qquad C_{\dot{1}\dot{1}\dot{1}\dot{2}} = 0 \Leftrightarrow C_{\dot{2}\dot{2}\dot{2}\dot{1}} = 0, \tag{2.67}$$

$$C_{1111} = 0 \Leftrightarrow C_{2222} = 0; \qquad C_{1112} = 0 \Leftrightarrow C_{2221} = 0. \tag{2.68}$$

The spinor $e_{\dot{1}}$ (e_1, resp.) such that (2.65) ((2.66), resp.) holds will be called the double $\dot{P}$-spinor (P-spinor, resp.) at the point $p \in M$. Recall also that the spinor $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ (C_{ABCD}, resp.) is said to be of type $[\dot{-}]$ ($[-]$, resp.) at the point $p \in M$ if $C_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0$ ($C_{ABCD} = 0$ resp.) at p.

We now have the following fundamental result (see [20])

Theorem 2.3.

Let U be an open subset of M such that

$$C_{AB\dot{C}\dot{D}} = R = 0 \tag{2.69}$$

on U (i.e., the vacuum Einstein equations are fulfilled on U). Then (*i*) If there exists a spinor frame $\{e_{\dot{A}}\}$ on U such that (2.38) holds on U then the spinor field $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ on U is algebraically special and in the case of $[2 \dot{-} 2]$ the spinor field $e_{\dot{1}}$ on U is the double $\dot{P}$-spinor field.

(*ii*) If the spinor field $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ on U is algebraically special of type $[2 \dot{-} 2]$ and $\{e_{\dot{A}}\}$ is a spinor frame on U such that the spinor field $e_{\dot{1}}$ on U is the double $\dot{P}$-spinor field then (2.38) holds on U.

If the spinor field $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ on U is algebraically special of type $[\dot{-}]$ (i.e., $C_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0$ on U) and U is simply connected, then there exists a spinor frame $\{e_{\dot{A}}\}$ on U such that (2.38) holds on U.

Proof:

Notice that the theorem 2.3. with some modification is known in General Relativity as the "Goldberg-Sachs theorem" (see [27–29]).

The proofs of both theorems are similar and hence we give here only the outline of the proof.

(*i*) Assume that (2.38) holds on U. Using then the definition of $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ one finds that this assumption yields the formula, $C_{\dot{1}\dot{1}\dot{1}\dot{1}} = 0$ on U. Using then some of the Bianchi identities and their intergrability conditions one finds that $C_{\dot{1}\dot{1}\dot{1}\dot{2}} = 0$ on U.

Hence, $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is algebraically special on U and if it is of type $[2 \dot{-} 2]$ then the spinor field $e_{\dot{1}}$ on U is the double $\dot{P}$-spinor field.

(*ii*) Suppose that (2.65) holds on U (the type $[2 \dot{-} 2]$). Then, from some of the Bianchi identities one finds immediately that (2.38) is fulfilled on U.

Assume now that $C_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0$ on U (the type $[\dot{-}]$). We also have (2.69). Hence, the curvature on the complex spinor bundle $\dot{S}$ over U vanishes identically (see (2.27)). Using also the assumption that U is simply connected we conclude that there exists a spinor frame $\{e_{\dot{A}}\}$ on U such that the connection matrix $(\Gamma^{\dot{A}}{}_{\dot{B}}) = 0$ on $\dot{S}$ (see [21], vol. I, pp. 92, 93). The proof is completed. ■

Consider now the non-vacuum case. One finds the following

Theorem 2.4.

Let U be an open subset of M such that for some spinor frame $\{e_{\dot{A}}\}$

$$DC^{AB}{}_{\dot{1}\dot{1}}(\partial_{B\dot{1}}) = 0 \tag{2.70}$$

on U, where $\partial_{A\dot{B}} \in \Gamma(T^c)$ for any A, $\dot{B}$ and

$$g^{A\dot{B}}(\partial_{C\dot{D}}) = \delta^{A}{}_{C}\delta^{\dot{B}}{}_{\dot{D}} \tag{2.71}$$

on U, (T^c is the complexified tangent bundle over U. D denotes the exterior covariant differentiation). Then

(*i*) If (2.38) holds on U then the spinor field $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ on U is algebraically special and in the case of $[2 \dot{-} 2]$ the spinor field $e_{\dot{1}}$ on U is the double $\dot{P}$-spinor field.

(*ii*) If the spinor field $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ on U is algebraically special of type $[2 \dot{-} 2]$ and $e_{\dot{1}}$ is the double $\dot{P}$-spinor field on U then (2.38) holds on U.

Proof:

One easily finds that the theorem 2.4 corresponds to the so called "Generalized Goldberg--Sachs theorem" well known in General Relativity (see [28–32]).

The proof is based on the non-vacuum Bianchi identities. With the assumption (2.70), some of these identities, precisely those which are necessary for the proof, are of the same form as the vacuum ones. Hence, using the same arguments as in the proof of the theorem 2.3 we find that our theorem holds. ■ (Notice that the relation (2.70) is fulfilled if e.g.,

$$C_{11\dot{1}\dot{1}} = C_{12\dot{1}\dot{1}} = C_{22\dot{1}\dot{1}} = 0 \quad \text{on } U). \tag{2.72}$$

By (2.36) we have also

$$C_{11\dot{1}\dot{1}} = C_{12\dot{1}\dot{1}} = C_{22\dot{1}\dot{1}} = 0 \Leftrightarrow C_{22\dot{2}\dot{2}} = C_{12\dot{2}\dot{2}} = C_{11\dot{2}\dot{2}} = 0. \tag{2.73}$$

We do not know yet, if there exists a non-vacuum version of the second part of the theorem 2.3 (*ii*).

However, using the results of the deformed twistor spaces (see [33]) we have

Conjecture 2.1.

If the spinor field $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ on an open subset U of M is of the type $[-\!\!\!\!\cdot\,]$ *(i.e., the Weyl tensor is self-dual on U) then for each point $p \in U$ there exist a neighbourhood V of p ($V \subset U$) and a spinor frame $\{e_{\dot{A}}\}$ on V such that (2.38) is fulfilled on V.* ■

Combining the theorem 2.1 with the theorems 2.3, 2.4 we have the following results

Proposition 2.1.

Let $N \subset M$ be an open submanifold of M such that the traceless Ricci tensor field and the scalar curvature vanish on N and the anti-self-dual part of the Weyl tensor field is algebraically special of the definite type on N, then N is a locally Hermitian manifold. ■

Proposition 2.2.

