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SOME REMARKS ON THE INTERNAL SYMMETRIES OF RELATIVISTIC WAVE EQUATIONS

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The internal symmetries of relativistic wave equations in the free field case are treated. It is shown that the additional invariance of wave equations is connected with the transformations of physical basis. The dyal symmetry is shown to be the transformation of Lorentz basis.

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1. *Introduction*

We deal in this paper with the internal symmetries of first order relativistic wave equations

$$(p_\mu\beta^\mu-\kappa)\Psi = 0, \tag{1}$$

where $p_\mu = i\partial_\mu$, $\kappa > 0$, and Ψ transforms according to some finite dimensional representation of the Lorentz group.

In some recent papers [1, 2] the additional invariance of equations (1) was analysed anew. The general internal symmetries are always the general linear transformations and give the groups GL(N, **C**) or direct products of such groups. It is shown that the additional invariance is connected with the spin projection and degeneracy due to spin. In this paper we demonstrate that the additional invariance can be easily interpreted if we use the physical basis. The existence of this invariance means that this basis admits arbitrary linear transformations.

There are also some other types of symmetries. In the recent papers [3–9] the new internal symmetry — the dyal symmetry was dealt with. This type of symmetry is allowed by a class of very special wave equations (1) where the β^μ matrices can be expressed with the help of Dirac γ-matrices in the following way:

$$\beta^\mu = \gamma^\mu \times I_n \tag{2}$$

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($\times$ means the direct product). The internal symmetry of equation (1) called the dyal symmetry is now trivially obtained. The wave function Ψ is represented as the direct product of two representations of the Lorentz group ψ_1 and ψ_2 : $\Psi = \psi_1 \times \psi_2$, where ψ_1 is the Dirac bispinor. From (2) it is obvious that in the second representation space ψ_2 one may perform arbitrary general linear transformations with operators $Q = I_4 \times U$, and the general dyal symmetry group is therefore $\mathrm{GL}(n, \mathbf{C})$. It has been established in [9] that the dyal symmetry group gives us the example of internal symmetry group where the transformations are given by tensorial parameters. It is true, but as we shall show later this symmetry must be interpreted as the transformation of the Lorentz basis in the ψ_2 space and the transformation of generators in this space.

In the following we use the concepts of physical states and physical basis. The physical states are labelled by mass, spin and spin projection. The additional invariance is a consequence of the fact that the relativistic wave equation (1) determines in general only the mass of physical states. In order to determine the spin and the spin projection one must introduce additional restrictions to the solutions of equation (1).

2. *Physical states*

As it is well known, the physical space-time symmetry group is the Poincaré group (inhomogeneous Lorentz group), since the invariants of the Poincaré group have a direct physical meaning [10]. The irreducible representations of the Poincaré group are labelled by mass and spin. The physical states in the momentum space are the following

$$|p, \sigma; (m, s)\rangle. \tag{3}$$

This basis is constructed by using the little group technique as follows: spin and spin projection are defined in the rest system of states $p_\mu p^\mu = m^2$ where $\hat{p}^\mu = (\varepsilon m, 0, 0, 0)$ and $\varepsilon = p^0/|p^0|$

$$\vec{S}^2 |\hat{p}, \sigma; (m, s)\rangle = s(s+1)\, |\hat{p}, \sigma; (m, s)\rangle, \tag{4}$$

$$S^3 |\hat{p}, \sigma; (m, s)\rangle = \sigma |\hat{p}, \sigma; (m, s)\rangle. \tag{5}$$

The physical states at arbitrary momentum p are defined by the boost transformation: $|p, \sigma; (m, s)\rangle = U(L(p)) |\hat{p}, \sigma; (m, s)\rangle$ where $p = L\hat{p}$.

Using the relativistic wave equations (1) we describe the representations of the Poincaré group by the help of the finite dimensional representations of the Lorentz group. The components of the wave function — ψ_α, where α is the Lorentz index, have no direct physical meaning, because α labels only the components of the Lorentz basis. In order to obtain the physical states with the given mass, spin and spin projection one must introduce the physical indices m, s and σ. The relativistic wave equation (1) gives us only the mass condition. In order to determine spin and spin projection one must also demand the relations (4) and (5).

