AN EXACTLY SOLVABLE MODEL FOR FERMIONIC GENERATIONS AND POINCARÉ STRESSES

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An exactly solvable quantum mechanical model is presented to illustrate the conjecture that leptons and quarks of higher generations may arise as excited states of lepton and quark internal charge distribution kept stable by Poincaré stresses. Excitations are then "quasiphonons" related to internal oscillations around the equilibrium provided by joint action of these stresses and internal electromagnetic forces. The model leads to an exponentially growing mass spectrum proposed previously on the phenomenological ground (predicting toponium at about 40 GeV and the next charged lepton at 28.5 GeV). When perturbed in a natural way, the model is able to describe Cabibbo-like mixing of generations, and, in particular, to predict the off-diagonal elements of Kobayashi-Maskawa matrix.

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1. *Introduction*

In this paper we conjecture that the modern fundamental puzzle of fermionic generations is closely related to the classical puzzle of Poincaré stresses [1]. As is well known, the latter are hypothetical non-electromagnetic forces, necessary in the structure of a charged particle with finite extension to keep stable its internal charge distribution against Coulomb repulsion. Our present argument may be outlined as follows. If it is true that leptons and quarks have some finite extension, Poincaré stresses should appear in their structure to provide its stability against internal electromagnetic forces. In the quantum theory, internal oscillations around the resulting equilibrium should imply the existence of discrete fermionic excited states which might be observed as leptons and quarks of higher generations. The required finite extension of leptons and quarks may or may not be connected with the existence of some subelementary point-like constituents usually called preons. So, Poincaré stresses may or may not be provided by a non-electromagnetic attraction between preons. At any rate, Poincaré stresses should describe in a more or less phenomenological way the effect of internal non-electromagnetic forces, at least in the neighbourhood of the resulting equilibrium.

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2. *The model*

In order to give a quantum-mechanical illustration for our conjecture (rather than to develop a quantum theory of Poincaré stresses) let us imagine a spherically symmetric extended particle whose internal oscillations around its equilibrium can be fully described by a radially oriented coordinate q and a corresponding momentum p, where

$$[q, p] = i\hbar. \tag{1}$$

Here, the internal angular momentum is always zero as it is connected with radial oscillations. We define q in such a way that formally $-\infty < q < +\infty$. For instance, $q = \frac{1}{2}(1+r_0/r)(r-r_0)$ where $0 \leqslant r < \infty$ denotes the radius of a spherical shell oscillating radially around its equilibrium characterized by r_0. Then in the case of small oscillations we have practically $r \simeq r_0$ and $q \simeq r-r_0$.

Let us *assume* that the mass operator of our particle times c^2 (equal to its hamiltonian in its centre-of-mass frame) is of the form corresponding to quasiharmonic *small oscillations*:

$$Mc^2 = \frac{1}{2\mu} p^2 + \frac{\mu\omega^2}{2} [q+lf(q, p)]^2 - \frac{\hbar\omega}{2} \tag{2}$$

with $p = -i\hbar\partial/\partial q$, where $\omega > 0$, l is a constant of length dimension and $f(q, p)$ denotes a dimensionless real function of the operators q and p. Here, the operator

$$V(q, p) = \frac{\mu\omega^2}{2} [q+lf(q, p)]^2 - \frac{\hbar\omega}{2} \tag{3}$$

describes the energy connected with quasiharmonic oscillatory forces representing (or rather replacing) in our model the joint effect of Poincaré stresses and internal electromagnetic forces considered in the neighbourhood of the resulting equilibrium. The mass operator (2) can be rewritten also as

$$Mc^2 = \hbar\omega \tfrac{1}{2}(a^+a+aa^+-1) = \hbar\omega[a^+a+\tfrac{1}{2}([a, a^+]-1)], \tag{4}$$

where

$$a = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{\mu\omega}{\hbar}}[q+lf(q, p)]+\frac{i}{\sqrt{\mu\hbar\omega}}p\right\} \tag{5}$$

is a quasiannihilation operator.

A characteristic property of Poincaré stresses in the quantum theory should be the proportionality $\langle M\rangle \sim \alpha Q^2$, where eQ denotes charge of the particle and $\alpha = e^2/4\pi\hbar c$. Here, the Q-independent (suppressed) proportionality coefficient may be large and dependent on internal quantum numbers other than Q as e.g. the colour[1]. In order to have in

[1] The property $\langle M\rangle \sim \alpha Q^2$ should be true for the *electromagnetic mode* of internal oscillations, connected with the equilibrium of internal electromagnetic forces and Poincaré stresses. One may imagine that there is also a higher, *colour mode* of internal oscillations, related to the equilibrium of internal colour forces and some novel non-colour Poincaré stresses being in this case an analogue of the former non-electromagnetic Poincaré stresses. If this hypothetical higher-frequency mode is not excited, it does not contribute to the mass operator M which is then given as in Eq. (2) with $\langle M\rangle \sim \alpha Q^2$.

our model this property we will *assume* that

$$\omega \sim \alpha Q^2 \tag{6}$$

with a physically adequate $f(q, p)$ maintaining $\langle a^+a \rangle$ and $\langle aa^+ \rangle$ independent of Q as they are in the limit of $l \to 0$ where our quasiharmonic oscillator becomes harmonic. Here, $Q \neq 0$ in order to have $\omega \neq 0$. The case of $Q = 0$ can be considered in the limit of $Q \to 0$ since the value Q may be arbitrarily chosen in our model.