Let $N \subset M$ be an open submanifold of M such that the anti-self-dual part of the Weyl tensor field is of type $[2 \dot{-} 2]$ on N and, for each point $p \in N$, let there exist an open neighbourhood U of $p (U \subset N)$ and a spinor frame $\{e_{\dot{A}}\}$ on U such that $DC^{AB}{}_{\dot{1}\dot{1}}(\partial_{B\dot{1}}) = 0$ on U and $e_{\dot{1}}$ is the double $\dot{P}$-spinor field on U, then N is a locally Hermitian manifold. ■ (Notice that in the "null tetrad language" we have [29, 32]

(*a*)
$$DC^{AB}{}_{\dot{1}\dot{1}}(\partial_{B\dot{1}}) = 0 \Leftrightarrow C_{4[4;1]} = C_{1[1;4]} = 0, \tag{2.74}$$

(*b*) $e_{\dot{1}}$ is the double $\dot{P}$-spinor at some point $p \in M$ iff

$$C_{4141} = C_{4121} + C_{4134} = 0, \quad C_{4132} \neq 0 \quad \text{at } p, \tag{2.75}$$

(*c*)
$$C_{11\dot{1}\dot{1}} = C_{12\dot{1}\dot{1}} = C_{22\dot{1}\dot{1}} = 0 \Leftrightarrow C_{44} = C_{41} = C_{11} = 0. \tag{2.76}$$

If the *conjecture* 2.1 is true then consequently the following *conjecture* is also true.

Conjecture 2.2.

If $N \subset M$ is an open submanifold of M such that the Weyl tensor field is self-dual on N then N is a locally Hermitian manifold. ■

Of course, the conjecture 2.2 is true for the vacuum case. Now assume that M is a locally Hermitian manifold and let $U \subset M$ be an open subset of M such that the formulae (2.43)–(2.46) hold. (We shall omit the words "on U", "over U" ... etc.).

Now it is very useful to extend the spinor group SU(2) to SL (2, *C*). This procedure corresponds to the extending of the SO (4, ***R***) group to SO (4, *C*) (see (2.15)). Then, we can construct two complex spinor bundles: $\tilde{S}$, which corresponds to SL (2, *C*) and $\tilde{\dot{S}}$, which corresponds to $\dot{\mathrm{S}}$L (2, *C*). Of course one has the natural isomorphisms

$$\Gamma(\tilde{S}) \cong \Gamma(S), \tag{2.77}$$

$$\Gamma(\tilde{\dot{S}}) \cong \Gamma(\dot{S}). \tag{2.78}$$

i.e., we can extend the spinorial objects introduced previously, to the SL (2, *C*) ($\dot{\mathrm{S}}$L (2, *C*)) — spinorial objects. One finds that the formulae (2.7), (2.35), (2,36) are not (generally) true for these extended objects but we still have

$$ds^2 = -\tfrac{1}{2}\, \tilde{g}_{A\dot{B}} \underset{s}{\otimes} \tilde{g}^{A\dot{B}}. \tag{2.79}$$

We find that if (2.43) holds then for some extended spinor frames

$$\tilde{g}^{1\dot{2}} = \sqrt{2}\, dz^1, \qquad \tilde{g}^{2\dot{2}} = \sqrt{2}\, dz^2,$$
$$\tilde{g}^{2\dot{1}} = \sqrt{2}\, g_{1\bar{\beta}} dz^{\bar{\beta}}, \qquad \tilde{g}^{1\dot{1}} = -\sqrt{2}\, g_{2\bar{\beta}} dz^{\bar{\beta}}. \tag{2.80}$$

From (2.71) and (2.80) one has

$$\tilde{\partial}_{1\dot{2}} = \frac{1}{\sqrt{2}} \frac{\partial}{\partial z^1}, \qquad \tilde{\partial}_{2\dot{2}} = \frac{1}{\sqrt{2}} \frac{\partial}{\partial z^2},$$
$$\tilde{\partial}_{2\dot{1}} = \frac{1}{\sqrt{2}} g^{\bar{\beta}1} \frac{\partial}{\partial z^{\bar{\beta}}}, \qquad \tilde{\partial}_{1\dot{1}} = -\frac{1}{\sqrt{2}} g^{\bar{\beta}2} \frac{\partial}{\partial z^{\bar{\beta}}}, \tag{2.81}$$

where the 2×2 matrix $(g^{\bar{\beta}\alpha})$ is the reciprocal matrix to $(g_{\alpha\bar{\beta}})$

$$g_{\alpha\bar{\beta}} g^{\bar{\beta}\gamma} = \delta^{\gamma}{}_{\alpha}. \tag{2.82}$$

Using (2.80), (2.21) and (2.23) we obtain

$$\tilde{S}^{\dot{1}\dot{1}} = -2g dz^{\bar{1}} \wedge dz^{\bar{2}}, \tag{2.83}$$

$$\tilde{S}^{\dot{1}\dot{2}} = g_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}}, \tag{2.84}$$

$$\tilde{S}^{\dot{2}\dot{2}} = 2dz^1 \wedge dz^2, \tag{2.85}$$

$$\tilde{S}^{11} = 2g_{2\bar{\beta}} dz^1 \wedge dz^{\bar{\beta}}, \tag{2.86}$$

$$\tilde{S}^{12} = g_{2\bar{\beta}} dz^2 \wedge dz^{\bar{\beta}} - g_{1\bar{\beta}} dz^1 \wedge dz^{\bar{\beta}}, \tag{2.87}$$

$$\tilde{S}^{22} = -2g_{1\bar{\beta}} dz^2 \wedge dz^{\bar{\beta}}, \tag{2.88}$$

where

$$g := \det(g_{\alpha\bar{\beta}}). \tag{2.89}$$

Observe now that with (2.43) and (2.46) one has

$$S^{\dot{1}\dot{2}} = g_{\alpha\bar{\beta}}dz^{\alpha} \wedge dz^{\bar{\beta}} = \tilde{S}^{\dot{1}\dot{2}}. \tag{2.90}$$

If J denotes the natural complex structure (2.48) then

$$J_{\alpha\bar{\beta}} = -J_{\bar{\beta}\alpha} = ig_{\alpha\bar{\beta}}, \quad J_{\alpha\beta} = J_{\bar{\alpha}\bar{\beta}} = 0 \tag{2.91}$$

and the fundamental 2-form (the Kähler form) defined according to the formula (see [21] vol. II, p. 147)

$$\Phi(X, Y) := ds^2(X, JY) \tag{2.92}$$

for any vector fields X, Y, can be found to be

$$\Phi = -ig_{\alpha\bar{\beta}}dz^{\alpha} \wedge dz^{\bar{\beta}} \tag{2.93}$$

(recall that we assume the convention according to which

$$dz^{\alpha} \wedge dz^{\bar{\beta}} = dz^{\alpha} \otimes dz^{\bar{\beta}} - dz^{\bar{\beta}} \otimes dz^{\alpha}).$$

Comparing (2.93) with (2.90) one has

$$-i\tilde{S}^{\dot{1}\dot{2}} = \Phi = -iS^{\dot{1}\dot{2}}. \tag{2.94}$$

Consequently, $U \subset M$ considered as an open submanifold of M with the induced metric and with the complex structure J defined as above is a Kählerian manifold iff

$$dS^{\dot{1}\dot{2}} = 0, \quad \text{or equivalently,} \quad d\tilde{S}^{\dot{1}\dot{2}} = 0. \tag{2.95}$$

(In the case of complex space-times this fact was observed by Flaherty [26]).