Without the loss of generality we may perform our analysis in the rest system where $\hat{p}^\mu = (\varepsilon m, 0, 0, 0)$. Then the equation (1) has a form

$$(\varepsilon m \beta^0 - \kappa)\Psi = 0. \tag{6}$$

We can treat it as the eigenvalue problem of operator β^0

$$\beta^0 \Psi = \varepsilon \frac{\kappa}{m} \Psi \qquad \text{or} \qquad \beta^0 \Psi_b = b \Psi_b. \tag{7}$$

The nonzero eigenvalues of $\beta^0 - \pm b_i$ determined the masses described by the equation (1) as follows: $m_i = \kappa / b_i$. Therefore the equation (1) introduces only the mass index $\Psi_b \equiv \Psi_m$.

In addition to the determination of mass one must determine spin and spin projection. β^0 commutes with the generators S^{kl} $(k, l = 1, 2, 3)$ of space rotations, therefore we demand that Ψ_m is also the eigenfunction of operators $\vec{S}^2 = (S^1)^2 + (S^2)^2 + (S^3)^2$ and S^3, where $2S^i = i\varepsilon^{ikl} S^{kl}$.

Therefore the physical state is defined as the eigenfunction of three commuting operators β^0, $\vec{S}^2$ and S^3

$$\beta^0 \Psi_{ms\sigma} = b \Psi_{ms\sigma}, \qquad \vec{S}^2 \Psi_{ms\sigma} = s(s+1) \Psi_{ms\sigma}, \qquad S^3 \Psi_{ms\sigma} = \sigma \Psi_{ms\sigma}. \tag{8}$$

The physical states at arbitrary momentum p are obtained by the boost transformation $U(L(p))$.

All the nonzero eigenfunctions of the system (8) $\Psi_{ms\sigma}$ form a basis which we call the physical basis. If there are many states with the given mass and spin one must introduce some additional commuting operators in order to determine the full physical basis. In Sect. 5 we demonstrate that in the case of the vector field of general type we add to β^0, $\vec{S}^2$ and S^3 also the parity operator Π. As it is well known, the physical basis $\Psi_{ms\sigma}$ can be used in the physical applications instead of the Lorentz basis Ψ_α. In quantization, for example, the field operators are expanded by the help of the plane wave solutions with a definite spin and spin projection, i.e. by the help of the physical basis $\Psi_{ms\sigma}$.

We have mentioned above that we may use the rest system without the loss of generality. The same analysis may be also performed in a more general way, using the authomorphisms gives in [11–13]

$$\beta^\mu(p) = \varepsilon^\mu{}_\nu \beta^\nu, \qquad S^{\mu\nu}(p) = \varepsilon^\mu{}_\varrho \varepsilon^\nu{}_\sigma S^{\varrho\sigma}$$

where $\varepsilon^0{}_\mu = p_\mu / \varepsilon m$ and $\varepsilon^\mu{}_\varrho \varepsilon^\nu{}_\sigma g^{\sigma\varrho} = g^{\mu\nu}$. Now the plane wave solutions of (1) at arbitrary momentum p are the eigenfunctions of operator $\beta^0(p)$. All the conditions (8) are the same, but instead of β^0, $\vec{S}^2$ and S^3 we use $\beta^0(p)$, $\vec{S}^2(p)$ and $S^3(p)$.

3. *Additional invariance*

The additional invariance of relativistic wave equation (1) can be easily understood by using the physical basis generated by the relativistic wave equation (1) and spin operators — $\Psi_{ms\sigma}$. From (8) one can see that there always exists a degeneracy due to spin projection and this means that for a given spin s all the linear combinations $\Psi_{ms} = \sum_\sigma a_\sigma \Psi_{ms\sigma}$ are the eigenvectors of β^0 with the same eigenvalue $b = \kappa/m$. These transformations generate the group of general linear transformations GL$(2s+1, \mathbf{C})$. The physical meaning of this internal symmetry is therefore the transformation of sub-basis of physical states $\Psi_{ms\sigma}$ with fixed m and s. If there are more spin states with the same mass, one can consider all the linear transformations of states $\Psi_{ms\sigma}$ with fixed m and the additional symmetry group is correspondingly larger (see also Sect. 5).

The Dirac equation has the additional symmetry group GL(2, **C**) × GL(2, **C**) which is a result of the existence of two spin projections $\sigma = \pm 1/2$ for $\varepsilon = +1$ and so for $\varepsilon = -1$.

The vector field of general type has the additional symmetry GL(8, **C**) × GL(8, **C**), since for $\varepsilon = +1$ and $\varepsilon = -1$ we have two spin 0 and two spin 1 states with the same mass m (see Sect. 5).