The function $f(q, p)$, introducing a correction to the harmonic oscillatory forces in the mass operator M, may a priori depend on the mass M itself. We will specifically *assume* that the rate of change in q of $f(q, p)$ is essentially equal to the excitation energy which is identical with Mc^2 if no additional additive constant appears on the right hand side of Eq. (2). Thus then

$$i\alpha Q^2[p, f(q, p)] = Mc \tag{7}$$

or

$$\alpha Q^2 \frac{\partial f(q, p)}{\partial q} = \frac{Mc}{\hbar}, \tag{8}$$

where M is given in Eq. (2) (this assumption jointly with Eq. (6) will determine $\langle a^+a \rangle$ and $\langle aa^+ \rangle$ independent of Q).

We can see from Eq. (8) that $f(q, p)$ satisfies a nonlinear differential equation. Fortunately, there is no need to try to solve Eq. (8) for $f(q, p)$ because the assumption (7) leads to the more convenient relation

$$[a, a^+] = \frac{1}{i\hbar}[q + lf(q, p), p] = 1 + l\frac{Mc}{\hbar\alpha Q^2} \tag{9}$$

with M as given in Eq. (4) [2]. Eqs. (4) and (9) solved for M or $[a, a^+]$ lead to

$$Mc^2 = \hbar\omega \tfrac{1}{2}(\lambda + 1)N \tag{10}$$

or

$$[a, a^+] = 1 + (\lambda^2 - 1)N, \tag{11}$$

respectively, where

$$N = a^+a = \frac{1}{\lambda^2}(aa^+ - 1) \tag{12}$$

[2] The *non-canonical* commutation relation

$$[q, p] = i\hbar(1 + lH/\hbar c)$$

with H and l being the hamiltonian and a length scale, respectively, was introduced by Saavedra and Utreras [2] as a possible bold generalization (in quark physics) of the usual canonical commutation relation (1). With such a new commutation relation they solved the "harmonic oscillator" described by the hamiltonian $H = p^2/2\mu + \mu\omega^2q^2/2$. So, in our case we can make use of their solution, taking the operator $\tilde{q} = q + lf(q, p)$ as their non-canonical coordinate q.

is a quasioccupation number operator and

$$\lambda^2 = \frac{1+\omega l/2c\alpha Q^2}{1-\omega l/2c\alpha Q^2} \tag{13}$$

is independent of Q since $\omega \sim \alpha Q^2$. Note that $|\lambda^2| > 1$ or < 1 if $l > 0$ or < 0, respectively, and $\lambda^2 > 0$ if $\omega|l|/Q^2 < 2c\alpha$ (we shall see later that $3.5 \lesssim \lambda \lesssim 4$ for charged leptons and quarks).

Thus, the model we propose in this paper is based on three assumptions (2), (6) and (7). Later on we will substitute in Eqs. (2) and (7) $M \to M - m_0$ with m_0 being a ground--state mass.

3. Exact solution

The eigenvalue equation

$$N|n\rangle = N_n|n\rangle, \quad \langle n|n\rangle = 1 \quad (n = 0, 1, 2, \ldots) \tag{14}$$

and the commutation relation (11) imply that

$$Na^+|n\rangle = (\lambda^2 N_n + 1)a^+|n\rangle, \quad a^+|n\rangle = \sqrt{\lambda^2 N_n + 1}\,|n+1\rangle,$$
$$Na|n\rangle = \frac{1}{\lambda^2}(N_n - 1)a|n\rangle, \quad a|n\rangle = \sqrt{N_n}|n-1\rangle. \tag{15}$$

From Eq. (15) we obtain the recurrence formula

$$N_{n+1} = \lambda^2 N_n + 1 \tag{16}$$

and, solving it, the spectrum

$$N_n = \frac{\lambda^{2n}-1}{\lambda^2 - 1}, \tag{17}$$

the latter when defining $|0\rangle$ through the condition $a|0\rangle = 0$ giving $N_0 = 0$. Obviously, Eq. (17) leads to

$$N_n = \begin{cases} 0 & \text{for} \quad n = 0, \\ 1 + \lambda^2 + \ldots + \lambda^{2n-2} & \text{for} \quad n \geqslant 1. \end{cases} \tag{18}$$

If $l \to 0$ then $\lambda^2 \to 1$ and we have $N_n \to n$.

From Eqs. (10), (16) and (17) we get for the eigenvalues m_n of the mass operator M in the eigenstates $|n\rangle$ the recurrence formula

$$m_{n+1} = \lambda^2 m_n + \varepsilon Q^2 \tag{19}$$

and the spectrum

$$m_n = \varepsilon Q^2 \frac{\lambda^{2n}-1}{\lambda^2-1}, \tag{20}$$

where

$$\varepsilon = \frac{\hbar\omega/c^2Q^2}{1-\omega l/2c\alpha Q^2} = \frac{\hbar\omega}{c^2Q^2}\,\tfrac{1}{2}(\lambda^2+1) = \frac{\alpha\hbar}{lc}(\lambda^2-1) \tag{21}$$

is independent of Q because $\omega \sim \alpha Q^2$. The spectrum (20) follows from the recurrence formula (19) if $m_0 = 0$ (the latter is a necessary consequence of $N_0 = 0$ and Eq. (10)). We can see from Eq. (20) that $m_n \sim \alpha Q^2$ and from Eq. (21) that $\hbar\omega = \varepsilon c^2 Q^2/\frac{1}{2}(\lambda^2+1)$ and $l = \alpha\hbar(\lambda^2-1)/\varepsilon c$ [3].