But this is the point to introduce the notion of a locally Kählerian manifold.

3. *Locally Kählerian manifolds*

Let $N \subset M$ be, as previously, an open submanifold of M with the induced metric. Then, we shall call N a locally Kählerian manifold if for each point $p \in N$ there exist an open neighbourhood $U \subset N$ of p and a complex structure J on U such that U is a Kählerian manifold, i.e.,

(*i*) $$ds^2(JX, JY) = ds^2(X, Y) \text{ for any vector fields } X, Y \text{ on } U \tag{3.1}$$

and the fundamental 2-form Φ defined by (2.92) is closed

(*ii*) $$d\Phi = 0. \tag{3.2}$$

Now, in the theory of the locally Kählerian manifolds we have a theorem similar to theorem 2 1.

Theorem 3.1.
An open submanifold $N \subset M$ of M is a locally Kählerian manifold iff for each point $p \in N$ there exist an open neighbourhood $V \subset N$ of p and spinor frames on V such that

$$\Gamma_{\dot{1}\dot{1}} = 0 \quad \text{or} \quad \Gamma_{11} = 0 \quad \text{on} \quad V. \tag{3.3}$$

Proof:
Assume that $\Gamma_{\dot{1}\dot{1}} = 0$ on V.

Hence, we find that there exists a local complex chart $\{U, \{z^{\alpha}\}\}$ of V such that the relations (2.43) are fulfilled on U (see the proof of the theorem 2.1). Then one has (see (2.94))

$$\Phi = -iS^{\dot{1}\dot{2}} \quad \text{on } U \tag{3.4}$$

where Φ is defined with respect to the complex structure J on U given by (2.48). From (3.4) it follows that U is a Kählerian manifold iff

$$dS^{\dot{1}\dot{2}} = 0. \tag{3.5}$$

But the first Cartan structure equation (2.25) implies the relation

$$DS^{\dot{A}\dot{B}} = 0. \tag{3.6}$$

Consequently

$$DS^{\dot{1}\dot{2}} = dS^{\dot{1}\dot{2}} + \Gamma_{\dot{1}\dot{1}} \wedge S^{\dot{1}\dot{1}} - \Gamma_{\dot{2}\dot{2}} \wedge S^{\dot{2}\dot{2}} = 0. \tag{3.7}$$

Now, with $\Gamma_{\dot{1}\dot{1}} = 0$ we also have $\Gamma_{\dot{2}\dot{2}} = 0$ (recall that $\Gamma_{\dot{1}\dot{1}} = \overline{\Gamma_{\dot{2}\dot{2}}}$, see (2.35)). Therefore, (3.7) yields (3.5) and one concludes that U is a Kählerian manifold. But p is an arbitrary point of N, hence, by definition, N is a locally Kählerian manifold.

If $\Gamma_{11} = 0$ on V then the proof is similar.

Therefore the proof of the "if part" of the theorem is completed.

Suppose now that N is a locally Kählerian manifold. This implies, of course, that N is a locally Hermitian manifold. Hence, for each point $p \in N$ there exist an open neighbourhood $V \subset N$ of p, complex coordinates $\{z^{\alpha}\}$ on V and spinor frames on V such that (2.43) or (2.50) hold on V. With (2.43) one has

$$S^{\dot{1}\dot{1}} = \bar{f} dz^{\bar{1}} \wedge dz^{\bar{2}} = \overline{S^{\dot{2}\dot{2}}}, \qquad S^{\dot{1}\dot{2}} = g_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}}; \tag{3.8}$$

with (2.50), the analogous relations for S^{AB} are fulfilled. Now, as N is a locally Kählerian manifold then $dS^{\dot{1}\dot{2}} = 0$ or $dS^{12} = 0$.

From (3.7) and (3.8) we have

$$dS^{\dot{1}\dot{2}} = 0 \Leftrightarrow \Gamma_{\dot{1}\dot{1}} = \Gamma_{\dot{2}\dot{2}} = 0 \tag{3.9}$$

analogously

$$dS^{12} = 0 \Leftrightarrow \Gamma_{11} = \Gamma_{22} = 0. \tag{3.10}$$

Therefore, the proof is completed. (Compare this with the "modified Kählerian space-times" of Flaherty [26], p. 202). ■

For future purposes it is important to remark that with (2.83), (2.85) and (2.90) one has

$$dS^{\dot{1}\dot{2}} = 0 \Leftrightarrow d\tilde{S}^{\dot{1}\dot{2}} = 0 \Leftrightarrow \tilde{\Gamma}_{\dot{1}\dot{1}} = \tilde{\Gamma}_{\dot{2}\dot{2}} = 0. \tag{3.11}$$

Analogously as it has been done in the previous section we can find the sufficient condition under which an open submanifold $N \subset M$ of M is a Kählerian manifold.

Theorem 3.2.

Let $N \subset M$ be an open submanifold of M such that there exist the (global) spinor frames on N such that

$$\Gamma_{\dot{1}\dot{1}} = 0 \quad \text{or} \quad \Gamma_{11} = 0 \tag{3.12}$$

on N, then N is a Kählerian manifold.

