COMMENTS ON THE RAINICH-MISNER-WHEELER THEORY IN EUCLIDEAN GRAVITY

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The "already unified field theory" of the electromagnetic and gravitational fields with a cosmological term on a four-dimensional Riemannian manifold with a positive definite metric is proposed.

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It has been shown recently that every locally Kähler gravitational instanton M_{LK} with the curvature scalar $R = const.$, admits, for each point, a local, real, source-free Maxwell field [1]. If $C_{AB\dot{C}\dot{D}}(p) \neq 0$, where p is some point of M_{LK}, then for some neighbourhood of p, the Maxwell field depends on a real constant parameter.

In the present paper we would like to study this dependence more deeply and from a general point of view, and we intend to give the analog of the "already unified field theory" of Rainich-Misner-Wheeler [2–5] for the Euclidean Gravity.

Our considerations are mainly local and they concern a four-dimensional Riemannian manifold M with a positive definite metric $g_{\mu\nu}$. All objects are differentiable of class C^∞.

Let us assume that

$$R = -4\Lambda = \text{const.}, \tag{1}$$

and

$$C^{\mu}{}_{\varrho}C^{\varrho}{}_{\nu} = \tfrac{1}{4}(C^{\alpha\beta}C_{\alpha\beta})\delta^{\mu}{}_{\nu} \tag{2}$$

where R — the curvature scalar, $C_{\mu\nu}$ — the traceless Ricci tensor field.

We assume also that (everywhere)

$$C^{\alpha\beta}C_{\alpha\beta} \neq 0. \tag{3}$$

Notice that in the case of positive definite metric

$$C^{\alpha\beta}C_{\alpha\beta} \neq 0 \Leftrightarrow C_{\alpha\beta} \neq 0. \tag{4}$$

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Consider the eigenvalue problem of $C^{\alpha}{}_{\beta}$

$$C^{\mu}{}_{\nu}V^{\nu} = CV^{\mu} \tag{5}$$

where, V^{ν} — the eigenvector field of $C^{\alpha}{}_{\beta}$, C — the eigenvalue of $C^{\alpha}{}_{\beta}$ corresponding to V^{ν}. Then using (2) and (5) one has, $\frac{1}{4}(C^{\alpha\beta}C_{\alpha\beta})V^{\nu} = C^2V^{\nu}$. Finally

$$C = \pm\tfrac{1}{2}\sqrt{C^{\alpha\beta}C_{\alpha\beta}}. \tag{6}$$

Therefore, we conclude that if the relations (2) and (3) are fulfilled then $C^{\alpha}{}_{\beta}$ is of type ((1, 1), (1, 1)), (see [6, 7]).

Then we can find the right-oriented orthonormal co-frame $\{E^a\}$, $a = 1, 2, 3, 4$, such that

$$C_{\mu\nu} = \tfrac{1}{2}\sqrt{C^{\alpha\beta}C_{\alpha\beta}}\,(-E^1_{\mu}E^1_{\nu} - E^2_{\mu}E^2_{\nu} + E^3_{\mu}E^3_{\nu} + E^4_{\mu}E^4_{\nu}). \tag{7}$$

With the use of the null tetrad $\{e^a\}$ defined according to the formulae [1]

$$e^1 = \overline{e^2} = \frac{1}{\sqrt{2}}(E^1 + iE^2), \qquad e^3 = \overline{e^4} = \frac{1}{\sqrt{2}}(E^3 + iE^4) \tag{8}$$

$C_{\mu\nu}$ is of the form

$$C_{\mu\nu} = \sqrt{C^{\alpha\beta}C_{\alpha\beta}}\,(-e^1_{(\mu}e^2_{\nu)} + e^3_{(\mu}e^4_{\nu)}). \tag{9}$$

For the spinorial image of $C_{\mu\nu}$ [1]

$$C_{AB\dot{C}\dot{D}} := \tfrac{1}{4}\, g_{A\dot{C}}{}^{\mu} g_{B\dot{D}}{}^{\nu} C_{\mu\nu} \tag{10}$$

we have

$$C_{AB\dot{C}\dot{D}} = -8 f^{(1)}_{AB} f^{(1)}_{\dot{C}\dot{D}}, \tag{11}$$

where

$$f^{(1)}_{AB} = \frac{i}{2}\left(\frac{1}{\sqrt{2}}\sqrt[4]{C^{\alpha\beta}C_{\alpha\beta}}\; k_{(A}l_{B)}\right), \qquad f^{(1)}_{\dot{A}\dot{B}} = -\frac{i}{2}\left(\frac{1}{\sqrt{2}}\sqrt[4]{C^{\alpha\beta}C_{\alpha\beta}}\; k_{(\dot{A}}l_{\dot{B})}\right) \tag{12}$$