As we have seen, the equation (1) always admits additional symmetries since it does not determine the physical basis. On the other hand, the meaning of additional symmetries is quite obvious, it simply means the transformation of physical basis. In our opinion one can not gain any deeper physical insight by using the additional symmetry of equation (1). If we introduce the interaction, the corresponding internal symmetry may be dynamical; in the free field case it is only kinematical. Also the physical basis $\Psi_{ms\sigma}$ is more easily obtained from the relations (8) than by using the transformations of corresponding additional symmetry group.

4. *Dyal symmetry*

As we have mentioned in the introduction the dyal symmetry is obtained in the case of special wave equations (1) where β^μ is represented as

$$\beta^\mu = \gamma^\mu \times I_n \tag{9}$$

and Ψ as a direct product

$$\Psi = \psi \times \phi. \tag{10}$$

In (10) ψ is a Dirac bispinor and ϕ transforms under some n-dimensional representation of the Lorentz group.

The generators of Lorentz transformations $S^{\mu\nu}$ have a general form

$$S^{\mu\nu} = S_1^{\mu\nu} \times I_n + I_4 \times S_2^{\mu\nu}. \tag{11}$$

In our case $S_1^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/4$ and $S_2^{\mu\nu}$ are the generators of ϕ representation.

The equations, where β^μ is represented as $\gamma^\mu \times I_n$ were recently treated in [14].

The internal symmetry, called in [3] the dyal symmetry, is quite trivially obtained using (9). From (9) it is obvious that β^μ commutes with operators $Q = I_4 \times U$, since the equation (1) imposes no restrictions to ϕ. Therefore one may perform arbitrary transformations in the representation space ϕ. Hence the most general dyal symmetry group is GL(n, **C**).

In order to establish the physical meaning of the dyal symmetry one must introduce the physical basis $\Psi_{ms\sigma}$. Because of the fact that the eigenvalues of $\beta^0 = \gamma^0 \times I_n$ are ± 1 one may in (1) set $\kappa = m$. It means that now all the states have the same mass m. In the rest system it means that the physical states $\Psi_{ms\sigma}$ are the eigenvectors of β^0. Spin and spin projection are determined as the eigenvalues of operators $\vec{S}^2$ and S^3. From (11) we see that for physical states $\Psi_{ms\sigma} = (\psi \times \phi)_{ms\sigma}$ is not arbitrary, but determined. The transformations of dyal symmetry act only on ϕ. The latter allows the following interpretation of dyal symmetry: in order to preserve the physical basis the dyal symmetry must be treated as the change of basis and generators in the ϕ space. Indeed, if we introduce $\Psi' = \psi \times U\phi$ and $(S^{\mu\nu})' = S_1^{\mu\nu} \times I_n + I_4 \times US_2^{\mu\nu}U^{-1}$ the physical basis remains the same.

5. *The vector field of general type*

We deal with the vector field of general type as an illustration of the considerations of the previous sections. This special field is treated in most of the above cited papers [1–9]. The vector field of general type is $\Psi = \psi \times \psi$, i.e. where both factors are Dirac bispinors. Now Ψ transforms according to $[(1/2, 0)+(0, 1/2)]^2 = (1, 0)+(0,1)+2(1/2, 1/2) +2(0, 0)$ and describes two spins: 1 and 0. The generators $S^{\mu\nu}$ are in form

$$S^{\mu\nu} = s_1^{\mu\nu} \times I_4 + I_4 \times s_1^{\mu\nu} \tag{12}$$

where $s_1^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/4$.

At first we constiuct the physical basis. The physical basis is formed from the eigenfunctions of operators β^0, $\vec{S}^2$ and S^3. In order to obtain the full physical basis one must add one more operator, since there are two spin 1 and two spin 0 states. This is the parity operator $\Pi = \gamma^0 \times \gamma^0$. Therefore the physical basis is determined by the following relations

$$\begin{aligned} \beta^0 \Psi^{\pm}_{ms\sigma} &= \varepsilon \Psi^{\pm}_{ms\sigma}, \\ \vec{S}^2 \Psi^{\pm}_{ms\sigma} &= s(s+1)\Psi^{\pm}_{ms\sigma}, \\ S^3 \Psi^{\pm}_{ms\sigma} &= \sigma \Psi^{\pm}_{ms\sigma}, \\ \Pi \Psi^{\pm}_{ms\sigma} &= \pm \Psi^{\pm}_{ms\sigma}. \end{aligned} \tag{13}$$