If (more generally) $m_0 \neq 0$, it is natural to *assume* that in the formula (2) for Mc^2 the constant m_0c^2 should be added to the right hand side. Then in *all* our subsequent equations we *should substitute* $M \to M-m_0$ and consequently $m_n \to m_n - m_0$. In particular, the recurrence formula (19) and the spectrum (20) take now the form

$$m_{n+1}-m_0 = \lambda^2(m_n-m_0)+\varepsilon Q^2 \tag{22}$$

and

$$m_n - m_0 = \varepsilon Q^2\,\frac{\lambda^{2n}-1}{\lambda^2-1}, \tag{23}$$

respectively. Here, the case of $Q = 0$ can be considered in the limit of $Q \to 0$, which gives $m_n = m_0$. The spectrum (20) or (23) leads generally to the mass relation

$$\frac{m_{n+2}-m_{n+1}}{m_{n+1}-m_n} = \text{const.} \equiv \lambda^2 \tag{24}$$

which shows that λ^2 can be determined from three consecutive masses (e.g. m_0, m_1 and m_2). On the other hand, for ε Eq. (23) gives $\varepsilon Q^2 = m_1 - m_0 = (m_2-m_0)(\lambda^2+1)^{-1}$.

The recurrence formula (22) and the exponential spectrum (23) are identical (excitingly enough!) to those introduced *phenomenologically* some years ago [3, 4][4] as "empirical"

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[3] The obtained exact solution for our quasiharmonic oscillator can be interpreted as clustering of n "phonons" ($n = 0, 1, 2, \ldots$) of energies $n\hbar\omega$ (corresponding to the harmonic oscillator with $l = 0$) into "*quasiphonons*" of energies $N_n\varepsilon c^2Q^2 = N_n\hbar\omega\frac{1}{2}(\lambda^2+1)$. Such a clustering is generated by the quasiharmonic correction $lf(q,p)$ to the coordinate q, if the specific assumption (7) on $f(q,p)$ is satisfied. Here, $N_0 = 0$, $N_1 = 1$ and e.g. $N_2 \simeq 17$, $N_3 \simeq 273, \ldots$ (as for charged leptons for which we shall see that $\lambda \simeq 4$). This specific quasiharmonic correction is an essential ingredient of our model, independent of the net idea of Poincaré stresses. It is closely related to the non-canonical commutation relation of Saavedra and Utreras [2], if the latter is boldly conjectured to hold inside leptons and quarks (rather than inside hadrons).

[4] In the earlier paper cited as Ref. [4] the recurrence formula (19) (which in Ref. [3] and in the present paper is valid only for $m_0 = 0$) is discussed in the general case of $m_0 \neq 0$, leading to the spectrum

$$m_n = [m_0+\varepsilon Q^2(\lambda^2-1)^{-1})^{-1}]\lambda^{2n}-\varepsilon Q^2(\lambda^2-1)^{-1}$$

with $\varepsilon Q^2 = m_1 - m_0\lambda^2$. When $Q \neq 0$, this spectrum is identical to the spectrum (23) but with $\varepsilon Q^2 = m_1 - m_0$ (and with the same m_0, m_1 and λ in both cases). Thus, when $Q \neq 0$, only ε's differ in Refs. [3] and [4] (by $m_0(\lambda^2-1)/Q^2$). However, in the case of $Q = 0$ one gets in these references $m_n = m_0$ and $m_n = m_0\lambda^{2n}$, respectively, so that *neutrino oscillations* may appear only in the second instance (if $m_0 \neq 0$ and $\lambda^2 \neq 1$).

formulae for masses of four fermionic families $f =$ ν, e, u, d, containing several generations numerated by $n = 0, 1, 2, \ldots$:

$$\nu_n = \nu_e, \nu_\mu, \nu_\tau, \ldots \qquad (Q = 0),$$

$$e_n = e^-, \mu^-, \tau^-, \ldots \qquad (Q = -1),$$

$$u_n = u, c, (t), \ldots \qquad (Q = 2/3),$$

$$d_n = d, s, b, \ldots \qquad (Q = -1/3).$$

Thus, following the idea of Poincaré stresses and making use of the solution found by Saavedra and Utreras [2] we did construct an exactly solvable quantum-mechanical *model* (defined by the *assumptions* (2), (6) and (7)) which in the case of $Q \neq 0$ as well as in the limit of $Q \to 0$ ıeproduces strictly these "empirical" formulae for lepton and quark masses.

4. *A fit to experimental masses*

For a more complete presentation let us quote from Ref. [3] the results of fitting the formulae (22) and (23) to empirical masses.

In the case of charged leptons ($Q = -1$) we determined the parameters λ and ε using the masses $m_{e_0} \equiv m_e$, $m_{e_1} \equiv m_\mu$ and and $m_{e_2} \equiv m_\tau = 1782^{+2}_{-7}$ MeV/c^2. Then we got

$$\lambda = 3.993^{+0.004}_{-0.009} \simeq 4, \qquad \varepsilon = m_\mu - m_e \tag{25}$$

and predicted the masses

$$m_{e_3} = 28.5^{+0.2}_{-0.5}\ \mathrm{GeV}/c^2, \qquad m_{e_4} = 455^{+3}_{-10}\ \mathrm{GeV}/c^2 \tag{26}$$

for two next (hypothetical) charged leptons. From Eqs. (21) and (25) we can calculate $\hbar\omega = (12.4\ \mathrm{MeV})Q^2$ and $l = 2.05 \times 10^{-14}$ cm (and $\omega l = (3.86 \times 10^8\ \mathrm{cm/sec})Q^2$), where $Q^2 = 1$. Note that $N_n = 0, 1, 17, 273, \ldots$ for $n = 0, 1, 2, 3, \ldots$ if $\lambda = 4$. In the case of neutrinos ($Q = 0$) Eq. (23) gives

$$m_{\nu_n} = m_{\nu_0} \tag{27}$$

so that all $m_{\nu_n} = 0$ if $m_{\nu_0} \equiv m_{\nu_e} = 0$. The mass degeneracy (27) implies the absence of neutrino oscillations even if $m_{\nu_e} \neq 0$ (cf., however, Ref. [4]).