Proof:

First, notice that by (2.31) we have

$$\begin{aligned} \Gamma_{\dot{1}\dot{1}} = 0 \Leftrightarrow \Gamma_{41} = 0, \qquad \Gamma_{11} = 0 \Leftrightarrow \Gamma_{42} = 0 \\ \Gamma_{\dot{2}\dot{2}} = 0 \Leftrightarrow \Gamma_{32} = 0, \qquad \Gamma_{22} = 0 \Leftrightarrow \Gamma_{31} = 0. \end{aligned} \tag{3.13}$$

Then, similarly to the proof of the theorem 2.2 one finds the complex structure on N (2.62) (or (2.64)). It is a straightforward matteı to show that if (3.12) holds on N then the respective fundamental 2-form is closed on N and consequently N is a Kählerian manifold. ■

Now if $N \subset M$ is a locally Kählerian manifold then (locally) for some complex coordinates $\{z^{\alpha}\}$

$$d(g_{\alpha\bar{\beta}}dz^{\alpha} \wedge dz^{\bar{\beta}}) = 0. \tag{3.14}$$

But it can be shown that by the Dolbeault-Grothendieck Lemma and Poincare's Lemma the formula (3.14) is equivalent to the statement that locally

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 K}{\partial z^{\alpha} \partial z^{\bar{\beta}}} \tag{3.15}$$

where

$$K = K(z^{\alpha}, z^{\bar{\alpha}}) \tag{3.16}$$

is a real-valued function (for proof see, e.g., [26]).

We would like to find the (local) expressions for the connection forms, the Weyl tensor field, the traceless Ricci tensor field and the curvature scalar on a locally Kählerian manifold. We assume that (2.80) and (3.14) hold. (This assumption corresponds to $\Gamma_{\dot{1}\dot{1}} = 0$; by interchanging $\tilde{g}^{1\dot{2}} \leftrightarrow \tilde{g}^{2\dot{1}}$ we obtain the case of $\Gamma_{11} = 0$).

Then from the first Cartan structure equation one finds (compare with (3.11))

$$\tilde{\Gamma}_{\dot{1}\dot{1}} = 0, \qquad \tilde{\Gamma}_{\dot{1}\dot{2}} = \tfrac{1}{2}(\ln g)_{,\alpha} dz^{\alpha}, \qquad \tilde{\Gamma}_{\dot{2}\dot{2}} = 0, \tag{3.17}$$

$$\tilde{\Gamma}_{11} = g^{\bar{\beta}2} g_{1\bar{\beta},\alpha} dz^{\alpha}, \quad \tilde{\Gamma}_{12} = \tfrac{1}{2}(g^{\bar{\beta}2} g_{2\bar{\beta},\alpha} - g^{\bar{\beta}1} g_{1\bar{\beta},\alpha}) dz^{\alpha},$$

$$\tilde{\Gamma}_{22} = -g^{\bar{\beta}1} g_{2\bar{\beta},\alpha} dz^{\alpha} \tag{3.18}$$

where the parenthesis "," denotes the partial derivative, e.g., $g_{1\bar{\beta},\alpha} := \dfrac{\partial g_{1\bar{\beta}}}{\partial z^{\alpha}}$. Applying (2.27), (2.28), (2.83) — (2.88) and (3.17), (3.18), we find

$$\tilde{C}_{\dot{1}\dot{1}\dot{1}\dot{1}} = \tilde{C}_{\dot{1}\dot{1}\dot{1}\dot{2}} = \tilde{C}_{\dot{1}\dot{2}\dot{2}\dot{2}} = \tilde{C}_{\dot{2}\dot{2}\dot{2}\dot{2}} = 0, \tag{3.19}$$

$$\dot{\tilde{C}}^{(3)} := 2\tilde{C}_{\dot{1}\dot{1}\dot{2}\dot{2}} = \frac{R}{6} = \tfrac{1}{3} g^{\bar{\beta}\alpha}(\ln g)_{,\alpha\bar{\beta}}, \tag{3.20}$$

$$\tilde{C}_{11\dot{1}\dot{1}} = \tilde{C}_{12\dot{1}\dot{1}} = \tilde{C}_{22\dot{1}\dot{1}} = \tilde{C}_{11\dot{2}\dot{2}} = \tilde{C}_{12\dot{2}\dot{2}} = \tilde{C}_{22\dot{2}\dot{2}} = 0, \tag{3.21}$$

$$\tilde{C}_{11\dot{1}\dot{2}} = -\tfrac{1}{2} g^{\bar{\beta}2}(\ln g)_{,1\bar{\beta}}, \tag{3.22}$$

$$\tilde{C}_{12\dot{1}\dot{2}} = \tfrac{1}{4}[g^{\bar{\beta}1}(\ln g)_{,1\bar{\beta}} - g^{\bar{\beta}2}(\ln g)_{,2\bar{\beta}}], \tag{3.23}$$

$$C_{22\dot{1}\dot{2}} = \tfrac{1}{2} g^{\bar{\beta}1}(\ln g)_{,2\bar{\beta}}, \tag{3.24}$$

$$\tfrac{1}{2}\tilde{C}^{(5)} := \tilde{C}_{1111} = g^{\bar{\beta}2}(g^{\bar{\gamma}2} g_{1\bar{\gamma},1})_{,\bar{\beta}}, \tag{3.25}$$

$$\tfrac{1}{2}\tilde{C}^{(4)} := \tilde{C}_{1112} = g^{\bar{\beta}2}(g^{\bar{\gamma}2} g_{2\bar{\gamma},1} - g^{\bar{\gamma}1} g_{1\bar{\gamma},1})_{,\bar{\beta}}, \tag{3.26}$$

$$\tfrac{1}{2}\tilde{C}^{(3)} := \tilde{C}_{1122} = \tfrac{1}{6} g^{\bar{\beta}\alpha}(\ln g)_{,\alpha\bar{\beta}} - g^{\bar{\beta}2}(g^{\bar{\gamma}1} g_{1\bar{\gamma},2})_{,\bar{\beta}}, \tag{3.27}$$

$$\tfrac{1}{2}\tilde{C}^{(2)} := \tilde{C}_{1222} = g^{\bar{\beta}1}(g^{\bar{\gamma}1} g_{1\bar{\gamma},2} - g^{\bar{\gamma}2} g_{2\bar{\gamma},2})_{,\bar{\beta}}, \tag{3.28}$$

$$\tfrac{1}{2}\tilde{C}^{(1)} := \tilde{C}_{2222} = g^{\bar{\beta}1}(g^{\bar{\gamma}1} g_{2\bar{\gamma},2})_{,\bar{\beta}}. \tag{3.29}$$

Now we are in a good position to study the locally Kähler gravitational instantons.