Operators $\vec{S}^2$ and S^3 are

$$\vec{S}^2 = -\tfrac{3}{2} I_{16} + \tfrac{1}{2} \sum_{i=1}^{3} s^i \times s^i, \qquad S^3 = s^3 \times I_4 + I_4 \times s^3$$

where $2s^i = i\varepsilon^{ikl}\gamma^k\gamma^l$. Using the following eigenfuctions of matrices β^0 and $s^3 = i\gamma^1\gamma^2/2$, which we denote as ψ_{++}, ψ_{+-}, ψ_{-+} and ψ_{--} (the first index shows the corresponding eigenvalue of γ^0 : ± 1, and the second one the eigenvalue of s^3 : $\sigma = \pm 1/2$) we get the following basis for $\varepsilon = +1$:

spin 0
$$\Psi^{+}_{m00} = \frac{1}{\sqrt{2}}(\psi_{++} \times \psi_{+-} - \psi_{+-} \times \psi_{++}),$$

$$\Psi^{-}_{m00} = \frac{1}{\sqrt{2}}(\psi_{++} \times \psi_{--} - \psi_{+-} \times \psi_{-+}),$$

spin 1
$$\Psi^{+}_{m11} = \psi_{++} \times \psi_{++},$$

$$\Psi^{+}_{m10} = \frac{1}{\sqrt{2}}(\psi_{++} \times \psi_{+-} + \psi_{+-} \times \psi_{++}),$$

$$\Psi^+_{m1-1} = \psi_{+-}\times\psi_{+-},$$

$$\Psi^-_{m11} = \psi_{++}\times\psi_{-+},$$

$$\Psi^-_{m10} = \frac{1}{\sqrt{2}}(\psi_{++}\times\psi_{--}+\psi_{+-}\times\psi_{-+}),$$

$$\Psi^-_{m1-1} = \psi_{+-}\times\psi_{--}. \tag{14}$$

We obtain similar 8 states for $\varepsilon = -1$, in Ψ^+ the first indices being $\psi_-\times\psi_-$ and in Ψ^- $\psi_-\times\psi_+$.

Now it is obvious that in the rest system all the linear combinations of 8 physical states (14) are the solutions of equation (1). The additional group is therefore GL(8, **C**). Since the $\varepsilon = -1$ states give also GL(8, **C**) we have the general additional symmetry GL(8, **C**) × GL(8, **C**).

The dyal symmetry is now generated by the operators $Q^A = I_4\times\gamma^A$, where γ^A, $A = 1, 2, \ldots, 16$, are the sixteen independent elements of the Dirac algebra. The transformations $Q = I_4\times U$, $U = \exp(a_A\gamma^A)$ with complex parameters a_A generate the transformations of dyal symmetry group GL(4, **C**). As we have mentioned above the dyal symmetry is not very useful physically if we take into consideration the fact that in physical applications one must use the physical basis $\Psi_{ms\sigma}$. Since $\Psi_{ms\sigma}$ is also the eigenfunction of operators $\vec{S}^2$ and S^3 which in spinor basis $\Psi_{\alpha\beta} = \psi_\alpha\times\psi_\beta$ acts also on the second spinor index β. This index is not arbitrary and does not admit arbitrary linear transformations (dyal transformations). In order to preserve the meaning of physical basis the dyal transformations must be interpreted as the change of spinor basis and generators in the second factor space in $\psi\times\psi$: the dyal transformations give us the new Lorentz basis $\Psi' = \psi\times U\phi$ with generators $S^{\mu\nu} = S_1^{\mu\nu}\times I_4 + I_4\times US_1^{\mu\nu}U^{-1}$. It means that in the first factor basis we use Dirac matrices γ^μ as opposed to the second factor basis we use Dirac matrices $(\gamma^\mu)' = U\gamma^\mu U^{-1}$. We have no physical interpretation why one must use different representations of Dirac matrices when operating with the direct product of two bispinors.

Concerning the basis (14), it has yet another advantage. If we define the Lorentz invaiiant scalar product by the help of matrix Λ

$$(\Psi, \Psi) = (\Psi)^+\Lambda\Psi \tag{15}$$

where $(\ )^+$ means hermitian conjugate, we can now take $\Lambda = \Pi = \gamma^0\times\gamma^0$. From (14) and (15) we obtain that for positive parity states $\Psi^+_{ms\sigma}$ the scalar product is positive and that for negative parity states $\Psi^-_{ms\sigma}$ the scalar product is negative. The parity operator Π in (13) takes off the degeneracy of two spin 1 and two spin 0 states. Because of the fact that the scalar product is indefinite one must use the indefinite metiic in quantization.