with the spinor fields $k^A, l^A, k^{\dot{A}}, l^{\dot{A}}$ defined as follows

$$e^1 = \frac{1}{\sqrt{2}}\, g^{A\dot{B}} k_A l_{\dot{B}}, \qquad e^2 = \frac{1}{\sqrt{2}}\, g^{A\dot{B}} l_A k_{\dot{B}},$$

$$e^3 = -\frac{1}{\sqrt{2}}\, g^{A\dot{B}} k_A k_{\dot{B}}, \qquad e^4 = \frac{1}{\sqrt{2}}\, g^{A\dot{B}} l_A l_{\dot{B}}, \tag{13}$$

$$k^A l_A = 1 = k^{\dot{A}} l_{\dot{A}}. \tag{14}$$

One can show that (13) and (14) yield

$$\overline{k^A} = l_A, \qquad \overline{k^{\dot{A}}} = l_{\dot{A}}; \tag{15}$$

this means that

$$\begin{pmatrix} k^1 & k^2 \\ l^1 & l^2 \end{pmatrix}, \quad \begin{pmatrix} k^{\dot 1} & k^{\dot 2} \\ l^{\dot 1} & l^{\dot 2} \end{pmatrix} \in \mathrm{SU}(2). \tag{16}$$

From (12) and (15) it follows that

$$\overline{f^{(1)}_{AB}} = f^{(1)AB}, \quad \overline{f^{(1)}_{\dot A \dot B}} = f^{(1)\dot A \dot B}. \tag{17}$$

Concluding, we have shown that if (2) and (3) hold then there exist symmetric nowhere vanishing spinor fields f_{AB} and $f_{\dot A \dot B}$ such that

$$\overline{f_{AB}} = f^{AB}, \quad \overline{f_{\dot A \dot B}} = f^{\dot A \dot B} \tag{18}$$

$$C_{AB\dot C \dot D} = -8 f_{AB} f_{\dot C \dot D}. \tag{19}$$

It is easy to show that the relations (19), (18) define the symmetric, nowhere vanishing spinor fields $f_{AB}, f_{\dot A \dot B}$ with the accuracy to the following transformation

$$f_{AB} \longmapsto h f_{AB}, \quad f_{\dot A \dot B} \longmapsto \frac{1}{h} f_{\dot A \dot B} \tag{20}$$

where h is an arbitrary (nowhere vanishing) real function.

Let us define the 2-form

$$F := f_{AB} S^{AB} + f_{AB} S^{AB} \tag{21}$$

where f_{AB} and $f_{\dot A \dot B}$ are any spinor fields satisfying the relations (19) and (18) (of course these fields are symmetric and, by (3), nowhere vanishing) and

$$S^{AB} := \tfrac{1}{2} \varepsilon_{\dot R \dot S} g^{A\dot R} \wedge g^{B\dot S}, \quad S^{\dot A \dot B} := \tfrac{1}{2} \varepsilon_{RS} g^{R\dot A} \wedge g^{S\dot B}. \tag{22}$$

Using (18) and the formulae (compare with [1]),

$$\overline{S^{AB}} = S_{AB}, \quad \overline{S^{\dot A \dot B}} = S_{\dot A \dot B} \tag{23}$$

one finds that the 2-form F defined by (21) is real

$$\bar F = F. \tag{24}$$

(It can be shown that the 2-form defined according to (21) with arbitrary symmetric spinor fields $f_{AB}, f_{\dot A \dot B}$ is real iff the relations (18) hold). Then, from (10), (19) and (21) we have

$$C_{\mu\nu} = -2(F^{\lambda}{}_{\mu} F_{\lambda\nu} - \tfrac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}) \tag{25}$$

where the antisymmetric tensor field $F_{\mu\nu} = F_{[\mu\nu]}$ is defined by the relation

$$F = \tfrac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu}. \tag{26}$$

Now, using the definition (21) and the well known formulae

$$* S^{AB} = S^{AB}, \quad * S^{\dot{A}\dot{B}} = -S^{\dot{A}\dot{B}} \tag{27}$$

one finds that the transformation (20) of the spinor fields f_{AB} and $f_{\dot{A}\dot{B}}$ leads to the following transformation of the 2-form F

$$F \mapsto \pm(F \cosh \varphi + * F \sinh \varphi) \tag{28}$$

with

$$h = \pm e^{\varphi}. \tag{29}$$

The transformation (28) for $\varphi = const.$ is the Euclidean Gravity version of a duality rotation [3, 4]. We'll call it (when $\varphi = const.$) a *duality hyper-rotation.* If φ is an arbitrary function then the transformation (28) is called a *general duality* (*g-duality*, for short) *hyper-rotation.* Assume now that the 2-form (21) is a real source-free Maxwell field. i.e.,