In the case of up and down quarks ($Q = 2/3$ and $Q = -1/3$), putting $m_{u_0} \equiv m_u \simeq 0$ and $m_{d_0} \equiv m_d \simeq 0$ (in comparison with m_{u_n} and m_{d_n}) we obtain from Eq. (23)

$$m_{u_n} \simeq 4m_{d_n} \simeq \tfrac{4}{9}\,\varepsilon\,\frac{\lambda^{2n}-1}{\lambda^2-1} \quad (n \geqslant 1). \tag{28}$$

Hence, using the masses $m_{u_1} \equiv m_c \simeq 1.5$ GeV/c^2 and $m_{d_2} \equiv m_b \simeq 5$ GeV/c^2 we got

$$\lambda \simeq 3.5, \qquad \varepsilon \simeq 3.4\ \mathrm{GeV}/c^2 \tag{29}$$

and predicted the masses

$$m_{d_1} \equiv m_s \simeq 0.38\ \mathrm{GeV}/c^2, \qquad m_{u_2} \equiv m_t \simeq 20\ \mathrm{GeV}/c^2 \tag{30}$$

for s and t quarks[5] as well as the masses

$$m_{u_3} \simeq 250\ \mathrm{GeV}/c^2, \qquad m_{u_4} \simeq 3100\ \mathrm{GeV}/c^2,$$
$$m_{d_3} \simeq 62\ \mathrm{GeV}/c^2, \qquad m_{d_4} \simeq 770\ \mathrm{GeV}/c^2 \tag{31}$$

for the next (hypothetical) quark doublets. So, toponium $t\bar{t}$ was predicted at about 40 GeV/c^2. From Eqs. (21) and (29) we can estimate $\hbar\omega \simeq (0.51\ \mathrm{GeV})Q^2$ and $l \simeq 4.8 \times 10^{-16}$ cm (and $\omega l \simeq (3.7\times 10^8\ \mathrm{cm/sec})Q^2$).

5. *Interpretation of the fit*

Because of colour carried by quarks it is not surprising that λ and ε (and consequently ω and l) result different for charged leptons and quarks. It is an extra bonus that λ values are not so different (they may become even closer if a bit higher values of m_c and/or m_b turn out to be better). It means that λ is independent not only of Q but, in some approximation, also of colour (while ε as well as ω and l are evidently colour-dependent, in contrast to $\omega l/Q^2$ which is approximately colour-independent as can be seen from Eq. (13)).

The considerably larger value of ε for quarks than for charged leptons may be a consequence of Coulombic colour interaction which in the case of quarks should intensify Poincaré stresses working against Coulomb electric repulsion. To support this point of view one may argue as follows. Since one can write $\varepsilon = \varepsilon_l + \frac{3}{4} C(\varepsilon_q - \varepsilon_l)$ where C is the quadratic Casimir operator of colour SU(3) which is equal to 0 for leptons and 4/3 for quarks, one obtains

$$\varepsilon = \varepsilon_l \left(1 + \frac{\tilde{\alpha}_s}{\alpha} C\right) \tag{32}$$

with a constant $\tilde{\alpha}_s$ defined through the relation

$$\tilde{\alpha}_s = \tfrac{3}{4}\alpha\left(\frac{\varepsilon_q}{\varepsilon_l} - 1\right). \tag{33}$$

From Eq. (33) one can estimate $\tilde{\alpha}_s \simeq 0.17$, so $\tilde{\alpha}_s$ is of the magnitude of the colour coupling constant $\alpha_s = g_s^2/4\pi\hbar c$ as determined from quarkonia. Using Eqs. (21) and (32) one gets

$$\omega = \frac{2\varepsilon\, c^2 Q^2}{\hbar(\lambda^2+1)}\left(1 + \frac{\tilde{\alpha}_s}{\alpha} C\right) \sim \alpha Q^2 \tag{34}$$

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[5] The prediction for m_s may be questionable as the formula (28) may not work well for the considerably light s quark where the ambiguity between the current and constituent mass is still acute. Note, however, that λ and ε are determined here from much heavier masses m_c and m_b.

and

$$l = \frac{\alpha\hbar(\lambda^2+1)}{\varepsilon_l c}\left(1+\frac{\tilde{\alpha}_s}{\alpha}C\right)^{-1}. \tag{35}$$

Eq. (34) shows that in the approximation, where λ can be considered as colour independent, ω is proportional to $(\alpha+\tilde{\alpha}_s C)Q^2$ with $\tilde{\alpha}_s C$ being of the magnitude of the coupling constant for Coulombic colour interaction. It suggests that this interaction is responsible for stronger Poincaré stresses in the case of quarks than in the case of charged leptons[6].