4. *Locally Kähler-Einstein gravitational instantons*

A gravitational instanton is said to be an Einstein gravitational instanton if

$$C_{AB\dot{C}\dot{D}} = 0. \tag{4.1}$$

From (4.1) and from the Bianchi identities it follows that

$$R = -4\Lambda = \text{const.} \tag{4.2}$$

An Einstein gravitational instanton which is, at the same time, a locally Kähler gravitational instanton will be called a *locally Kähler-Einstein gravitational instanton.*

From (4.1), (4.2), (3.20)–(3.24) and (3.15), (3.16) one finds that the metric of any locally Kähleı-Einstein gravitational instanton is defined locally by the relation

$$g_{\alpha\bar{\beta}} = K_{,\alpha\bar{\beta}} \tag{4.3}$$

where the real-valued function $K = K(z^\alpha, z^{\bar\alpha})$ is a solution of the following differential equation

$$K_{,1\bar1}K_{,2\bar2}-K_{,1\bar2}K_{,2\bar1} = |H|^2 e^{-\Lambda K}, \tag{4.4}$$

$H = H(z^\alpha)$ is a holomorphic function.

Performing the transformation of the local coordinates (see [26], p. 344)

$$z^1 \mapsto F(z^1, z^2), \qquad z^2 \mapsto z^2 \tag{4.5}$$

where $F(z^1, z^2)$ is any holomorphic function such that

$$F_{,1} = H \tag{4.6}$$

we find that in the new coordinates the equation (4.4) is of the form (see [8])

$$K_{,1\bar1}K_{,2\bar2}-K_{,1\bar2}K_{,2\bar1} = e^{-\Lambda K}. \tag{4.7}$$

This is our fundamental result. (For complex space-times this result has been given in [34]). Using then (3.20) and (4.2) one has

$$\tilde{\dot{C}}^{(3)} = -\tfrac{2}{3}\Lambda = \text{const.} \tag{4.8}$$

Then from (2.27), (2.28), (4.1), (4.2) and (3.19) we find that

$$\tilde{\dot{C}}^{(3)} := 2\tilde{C}_{\dot1\dot1\dot2\dot2} = \frac{R}{6} = -\tfrac{2}{3}\Lambda = \text{const.} \tag{4.9}$$

for any extended spinor frame such that $\tilde{\Gamma}_{\dot1\dot1} = 0$ or $\tilde{\Gamma}_{\dot2\dot2} = 0$. Therefore we have

Proposition 4.1.

(*i*) For any locally Kähler-Einstein gravitational instanton

$$\left(\tilde{\dot{C}}^{(3)} = \frac{R}{6} = \text{const.}\right) \quad \text{or} \quad \left(\tilde{C}^{(3)} = \frac{R}{6} = \text{const.}\right)$$

for an arbitrary extended spinor frame such that

$$(\tilde{\Gamma}_{\dot1\dot1} = 0 \quad \text{or} \quad \tilde{\Gamma}_{\dot2\dot2} = 0) \quad \text{or} \quad (\tilde{\Gamma}_{11} = 0 \quad \text{or} \quad \tilde{\Gamma}_{22} = 0, \text{resp.})$$

(*ii*) If a locally Kähler-Einstein gravitational instanton M is such that $R \neq 0$, then for each point $p \in M$ the Weyl tensor is non-zero.

(*iii*) A locally Kähler-Einstein gravitational instanton M is a vacuum one iff for each point $p \in M$ there exists a neighbourhood U of p such that the Weyl tensor field on U is self-dual or anti-self-dual. ∎

Now, one can easily show that every vacuum gravitational instanton such that for each point p there exists a neighbourhood U of p such that the Weyl tensor field on U is self-dual or anti-self-dual, is a locally Kähler-Einstein gravitational instanton (see, proof of the theorem 2.3 (*ii*)); consequently it is defined locally by the solution of the following differential equation (Eq. (4.7) with $\Lambda = 0$)

$$K_{,1\bar{1}}K_{,2\bar{2}} - K_{,1\bar{2}}K_{,2\bar{1}} = 1. \tag{4.10}$$

This is the equation given by S. Hacyan [12].

We can find the sufficient condition under which a locally Hermit-Einstein gravitational instanton (i.e., an Einstein gravitational instanton which is, at the same time, a locally Hermit gravitational instanton) is a locally Kähler-Einstein gravitational instanton.

Proposition 4.2.

Let M be a locally Hermit-Einstein gravitational instanton such that for each point p of M there exists a neighbourhood U of p such that

$$\dot{C}^{(3)} = \text{const} \neq 0 \qquad \text{or} \qquad C^{(3)} = \text{const} \neq 0$$

on U for some spinor frame such that

$$\Gamma_{\dot{1}\dot{1}1\dot{1}} = \Gamma_{\dot{1}\dot{1}2\dot{1}} = 0 \qquad \text{or} \qquad \Gamma_{111\dot{1}} = \Gamma_{111\dot{2}} = 0 \qquad \text{respectively,}$$

then M is a locally Kähler-Einstein gravitational instanton and

$$\dot{C}^{(3)} = \frac{R}{6} \neq 0 \qquad \text{or} \qquad C^{(3)} = \frac{R}{6} \neq 0 \qquad \text{respectively.}$$

Proof:

Let p be any point of M and let U be some neighbourhood of p such that the formula (2.38) holds for some spinor frame on U (M is a locally Hermitian manifold!) and moreover

$$\dot{C}^{(3)} = \text{const} \neq 0 \qquad \text{on } U. \tag{4.11}$$

Now as $C_{AB\dot{C}\dot{D}} = 0$, then with (2.38), (4.11), using the theorem 2.4 (*i*) one finds that

$$C_{\dot{1}\dot{1}\dot{1}\dot{1}} = C_{\dot{1}\dot{1}\dot{1}\dot{2}} = 0 \qquad \text{on } U \tag{4.12}$$

The Bianchi identities

$$DC^{\cdot\dot{D}}{}_{\dot{A}\dot{B}\dot{C}}(\partial_{A\dot{D}}) = 0 \tag{4.13}$$

(for the definitions of D, $\partial_{A\dot{D}}$, see the theorem 2.4) for $\dot{A} = \dot{2}$, $\dot{B} = \dot{1}$, $\dot{C} = \dot{1}$, with (2.38), (4.11), (4.12) imply

$$\Gamma_{\dot{1}\dot{1}1\dot{2}} = \Gamma_{\dot{1}\dot{1}2\dot{2}} = 0 \qquad \text{on } U. \tag{4.14}$$

Eqs. (2.38) and (4.14) can be written compactly in the form, $\Gamma_{\dot{1}\dot{1}} = 0$ on U. Hence, M is a locally Kähler-Einstein gravitational instanton and, by proposition 4.1 (*i*), $\dot{C}^{(3)} = \frac{R}{6} \neq 0$.

If $\Gamma_{111\dot{1}} = \Gamma_{111\dot{2}} = 0$, the proof is analogous. ■

Maybe, the simplest, locally Kähler-Einstein gravitational instanton with $\Lambda \neq 0$ is $P_2(C)$ [1].