$$dF = 0 = d * F. \tag{30}$$

One has (compare with [8])

$$(dF = 0 = d * F) \Leftrightarrow (\nabla^{A\dot{C}} f_{AB} = 0 = \nabla^{A\dot{C}} f_{\dot{C}\dot{D}}) \tag{31}$$

where

$$\nabla_{A\dot{B}} := g_{A\dot{B}}{}^{\mu} \nabla_{\mu}. \tag{32}$$

Let F' be a 2-form obtained from F by a g-duality hyper-rotation according to (28). Now the question is if F' is also a real, source-free Maxwell field. Of course $\overline{F'} = F'$, but

$$(dF' = 0 = d * F') \Leftrightarrow \left(f_{AB} \nabla^{A\dot{C}} h = 0 = f_{\dot{C}\dot{D}} \nabla^{A\dot{C}} \left(\frac{1}{h} \right) \right) \tag{33}$$

where h is defined by (29).

From (3) (see also (43) and (44)) it follows that

$$\left(f_{AB} \nabla^{A\dot{C}} h = 0 = f_{\dot{C}\dot{D}} \nabla^{A\dot{C}} \left(\frac{1}{h} \right) \right) \Leftrightarrow \nabla^{A\dot{C}} h = 0. \tag{34}$$

Therefore (we assume that the real, source-free Maxwell field F is defined on an open, connected, oriented subset of M!) from (33) and (34) we find that F' is a real, source-free Maxwell field, too, iff

$$h = \text{const.} \tag{35}$$

or, in other words, iff

$$\varphi = \text{const}; \tag{36}$$

that is, iff F' is related to F by a duality hyper-rotation. Now the fundamental problem arises if one can formulate, in a geometrical language, the conditions equivalent to the existence of a nowhere vanishing real source-free Maxwell field such that moreover

$$R_{\mu\nu}-\tfrac{1}{2}\, g_{\mu\nu}R = -8\pi T_{\mu\nu}+\Lambda g_{\mu\nu} \tag{37}$$

where, $\Lambda = const.$, (the "cosmological constant") and

$$T_{\mu\nu} = \frac{1}{4\pi}\,(F^{\lambda}{}_{\mu}F_{\lambda\nu}-\tfrac{1}{4}\, g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}). \tag{38}$$

From our previous considerations one finds that the relations {(1), (2), (3)} are equivalent to (37), with nowhere vanishing real $F_{\mu\nu} = F_{[\mu\nu]}$. But what about the Maxwell equations?

First, notice that the contracted Bianchi identities (see [7, 8, 9]) with $R = -4\Lambda = const.$,

$$\nabla^{A\dot{C}}C_{AB\dot{C}\dot{D}} = 0 \tag{39}$$

and the equations

$$\nabla^{A\dot{C}}f_{AB} = 0, \tag{40}$$

imply

$$\nabla^{A\dot{C}}f_{\dot{C}\dot{D}} = 0. \tag{41}$$

Indeed, (39), (40) and (19) yield the formula

$$f_{AB}\nabla^{A\dot{C}}f_{\dot{C}\dot{D}} = 0. \tag{42}$$

But

$$C^{\alpha\beta}C_{\alpha\beta} = 4C^{AB\dot{C}\dot{D}}C_{AB\dot{C}\dot{D}} = 256 f^{AB}f_{AB}f^{\dot{C}\dot{D}}f_{\dot{C}\dot{D}}, \tag{43}$$

hence

$$C^{\alpha\beta}C_{\alpha\beta} \neq 0 \Leftrightarrow (f_{AB}f^{AB} \neq 0 \quad \text{and} \quad f_{\dot{C}\dot{D}}f^{\dot{C}\dot{D}} \neq 0). \tag{44}$$

From (3), (44) and (42) we have (41). Analogously (39) and (41), of course together with (3), imply (40). The conclusion is that four of the Maxwell equations are "contained" in the contracted Bianchi identities (compare with [3–5]).

Let F be some real 2-form defined by (21) with $f_{AB}, f_{\dot{A}\dot{B}}$ satisfying (19). By analogy to the Rainich-Misner-Wheeler theory in General Relativity [3, 4] we define the *complexion* of F as a real function α such that for the 2-form ξ

$$\xi := \pm(F\cosh\alpha + *F\sinh\alpha) \tag{45}$$

the invariant $\frac{1}{4}\,\xi_{\mu\nu} * \xi^{\mu\nu}$ vanishes.
We have

$$\xi = \xi_{AB}S^{AB}+f_{\dot{A}\dot{B}}S^{\dot{A}\dot{B}} \tag{46}$$

where (compare with (29))

$$\xi_{AB} = \pm e^{\alpha} f_{AB}, \quad \xi_{\dot{A}\dot{B}} = \pm e^{-\alpha} f_{\dot{A}\dot{B}}. \tag{47}$$