6. *Consistency of the fit*

Having fitted our model to lepton and quark experimental masses (and so determined the constants ω and l) we can check the consistency of this fit with the assumption of *small oscillations* (explicit in Eq. (2)). To this end let us observe that the magnitude of our quasi-harmonic oscillations can be estimated from the following formula for $\tilde{q} = q+lf(q,p)$:

$$\langle n|\tilde{q}^2|n\rangle = \frac{\hbar}{\mu\omega}\frac{1}{2}\langle n|a^+a+aa^+|n\rangle = \frac{\hbar}{\mu\omega}\frac{1}{2}[(\lambda^2+1)N_n+1] \tag{36}$$

which is a consequence of Eq. (12) implying

$$aa^+ \mp a^+a = (\lambda^2 \mp 1)N+1. \tag{37}$$

The assumption of small oscillations requires that $\langle n|\tilde{q}^2|n\rangle \ll r_0^2$, where r_0 is an experimental radius of the ground state $n = 0$ (for charged leptons $r_0 \lesssim 10^{-16}$ cm is a radius of the electron). Thus, from Eq. (36) we obtain the condition

$$\frac{\hbar}{\mu c}\frac{c}{\omega}\frac{1}{2}[(\lambda^2+1)N_n+1] \ll r_0^2. \tag{38}$$

Three length scales appearing in Eq. (38), $\hbar/\mu c$, $c/\omega = l(\lambda^2+1)\,[2\alpha Q^2(\lambda^2-1)]^{-1}$ and r_0, are in our model uncorrelated and so independent. Of them, only the scale c/ω related to the oscillator frequency ω is experimentally known as determined by masses: $c/\omega = 1.59 \times 10^{-12}\,\mathrm{cm}/Q^2$ for charged leptons ($Q^2 = 1$) and $c/\omega \simeq 3.8\times 10^{-14}\,\mathrm{cm}/Q^2$ for quarks. Thus, in the present experimental situation, the scale $\hbar/\mu c$ connected with the oscillator inertia μ can be chosen as small as desired in order to satisfy the inequality (38) (with fixed ω, l and r_0). Since $N_n = (\lambda^{2n}-1)(\lambda^2-1)^{-1}$ grows rapidly with $n = 0, 1, 2, \ldots$, the condition (38) (with fixed μ, ω, l and r_0) gives also a *limitation* for the number of fermionic generations n which could be described by our model based on small oscillations.

At this point a comment is due on the *radiation stability* of the excited n-states ($n \geqslant 1$).

[6] It should be emphasized that the proportionality of ω to αQ^2 means that n-states, both for leptons and quarks, corresponds to the *electromagnetic mode* of internal oscillations. For neutrinos it is true in the limit of $Q \to 0$. The hypothetical *colour mode* with a higher frequency ω_s proportional to $\alpha_s C$ is apparently not excited for quarks, at least at present energies.

As long as in our model of a spherically symmetric extended particle there are no internal angular-momentum excitations, the internal n-states are always pure S states. It forbids the real transitions $n \to n'$ accompanied by the emission or absorption of a vector particle e.g. a photon or gluon. This conclusion is independent of the length scale connected with the radius r_0 of the ground state $n = 0$. Two-photon or two-gluon emission or absorption is not forbidden but depends on the small radius r_0.

7. *Initial-stress perturbation*

Hitherto, we did not discuss any initial conditions for $f(q, p)\,|n\rangle$, though the initial values $f(0, p)\,|n\rangle_{q=0}$ should be relevant from the physical point of view as determining the *initial stress* at $q = 0$ inside our particle (the related potential energy when acting on $|n\rangle$ gives $V(0, p)\,|n\rangle_{q=0} = [\frac{1}{2}\,\mu\omega^2 l^2 f^2(0, p) - \frac{1}{2}\,\hbar\omega]\,|n\rangle_{q=0}$).

One can see from the assumptions (2) and (7) that $f(q, p)$ has the property of $f(-q, -p) = -f(q, p)$. Since for the ground state $p|0\rangle_{q=0} = 0$ (or more explicitly $(-i\hbar\partial\langle q|0\rangle/\partial q)_{q=0} = 0$), this property implies that

$$f(0, p)\,|0\rangle_{q=0} = 0. \tag{39}$$

Thus the initial stress vanishes for ground state $n = 0$. Note that $[\partial f(q, p)/\partial q]\,|0\rangle = 0$ from Eq. (8) (with $M \to M - m_0$). So, $f(q, p)\,|0\rangle = 0$ for any q in consequence of Eq. (39).

Now, the quantum fields (say, the electromagnetic and colour), when dressing our so far *bare* particle, modify its equilibrium (for an analogy recall the dressing of a hydrogen atom by the electromagnetic field). So, in our model, they should introduce a ground-state initial stress into the mass operator (by changing in the simplest case the particle radius r_0: $r_0 \to r_0 + \delta r_0$ then $\tilde{q} \to \tilde{q} - \delta r_0$, where one can write $\delta r_0 = -lf_0$). Thus, in the simplest case, the resulting dressed mass operator may have the form as given in Eq. (12) but with $(q, p) \to f(q, p) + f_0$ (and $M \to M - m_0$),

$$(M - m_0)c^2 = \frac{1}{2\mu}\,p^2 + \frac{\mu\omega^2}{2}\,[q + lf(q, p) + lf_0]^2 - \frac{\hbar\omega}{2}, \tag{40}$$

where f_0 is a dimensionless constant determining the ground-state initial stress, whereas $f(q, p)$ remains unchanged satisfying Eq. (7) (with $M_{f_0=0} \to M_{f_0=0} - m_0$):

$$i\alpha Q^2[p, f(q, p)] = (M_{f_0=0} - m_0)c. \tag{41}$$

Hence, one gets for M the formula

$$(M - m_0)c^2 = \hbar\omega[\tfrac{1}{2}(\lambda^2 + 1)N + g(a + a^+) + g^2] \tag{42}$$

with

$$g = f_0\alpha Q^2\,\frac{\lambda^2 - 1}{\lambda^2 + 1}\sqrt{\frac{2\mu c^2}{\hbar\omega}} = f_0\alpha(\lambda^2 - 1)\sqrt{\frac{\mu Q^2}{\varepsilon(\lambda^2 + 1)}} \tag{43}$$

playing the role of a "quasiphonon" coupling constant whose unknown magnitude depends on f_0. For a and a^+ one has still the commutation relations (11).