5. *Locally Kähler gravitational instantons with a Maxwell field*

Consider a locally Kähler gravitational instanton M for the general case. Assume that U is an open, connected subset of M such that (2.80) and (3.14) hold on U. Hence, by (3.21), one finds that for an arbitrary point $p \in U$ the only non-zero components of $C_{AB\dot{C}\dot{D}}$ are, maybe, $C_{11\dot{1}\dot{2}}, C_{12\dot{1}\dot{2}}, C_{22\dot{1}\dot{2}}$. Now, the eigenvalue problem

$$\tilde{C}^{A}{}_{B}{}^{\dot{C}}{}_{\dot{D}}\tilde{g}^{B\dot{D}} = C'\tilde{g}^{A\dot{C}} \tag{5.1}$$

gives

$$C' = \pm 2\sqrt{(\tilde{C}_{12\dot{1}\dot{2}})^2 - \tilde{C}_{11\dot{1}\dot{2}}\tilde{C}_{22\dot{1}\dot{2}}} \tag{5.2}$$

(in the null tetrad language

$$C' = \pm\sqrt{(\tilde{C}_{12})^2 + \tilde{C}_{31}\tilde{C}_{42}}). \tag{5.3}$$

Consequently, for any point $p \in U$ the traceless Ricci tensor is one of the types, ((1,1), (1, 1)) or ((1, 1, 1, 1)) (see [35]). Hence

Proposition 5.1.
For each point of a locally Kähler gravitational instanton the traceless Ricci tensor is one of the types ((1, 1), (1, 1)) or ((1, 1, 1, 1)). ∎

Then, with (3.21), it follows that we can write

$$\tilde{C}_{AB\dot{C}\dot{D}} = -8\tilde{f}_{AB}\tilde{f}_{\dot{C}\dot{D}}, \tag{5.4}$$

where $\tilde{f}_{AB}, \tilde{f}_{\dot{C}\dot{D}}$ are symmetric, extended spinor fields

$$\tilde{f}_{AB} = \tilde{f}_{(AB)}, \quad \tilde{f}_{\dot{A}\dot{B}} = \tilde{f}_{(\dot{A}\dot{B})} \tag{5.5}$$

such that for each point $p \in U$

$$\tilde{f}_{\dot{1}\dot{1}} = \tilde{f}_{\dot{2}\dot{2}} = 0, \tag{5.6}$$

$$\tilde{f}_{\dot{1}\dot{2}} \neq 0. \tag{5.7}$$

Let us define the 2-form (compare [29, 35])

$$F := \tilde{f}_{AB}\tilde{S}^{AB} + \tilde{f}_{\dot{A}\dot{B}}\tilde{S}^{\dot{A}\dot{B}}. \tag{5.8}$$

One has

$$\tfrac{1}{2}(F - *F) = \tilde{f}_{\dot{A}\dot{B}}\tilde{S}^{\dot{A}\dot{B}}, \quad \tfrac{1}{2}(F + *F) = \tilde{f}_{AB}\tilde{S}^{AB}. \tag{5.9}$$

With (2.84), (5.6) and (3.14)

$$d(\tilde{f}_{\dot{A}\dot{B}}\tilde{S}^{\dot{A}\dot{B}}) = 0 \Leftrightarrow \tilde{f}_{\dot{1}\dot{2}} = \text{const}. \tag{5.10}$$

Hence, suppose that $\tilde{f}_{\dot{1}\dot{2}} = \text{const} \neq 0$. The question is, if $d(\tilde{f}_{AB}\tilde{S}^{AB}) = 0$.

From (2.27), (2.28), (3.17), (3.19), (3.20) and (5.4) it follows that

$$d\tilde{\Gamma}_{\dot{1}\dot{2}} = -\frac{R}{8}\tilde{S}^{\dot{1}\dot{2}} - 4\tilde{f}_{\dot{1}\dot{2}}\tilde{f}_{AB}\tilde{S}^{AB}. \tag{5.11}$$

Performing the exterior differentiation in (5.11) and using (2.84), (3.14) as well as the assumption $\tilde{f}_{\dot{1}\dot{2}} = \text{const} \neq 0$, we find that

$$d(\tilde{f}_{AB}\tilde{S}^{AB}) = 0 \Leftrightarrow R = \text{const}. \tag{5.12}$$

Therefore, with (5.6), (5.7), one can assert that $dF = 0 = d * F$ i.e., F is a (complex) source-free Maxwell field iff

$$\tilde{f}_{\dot{1}\dot{2}} = \text{const.} \neq 0 \quad \text{and} \quad R = \text{const}. \tag{5.13}$$

Now the problem arises if there exists $\tilde{f}_{\dot{1}\dot{2}} = \text{const} \neq 0$ such that (5.4) holds and F is a real, source-free Maxwell field. We have

$$\begin{aligned} F &= \tilde{f}_{AB}\tilde{S}^{AB} + \tilde{f}_{\dot{A}\dot{B}}\tilde{S}^{\dot{A}\dot{B}} = \tilde{f}_{AB}\tilde{S}^{AB} + \tilde{f}_{\dot{1}\dot{2}}\tilde{S}^{\dot{1}\dot{2}} \\ &= \frac{1}{\tilde{f}_{\dot{1}\dot{2}}}\left[-\tfrac{1}{8}\tilde{C}_{AB\dot{1}\dot{2}}\tilde{S}^{AB} + (\tilde{f}_{\dot{1}\dot{2}})^2\tilde{S}^{\dot{1}\dot{2}}\right]. \end{aligned} \tag{5.14}$$

Using (5.11) and (3.17) one finds

$$F = \frac{1}{8\tilde{f}_{\dot{1}\dot{2}}}\left\{(\ln g)_{,\alpha\bar{\beta}}dz^{\alpha} \wedge dz^{\bar{\beta}} + \left[8(\tilde{f}_{\dot{1}\dot{2}})^2 - \frac{R}{4}\right]\tilde{S}^{\dot{1}\dot{2}}\right\}. \tag{5.15}$$

Finally, using (2.84) we obtain

$$F = \frac{1}{8\tilde{f}_{\dot{1}\dot{2}}}\left\{(\ln g)_{,\alpha\bar{\beta}} + \left[8(\tilde{f}_{\dot{1}\dot{2}})^2 - \frac{R}{4}\right]g_{\alpha\bar{\beta}}\right\}dz^{\alpha} \wedge dz^{\bar{\beta}}. \tag{5.16}$$