It is possible to introduce the positive definite scalar product by using the physical basis $\Psi_{ms\sigma}$. Following [15, 16], one defines the new scalar product in the rest system by the relation

$$(\Psi, \Psi)_{\text{new}} = (\Psi)^+\Psi. \tag{16}$$

At arbitrary momentum p

$$(\Psi(p), \Psi(p))_{new} = (\Psi(p))^{+} U^{2}(L^{-1}(p))\Psi(p). \tag{17}$$

This procedure works well in the free field case, but in interactions we may have the instability of Poincaré states [17].

To conclude this section we have some remarks concerning the vector field of general type:

1) The wave equation of the vector field of general type has many equivalent formulations. The tensorial formulation (see, for example [9]) uses antisymmetric tensors $F_{\mu\nu}$, $\tilde{F}_{\mu\nu}$, vectors A_μ, B_μ and scalars φ, $\tilde{\varphi}$. This formulation is used to give the quaternionic form of the wave equation for this type of field (see, for example [6]). It should be remarked that the quaternionic form is a quite formal rewriting of the wave equation. It is not easy to get the physical basis, since the spin and spin projection operators are not presented in the quaternionic form. Also it is problematical whether the symmetry of quaternionic equation is really the dyal symmetry.

In [9] the wave equation is written in the Gel'fand-Yaglom basis, but the transformations of dyal symmetry are not formulated.

As we have shown the dyal symmetry is more easily seen in the direct product basis (10) and it is a quite trivial symmetry of a corresponding wave equation.

2) As it is well known [18], the wave equations may have the acausality defects. The wave equation of the vector field of general type is causal, since the β^μ-matrices are diagonalizable $((\beta^\mu)^2 = I)$ [19].

3) The equation of the vector field of general type is quite simple, it describes two spin 1 and two spin 0 particles with the same mass m. In most papers it is interpreted as an object with multispin 0–1.

In some way the equation of the vector field of general type is connected with the well known Bargmann-Wigner equation for spin 1 [20]

$$\begin{aligned} (p_\mu\gamma^\mu \times I_4 - m)\Psi &= 0, \\ (I_4 \times p_\mu\gamma^\mu - m)\Psi &= 0, \end{aligned} \tag{18}$$

where $\Psi_{\alpha\beta} = \psi_\alpha \times \psi_\beta$ is symmetrical with respect to α and β.

From (18) we see that the vector field of general type is described by the first of the equations (18), where no symmetry conditions on $\Psi_{\alpha\beta}$ are imposed.

The system (18) may be written in form [11, 12]:

$$\tfrac{1}{2}(p_\mu\gamma^\mu \times I_4 + I_4 \times p_\mu\gamma^\mu)\Psi = m\Psi, \tag{19a}$$

$$(p_\mu\gamma^\mu \times I_4 - I_4 \times p_\mu\gamma^\mu)\Psi = 0. \tag{19b}$$

The matrices

$$\beta^\mu = \tfrac{1}{2}(\gamma^\mu \times I_4 + I_4 \times \gamma^\mu) \tag{20}$$

generate the algebra of the SO(1, 4) group.

It is easy to verify that the conditions (19b) are automatically satisfied and the equation (19a) is equivalent to (18). The equation (19a) is the well known Kemmer-Duffin equation. In the case of symmetric $\Psi_{\alpha\beta}$ we get the spin 1 Kemmer-Duffin equation and in the case of antisymmetric $\Psi_{\alpha\beta}$ the spin 0 Kemmer-Duffin equation. So the Bargmann-Wigner equation for spin 1 reduces to the Kemmer-Duffin spin 1 equation. Because of the fact that β^μ generates the SO(1, 4) algebra, the Bargmann-Wigner equation belongs to the class of SO(1, 4) type equations [11–13].

6. *Conclusions*

We have dealt in this paper with the additional invariance and the internal symmetry, called the dyal symmetry of relativistic wave equations. The additional invariance is easily interpreted by using the concept of physical states and physical basis: it reduces to general linear transformations of the physical basis. The additional invariance of relativistic wave equation is a consequence of the fact that the physical basis is not determined by the wave equation only, and one must impose some additional conditions to determine spin and spin projection.

The internal symmetry — the dyal symmetry is also a tiivial symmetry of a special class of relativistic wave equations. The dyal symmetry must be interpreted as the partial transformation of the Lorentz basis and Lorentz generators.

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