In the representation defined by n-states of the bare particle one obtains from Eq. (42) the non-diagonal mass matrix $M = (\langle n'|M|n\rangle)$ with the elements

$$\langle n'|M|n\rangle = (m_n + g^2\kappa)\delta_{n'n} + g\kappa(\sqrt{N_n}\,\delta_{n'n-1} + \sqrt{N_{n+1}}\,\delta_{n'n+1}) \tag{44}$$

(cf. Eqs. (15) and (16)), where m_n and N_n are given in Eqs. (23) and (17), respectively, and

$$\kappa = \frac{\hbar\omega}{c^2} = \frac{2\varepsilon Q^2}{\lambda^2+1} = \frac{2(m_1-m_0)}{\lambda^2+1} \tag{45}$$

(cf. Eq. (21)). For charged leptons $\kappa = (12.4\ \text{MeV}/c^2)Q^2 + O(g^2)$ with $Q^2 = 1$ and for quarks $\kappa \simeq (0.51\ \text{GeV}/c^2)Q^2 + O(g^2)$. Note that

$$g^2\kappa = \left(f_0\alpha Q\,\frac{\lambda^2-1}{\lambda^2+1}\right)^2 2\mu. \tag{46}$$

If only three generations $n = 0, 1, 2$ are relevant, Eq. (44) gives the nondiagonal mass matrix of the form

$$M = \begin{pmatrix} m_0+g^2\kappa, & g\kappa & , & 0 \\ g\kappa & , & m_1+g^2\kappa, & , & g\kappa\sqrt{\lambda^2+1} \\ 0 & , & g\kappa\sqrt{\lambda^2+1}, & m_2+g^2\kappa \end{pmatrix}, \tag{47}$$

where $m_1 = m_0 + \varepsilon Q^2$ and $m_2 = m_0 + \varepsilon Q^2(\lambda^2+1)$. For the matrix (47) one finds the following eigenvalues:

$$\left.\begin{aligned} M_0 &= m_0 + g^2\kappa\left(1 - \frac{\kappa}{m_1-m_0}\right) \\ M_1 &= m_1 + g^2\kappa\left[1 + \frac{1}{2}\,\frac{\kappa}{m_1-m_0}\left(1 - \frac{m_2+m_1-2m_0}{m_2-m_1}\right)\right] \\ M_2 &= m_2 + g^2\kappa\left[1 + \frac{1}{2}\,\frac{\kappa}{m_1-m_0}\left(1 + \frac{m_2+m_1-2m_0}{m_2-m_1}\right)\right] \end{aligned}\right\} + O(g^4), \tag{48}$$

where $\kappa(m_1-m_0)^{-1} = 2(\lambda^2+1)^{-1}$ and $(m_2+m_1-2m_0)\,(m_2-m_1)^{-1} = (\lambda^2+2)\lambda^{-2}$. If in the mass matrix M only two generations $n = 0, 1$ were relevant, one would obtain

$$\left.\begin{aligned} M_0 &= m_0 + g^2\kappa\left(1 - \frac{\kappa}{m_1-m_0}\right) \\ M_1 &= m_1 + g^2\kappa\left(1 + \frac{\kappa}{m_1-m_0}\right) \end{aligned}\right\} + O(g^4). \tag{49}$$

Eqs. (48) and (49) display an evident difference for M_1 already in the second order in g.

One can see from Eq. (44) that $m_n = M_n + O(g^2)$, where M_n are eigenstates of M. Since $M_n \equiv m_{n\,\mathrm{exp}}$ one gets consequently $\lambda = \lambda_{\mathrm{exp}} + O(g^2)$ and $\kappa = \kappa_{\mathrm{exp}} + O(g^2)$, where λ_{exp} and κ_{exp} are equal to λ and κ as they were determined for leptons and quarks before (when g was zero):

$$\lambda^2_{\mathrm{exp}} \equiv \begin{cases} (m_\tau - m_\mu)(m_\mu - m_e)^{-1} \\ \text{or} \\ (4m_b - m_d)(m_c - m_u)^{-1} \end{cases}, \tag{50}$$

$$\kappa_{\mathrm{exp}} \equiv \frac{2\varepsilon_{\mathrm{exp}} Q^2}{\lambda^2_{\mathrm{exp}} + 1} \equiv \begin{cases} 2(m_\mu - m_e)(\lambda^2_{\mathrm{exp}} + 1)^{-1} \\ \text{or} \\ 2(m_b - m_d)(\lambda^2_{\mathrm{exp}} + 1)^{-2} \end{cases}$$