Then

$$* \xi = \xi_{AB} S^{AB} - \xi_{\dot{A}\dot{B}} S^{\dot{A}\dot{B}}. \tag{48}$$

Using (46), (48) and the well known properties of S^{AB}, $S^{\dot{A}\dot{B}}$ (see e.g., [8]) one finds directly

$$\tfrac{1}{4}\, \xi_{\mu\nu} * \xi^{\mu\nu} = 0 \Leftrightarrow \xi_{AB}\xi^{AB} - \xi_{\dot{A}\dot{B}}\xi^{\dot{A}\dot{B}} = 0. \tag{49}$$

Therefore, from (47) and (49) we obtain the complexion of F

$$\alpha = \ln \left(\sqrt[4]{\frac{f_{\dot{A}\dot{B}} f^{\dot{A}\dot{B}}}{f_{AB} f^{AB}}} \right). \tag{50}$$

Now, with (31), (47), using also the contracted Bianchi identities (39) one finds that F is a real, source-free Maxwell field iff

$$\nabla^{A\dot{C}}(e^{-\alpha}\xi_{AB}) = 0. \tag{51}$$

Applying then the spinoiial bases for which

$$\xi_{11} = \xi_{22} = 0 = \xi_{\dot{1}\dot{1}} = \xi_{\dot{2}\dot{2}} \tag{52}$$

and hence, by (49),

$$\xi_{12} = \pm \xi_{\dot{1}\dot{2}}; \tag{53}$$

using also the contracted Bianchi identities, one finds that

$$\nabla^{A\dot{B}}(e^{-\alpha}\xi_{AB}) = 0 \Leftrightarrow \frac{\partial \alpha}{\partial x^{\mu}} = -\sqrt{g}\, \varepsilon_{\mu\nu\varrho\sigma} \frac{C^{\gamma\nu;\varrho} C^{\sigma}{}_{\gamma}}{C^{\beta\delta} C_{\beta\delta}} \tag{54}$$

where

$$g := \det(g_{\mu\nu}). \tag{55}$$

Hence, we conclude that if the 2-form ξ is defined on some open, oriented set $U \subset M$, then there exists a real, source-free Maxwell field on U iff for each *1*-cycle $c_1 \in Z_1$ in U

$$\int_{c_1} \overset{(1)}{\alpha} = 0 \tag{56}$$

where

$$\overset{(1)}{\alpha} = \alpha_{\mu} dx^{\mu} := -\sqrt{g}\, \varepsilon_{\mu\nu\varrho\sigma} \frac{C^{\gamma\nu;\varrho} C^{\sigma}{}_{\gamma}}{C^{\beta\delta} C_{\beta\delta}}\, dx^{\mu}. \tag{57}$$

If the 1-cohomology group

$$H^1(U, \boldsymbol{R}) = 0 \tag{58}$$

then the condition (56) is equivalent to the assertion that the 1-form $\overset{(1)}{\alpha}$ is closed

$$d\overset{(1)}{\alpha} = 0. \tag{59}$$

Summing up, we have

Theorem

Let M be a four-dimensional Riemannian manifold with a positive definite metric $g_{\mu\nu}$. Then

(*i*) For each point $p \in M$ there exists an open neighbourhood U of p such that (37) and (38) hold on U with a nowhere vanishing, antisymmetric, real tensor field $F_{\mu\nu}$ on U and $\Lambda = const.$ on M iff the formulae (1), (2) and (3) hold for every point of M.

If U is connected and oriented then $F_{\mu\nu}$ is defined by (37) and (38) up to a general duality hyper-rotation defined by (28) with (26), where φ is any real function on U.

(*ii*) If for each point $p \in M$ there exists an open, oriented neighbourhood U of p such that (37), (38) hold on U and $F_{\mu\nu} = F_{[\mu\nu]}$ is a nowhere vanishing, real, source-free Maxwell field on U, $\Lambda = const.$ on M, then the relations (1), (2), (3) are fulfilled for every point of M and, for each 1-cycle $c_1 \in Z_1$ in U, (56) and (57) hold on U.

(*iii*) If for some point $p \in M$, the relations (1), (2), (3) hold and moreover, there exists an open, oriented neighbourhood U_1 of p such that the relations (56) and (57) are fulfilled for any 1-cycle $c_1' \in Z_1$ in U_1, then there exists an open neighbourhood of p $U \subset U_1$ such that (37) and (38) hold on U and $F_{\mu\nu} = F_{[\mu\nu]}$ is a nowhere vanishing, real, source--free Maxwell field on U.

If U is connected then this Maxwell field is defined up to a duality hyper-rotation defined by (28) with

$$\varphi = \text{const.} \quad \text{on } U. \blacksquare$$

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