Then it follows that $\bar{F} = F$ iff

$$\left(\frac{1}{8\tilde{f}_{\dot{1}\dot{2}}} + \overline{\frac{1}{8\tilde{f}_{\dot{1}\dot{2}}}}\right)\left[(\ln g)_{,\alpha\bar{\beta}} - \frac{R}{4}g_{\alpha\bar{\beta}}\right]dz^{\alpha} \wedge dz^{\bar{\beta}} + (\tilde{f}_{\dot{1}\dot{2}} + \overline{\tilde{f}_{\dot{1}\dot{2}}})g_{\alpha\bar{\beta}}dz^{\alpha} \wedge dz^{\bar{\beta}} = 0. \tag{5.17}$$

Applying Hodge's star to (5.17) one finds (compare (5.11))

$$\begin{aligned} &\left(\frac{1}{8\tilde{f}_{\dot{1}\dot{2}}} + \overline{\frac{1}{8\tilde{f}_{\dot{1}\dot{2}}}}\right)\left[(\ln g)_{,\alpha\bar{\beta}} - \frac{R}{4}g_{\alpha\bar{\beta}}\right]dz^{\alpha} \wedge dz^{\bar{\beta}} \\ &\quad -(\tilde{f}_{\dot{1}\dot{2}} + \overline{\tilde{f}_{\dot{1}\dot{2}}})g_{\alpha\bar{\beta}}dz^{\alpha} \wedge dz^{\bar{\beta}} = 0. \end{aligned} \tag{5.18}$$

Comparing (5.17) with (5.18) we obtain the result

$$\bar{F} = F \quad \text{iff} \quad \operatorname{Re}\tilde{f}_{\dot{1}\dot{2}} = 0. \tag{5.19}$$

Therefore, substituting

$$\tilde{f}_{1\dot{2}} = ir \tag{5.20}$$

where, r, is an arbitrary real number ($\neq 0$) and

$$R = -4\Lambda = \text{const.}, \tag{5.21}$$

we find the real, source-free Maxwell field in the form

$$F = \frac{1}{8ir}[(\ln g)_{,\alpha\bar{\beta}} + (\Lambda - 8r^2)g_{\alpha\bar{\beta}}]dz^\alpha \wedge dz^{\bar{\beta}}. \tag{5.22}$$

With the use of components

$$F = \tfrac{1}{2} f_{ab} e^a \wedge e^b, \quad f_{ab} = f_{[ab]} \tag{5.23}$$

one easily finds that the traceless Ricci tensor field on U is of the form (see (2.29) and also [29, 35])

$$C_{ab} = -2(f^c{}_a f_{cb} - \tfrac{1}{4} g_{ab} f^{cd} f_{cd}) \tag{5.24}$$

and the Einstein equations with Λ

$$R_{ab} - \tfrac{1}{2} R g_{ab} - \Lambda g_{ab} = -8\pi T_{ab}, \tag{5.25}$$

where

$$T_{ab} := \frac{1}{4\pi}(f^c{}_a f_{cb} - \tfrac{1}{4} g_{ab} f^{cd} f_{cd}) \tag{5.26}$$

is the "energy-momentum" tensor field of a Maxwell field. With the use of our complex coordinates $\{z^\alpha\}$

$$f_{\alpha\bar{\beta}} = \frac{1}{8ir}[(\ln g)_{,\alpha\bar{\beta}} + (\Lambda - 8r^2)g_{\alpha\bar{\beta}}] = -f_{\bar{\beta}\alpha}, \tag{5.27}$$

$$f_{\alpha\beta} = f_{\bar{\alpha}\bar{\beta}} = 0, \tag{5.28}$$

$$C_{\alpha\bar{\beta}} = -2(f^\gamma{}_\alpha f_{\gamma\bar{\beta}} - \tfrac{1}{4} g_{\alpha\bar{\beta}} 2 f^{\gamma\bar{\delta}} f_{\gamma\bar{\delta}}) = C_{\bar{\beta}\alpha}, \tag{5.29}$$

$$C_{\alpha\beta} = C_{\bar{\alpha}\bar{\beta}} = 0, \tag{5.30}$$

$$T_{\alpha\bar{\beta}} = \frac{1}{4\pi}(f^\gamma{}_\alpha f_{\gamma\bar{\beta}} - \tfrac{1}{4} g_{\alpha\bar{\beta}} 2 f^{\gamma\bar{\delta}} f_{\gamma\bar{\delta}}) = T_{\bar{\beta}\alpha}, \tag{5.31}$$

$$T_{\alpha\beta} = T_{\bar{\alpha}\bar{\beta}} = 0. \tag{5.32}$$

Concluding, any locally Kähler gravitational instanton admits locally the real, source--free Maxwell field F defined by (5.22) iff $R = const. = -4\Lambda$.

Now, one can easily show that if for each point of U, $C_{AB\dot{C}\dot{D}}$ is non-zero, (U is a connected set!), then the general form of our "induced" real, source-free Maxwell field is defined by the formula (5.22). This is also true in the more general case, i.e., if there does not exist any open subset V of U such that $C_{AB\dot{C}\dot{D}}$ vanishes identically on V and the set $U-P$, where $P := \{p \in U: C_{AB\dot{C}\dot{D}}(p) = 0\}$, possesses a finite number of connected components, then the general form of the "induced" real source-free Maxwell field is (5.22).

(The fact that our real, source-free Maxwell field (5.22) depends on an arbitrary real constant, r, is not accidental. Indeed, one has a general result which is analogous to the "already unified field theory" of Rainich-Misner-Wheeler in General Relativity [36–38]. It will be published elsewhere). Now, if M is a Kähler gravitational instanton with $R = -4\Lambda = const.$ on M, then the real, source-free Maxwell field (5.22) is defined globally. Moreover, we can find that if

$$(i) \qquad \text{the set } P := \{p \in M : C_{AB\dot{C}\dot{D}}(p) = 0\} \tag{5.33}$$

does not contain any open subset of M, (*ii*) the set $M-P$ possesses a finite number of connected components, then every global, real, source-free Maxwell field is of the form (5.22) (recall that M is connected).

In the case of a locally Kähler gravitational instanton one can speak only about a local, real, source-free Maxwell field. The (local) Einstein-Maxwell equations are reduced to one differential equation, see (3.20),

$$(\ln g)_{,\alpha\bar{\beta}}g^{\bar{\beta}\alpha} = -2\Lambda, \tag{5.34}$$

where, according to (3.15), $g_{\alpha\bar{\beta}} = K_{,\alpha\bar{\beta}}$.
Now, if M is a Kähler gravitational instanton and $R = 0$ on M then M is self-dual in the natural orientation (see [21], vol. II, p. 121).