(in the latter expression $(m_b - m_d)(\lambda^2_{\mathrm{exp}} + 1)^{-1} = m_s - m_d$ defines $m_s \simeq 0.38\ \mathrm{GeV}/c^2$ if $m_s \gg m_d$). Thus, one can conclude from Eq. (48) that

$$\left.\begin{aligned} g^2 &= \frac{M_0 - m_0}{\kappa_{\mathrm{exp}}\left(1 - \dfrac{2}{\lambda^2_{\mathrm{exp}} + 1}\right)} \\ m_1 &= M_1 - g^2 \kappa_{\mathrm{exp}}\left[1 + \frac{1}{\lambda^2_{\mathrm{exp}} + 1}\left(1 - \frac{\lambda^2_{\mathrm{exp}} + 2}{\lambda^2_{\mathrm{exp}}}\right)\right] \\ m_2 &= M_2 - g^2 \kappa_{\mathrm{exp}}\left[1 + \frac{1}{\lambda^2_{\mathrm{exp}} + 1}\left(1 + \frac{\lambda^2_{\mathrm{exp}} + 2}{\lambda^2_{\mathrm{exp}}}\right)\right] \end{aligned}\right\} + O(g^4). \tag{51}$$

From Eqs. (50) and (51) one may calculate g^2, m_1, m_2 as well as λ and κ (all up to $O(g^2)$) in terms of the experimental values of M_0, M_1 and M_2, if m_0 is known. Henceforth we will assume $m_0 = 0$ in accordance with the original formulation of our model, where Eqs. (2) and (7) hold (without the substitution $M_{f_0=0} \to M_{f_0=0} - m_0$).

In the case of charged leptons (where $M_0 \equiv m_e$, $M_1 \equiv m_\mu$ and $M_2 \equiv m_\tau$), one gets for λ_{exp} and $\varepsilon_{\mathrm{exp}}$ the values (25) (giving $\kappa_{\mathrm{exp}} = (12.4\ \mathrm{MeV}/c^2)Q^2$ with $Q^2 = 1$), and for the "quasiphonon" coupling constant g the reasonably small value

$$g^2 = 0.0467 = \frac{1}{21.4}. \tag{52}$$

In the case of quarks (where $M_0 = m_u$ or m_d, $M_1 \equiv m_c \simeq 1.5\ \mathrm{GeV}/c^2$ or m_s and $M_2 \equiv m_t$ or $m_b \simeq 5\ \mathrm{GeV}/c^2$), one obtains for λ_{exp} and $\varepsilon_{\mathrm{exp}}$ the values (29) (implying $\kappa_{\mathrm{exp}} \simeq (0.51\ \mathrm{GeV}/c^2)Q^2$) and for g the values

$$g^2 \simeq 0.021 = \tfrac{1}{48} \tag{53}$$

or

$$g^2 \simeq 0.14 = \frac{1}{6.9} \tag{54}$$

for up or down quarks, respectively. Here we used the popular current masses $m_u \simeq 4\ \mathrm{MeV}/c^2$ and $m_d \simeq 7\ \mathrm{MeV}/c^2$.

The n-states of the *dressed* particle, $|n\rangle^D$, when expanded into the n-states of the bare particle, $|n\rangle$, define the transformation coefficients $\langle n'|n\rangle^D = \langle n'|U|n\rangle$. They are matrix elements of the unitary transformation operator U which mixes the *bare* n'-states in order to produce the *dressed* n-states:

$$|n\rangle^D = \sum_{n'} |n'\rangle\langle n'|U|n\rangle = U|n\rangle, \tag{55}$$

where

$$M|n\rangle^D = M_n|n\rangle^D \quad (n = 0, 1, 2, \ldots) \tag{56}$$

or

$$\sum_{n''} \langle n'|M|n''\rangle\langle n''|U|n\rangle = M_n\langle n'|U|n\rangle \tag{57}$$

is the eigenvalue equation for the mass operator M.

In the case of only three relevant generations $n = 0, 1, 2$, where the mass matrix M has the form (47) implying the eigenvalues (48), one finds the following orthogonal transformation operator:

$$U = \begin{bmatrix} 1-\frac{1}{2}\frac{g^2\kappa^2}{m_1} & , \frac{g\kappa}{m_1} & , \frac{g^2\kappa^2}{m_1^2\lambda^2\sqrt{\lambda^2+1}} \\ -\frac{g\kappa}{m_1} & , 1-\frac{1}{2}\frac{g^2\kappa^2}{m_1^2}\left(1+\frac{\lambda^2+1}{\lambda^4}\right), & \frac{g\kappa\sqrt{\lambda^2+1}}{m_1\lambda^2} \\ \frac{g^2\kappa^2}{m_1^2\sqrt{\lambda^2+1}}, & -\frac{g\kappa\sqrt{\lambda^2+1}}{m_1\lambda^2} & , 1-\frac{1}{2}\frac{g^2\kappa^2(\lambda^2+1)}{m_1^2\lambda^4} \end{bmatrix} + O(g^3), \tag{58}$$

where $m_2 = (\lambda^2+1)m_1$ was used. Here, the columns describe three eigenvectors $|n\rangle^D = (\langle n'|U|n\rangle)$ $(n = 0, 1, 2)$ of M in the basis of $|n'\rangle$ $(n' = 0, 1, 2)$.

Since all terms in Eq. (58) (except for 1's) are of the order of g or g^2, one can here replace $m_1 \to M_1$, $\lambda \to \lambda_{\mathrm{exp}}$ and $\kappa \to \kappa_{\mathrm{exp}}$, not changing the result up to the second order in g. Denoting $s = g\kappa_{\mathrm{exp}}/M_1$ and putting in the case of quarks $\lambda_{\mathrm{exp}} \simeq 3.5$ as determined from $m_c \simeq 1.5\ \mathrm{MeV}/c^2$ and $m_b \simeq 5\ \mathrm{GeV}/c^2$ (cf Eq. (29)), one obtains the estimation valid up to the second order in g:

$$U \simeq \begin{bmatrix} 1-\frac{1}{2}s^2, & s & , 0.002s^2 \\ -s & , 1-\frac{1\cdot 1}{2}s^2, & 0.30s \\ 0.27s^2 & , -0.30s & , 1-\frac{0.088}{2}s^2 \end{bmatrix}. \tag{59}$$

Here, $M_1 \equiv m_c \simeq 1.5\ \text{GeV}/c^2$ or $m_s \simeq 0.38\ \text{GeV}/c^2$ and $\kappa_{\text{exp}} \simeq (0.51\ \text{GeV}/c^2)Q^2$ with $Q = 2/3$ or $-1/3$. Thus, if taking the previously estimated $g^2 \simeq 0.021$ or 0.14, one gets $s \simeq \pm 0.022$ or ± 0.057 for up or down quarks, respectively.