Hence, every Kähler gravitational instanton with $R = 0$ is self-dual in the natural orientation, admits the (global) real, source-free Maxwell field (5.22) and is defined (locally) by the differential equation

$$(\ln g)_{,\alpha\bar{\beta}}g^{\bar{\beta}\alpha} = 0 \tag{5.35}$$

with $g_{\alpha\bar{\beta}} = K_{,\alpha\bar{\beta}}$ (compare with [9], p. 428).

For complex space-times similar results have been given by Boyer, Finley, and Plebański [34].

6. *Conclusions*

We list some problems which are closely related to our considerations:

1) Study the conditions under which a locally Hermitian (Kählerian) manifold proves to be Hermitian (Kählerian, respectively).

2) We would like to reduce the vacuum Euclidean Einstein equations for a locally Hermit gravitational instanton to a single differential equation, analogously as in the case of a locally Kähler gravitational instanton.

3) Does there exist any analog of Plebański's "second key function" [15, 16] for the locally Hermit gravitational instantons?

4) Find interesting unknown solutions of the equations, (4.7) or (5.34).

5) Does there exist any general method of generating physical space-times from gravitational instantons?

REFERENCES

[1] T. Eguchi, P. G. O. Freund, *Phys. Rev. Lett.* **37**, 1251 (1976).
[2] S. W. Hawking, *Phys. Lett.* **A60**, 81 (1977).
[3] A. A. Belavin, A. M. Polyakov, A. S. Schwarz, Yu. S. Tyupkin, *Phys. Lett.* **B59**, 85 (1975).
[4] T. Eguchi, A. J. Hanson, *Phys. Lett.* **B74**, 249 (1978).
[5] V. A. Belinskii, G. W. Gibbons, D. N. Page, C. N. Pope, *Phys. Lett.* **B76**, 433 (1978).
[6] G. W. Gibbons, S. W. Hawking, *Phys. Lett.* **B78**, 430 (1978).
[7] G. W. Gibbons, S. W. Hawking, *Commun. Math. Phys.* **66**, 291 (1979).
[8] G. W. Gibbons, C. N. Pope, *Commun. Math. Phys.* **66**, 267 (1979).
[9] M. Atiyah, N. Hitchin, I. M. Singer, *Proc. Roy. Soc. Lond.* **A362**, 425 (1978).
[10] R. S. Ward, *The Self-Dual Yang-Mills and Einstein Equations*, in *Complex Manifold Techniques in Theoretical Physics*, D. E. Lerner, P. D. Sommers (Editors), Pitman Advanced Publishing Program, San Francisco, London, Melbourne 1979.
[11] T. Eguchi, A. J. Hanson, *Ann. Phys.* **120**, 82 (1979).
[12] S. Hacyan, *Phys. Lett.* **A75**, 23 (1979).
[13] G. W. Gibbons, *Phys. Rev.* **D22**, 313 (1980).
[14] T. Eguchi, A. J. Hanson, *Gen. Relativ. Gravitation* **11**, 315 (1979).
[15] J. F. Plebański, I. Robinson, *Phys. Rev. Lett.* **37**, 493 (1976).
[16] J. F. Plebański, *J. Math. Phys.* **16**, 2395 (1975).
[17] J. F. Plebański, M. Przanowski, *The Action Principles of Riemannian Structures in Four Complex or Real Dimensions in the Helicity Formalism*, to be published.
[18] R. O. Wells, *Differential Analysis on Complex Manifolds*, Prentice-Hall, Inc., Englewood Cliffs, N. J. 1973.
[19] J. F. Plebański, *J. Math. Phys.* **18**, 2511 (1977).
[20] J. F. Plebański, M. Przanowski, *The Spinorial and the Helicity Formalisms for all Riemannian Structures in Complex and Real Four Dimensions*, to be published.
[21] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, New York-London, Vol. I, 1963, Vol. II, 1969.
[22] R. Sulanke, P. Wintgen, *Differentialgeometrie und Faserbündel*, VEB Deutscher Verlag der Wissenschaften, Berlin 1972.
[23] W. Ślebodziński, *Exterior Forms and Their Applications*, Monografie Matematyczne, PWN, Warszawa 1970.
[24] J. Morrow, K. Kodaira, *Complex Manifolds*, Holt, Rinehart and Winston, New York 1971.
[25] S. S. Chern, *Bull. Am. Math. Soc.* **62**, 101 (1956).
[26] E. J. Flaherty, *Hermitian and Kählerian Geometry in Relativity*, Springer-Verlag, Berlin, Heidelberg, New York 1976.
[27] J. N. Goldberg, R. K. Sachs, *Acta Phys. Pol.* **22**, Suppl. 13 (1962).
[28] F. A. E. Pirani, *Introduction to Gravitational Radiation Theory*, in *Lectures on General Relativity*, Brandeis Summer Institute, Prentice Hall 1964.
[29] J. F. Plebański, *Spinors, Tetrads and Forms*, unpublished monograph of Centro de Investigación y de Estudios Avanzados del I.P.N., México 14, D. F., 1974.
[30] W. Kundt, A. Thompson, *Compt. Rend. Acad. Sci. Paris* **254**, 4257 (1962).
[31] I. Robinson, A. Schild, *J. Math. Phys.* **4**, 484 (1963).
[32] M. Przanowski, J. F. Plebański, *Acta Phys. Pol.* **B10**, 573 (1979).
[33] R. Penrose, *Gen. Relativ. Gravitation* **7**, 31 (1976).
[34] C. P. Boyer, J. D. Finley, III, J. F. Plebański, *Complex Relativity, H and HH Spaces — A Survey*,

Com. Tec. Instituto de Investigaciónes en Matematicas Aplicadas y en Sistemas. Universidad Nacional Autonoma de México, México 20, D.F., 9, No 174 (1978).

[35] M. Przanowski, J. F. Plebański, *Acta Phys. Pol.* **B10**, 485 (1979).

[36] G. Y. Rainich, *Trans. Am. Math. Soc.* **27**, 106 (1925).

[37] C. W. Misner, J. A. Wheeler, *Ann. Phys.* **2**, 525 (1957).

[38] J. A. Wheeler, *Ann. Phys.* **2**, 604 (1957).

[39] A. A. Belavin, D. E. Burlankov, *Phys. Lett.* **A58**, 7 (1976).

[40] W. Marciano, H. Pagels, Z. Parsa, *Phys. Rev.* **D15**, 1044 (1977).

[41] J. F. Plebański, *The Spinorial and Helicity Formalisms of Riemannian Structures in Complex or Real Four Dimensions*, preprint of Centro de Investigación y de Estudios Avanzados del I.P.N., México 14, D.F., 1980.