However, Eq. (48) may be expected to fail in giving the "true mass" of s quark which is the lightest excited quark ($n = 1$ state in d family). In order to repair the value of s for down quarks we will introduce an unknown parameter ξ (a "parameter of ignorance") such that $s = \xi g\kappa_{\text{exp}}/M_1 \simeq \pm 0.057\,\xi$ for d family (where $M_1 \equiv m_s \simeq 0.38\ \text{GeV}/c^2$), whilst still $s = g\kappa_{\text{exp}}/m_1 \simeq \pm 0.022$ for u family (where the lowest excited mass $M_1 \equiv m_c \simeq 1.5$ GeV/c^2 is considerably heavier and so approaches the "true mass"). Then $M_1/\xi \equiv m_s/\xi$ may be interpreted as a "corrected mass" of s quark.

8. *Cabibbo-like mixing*

Four fermionic families $f = \nu$, e, u, d (each containing several generations $n = 0, 1, 2, \ldots$) can be labelled by charge and colour: $f = (Q, C)$ with $f = (0, 0), (-1, 0), (2/3, 4/3), (-1/3, 4/3)$, respectively. In our model, the quantum number n is defined by the eigenvalue equation $N|n\rangle = N_n|n\rangle$ for the operator $N = a^+a$, where a and a^+ satisfy the commutation relation (11). The latter involves the Q-independent parameter λ which in an approximation is also C-independent: $\lambda = \lambda(C)$ is a gentle function. Thus we get the gentle functions of C: $|n\rangle = |n\rangle_C$ and $N_n = N_{nC}$, the latter given in Eq. (17). The unperturbed mass operator $(M_{f_0=0} - m_0)c^2 = \hbar\omega\,\frac{1}{2}(\lambda^2+1)N$, besides on λ, depends also on the parameter ω depending in turn both on Q and C: $\omega = \omega(Q, C)$. So we have

$$M_{f_0=0}|nQC\rangle = m_{nQC}|nQC\rangle, \tag{60}$$

where

$$|nQC\rangle = |n\rangle_C|Q\rangle\,|C\rangle \tag{61}$$

are simultaneous eigenstates of N, charge and colour (the latter described by the quadratic Casimir operator of colour SU(3)), and

$$m_n = m_{nQC} = m_{0QC} + \frac{\hbar\omega(Q, C)}{c^2}\,\frac{1}{2}\,[\lambda^2(C)+1]N_{nC}. \tag{62}$$

In conclusion we can see that the eigenstates $|n\rangle = |n\rangle_C$ of N can be considered as *identical* for both lepton families ν and e ($C = 0$) and for both quark families u and d ($C = 4/3$). In an approximation (which may be of a fundamental physical importance), they are identical also for lepton and quark families.

When the bare mass operator $M_{f_0=0}$ is perturbed by the initial stress determined by the parameter f_0, passing into the dressed mass operator M [7], we obtain

$$M|nQC\rangle^{\text{D}} = M_{nQC}|nQC\rangle^{\text{D}}, \tag{63}$$

[7] Other possible effects of dressing our bare particle on its mass operator and its wave function are not discussed in this paper.

where

$$|nQC\rangle^{\mathrm{D}} = |n\rangle^{\mathrm{D}}_{QC}|Q\rangle\,|C\rangle \tag{64}$$

and $M_n = M_{nQC}$ is given, in the case of three generations, in Eq. (48). The functions $|n\rangle^{\mathrm{D}} = |n\rangle^{\mathrm{D}}_{QC}$ may be gentle with respect to both indices Q and C because the function $|n\rangle = |n\rangle_C$ is gentle and $|n\rangle^{\mathrm{D}} = |n\rangle + O(g)$ where $g = g(f_0, Q, C)$ may be small enough.

The wave function (or the quantum field) of our bare extended particle can be written down as

$$\psi_{QC}(x, q) = \sum_n \langle q|n\rangle_C \psi_{nQC}(x). \tag{65}$$

Similarly, for the dressed particle

$$\psi^{\mathrm{D}}_{QC}(x, q) = \sum_n \langle q|n\rangle^{\mathrm{D}}_{QC} \psi^{\mathrm{D}}_{nQC}(x). \tag{66}$$

Since $|n\rangle^{\mathrm{D}}_{QC} = U_{QC}|n\rangle_C$, the wave functions (or the quantum fields) $\psi_{nQC}(x)$ and $\psi^{\mathrm{D}}_{nQC}(x)$ are connected through the unitary transformation

$$\psi_{nQC}(x) = \sum_{n'} {}_C\langle n|U_{QC}|n'\rangle_C \psi^{\mathrm{D}}_{n'QC}(x). \tag{67}$$

Thus, the charge-changing current for interfamily transitions $\mathrm{e} \to \nu$ or $\mathrm{u} \to \mathrm{d}$ (where $Q \mp 1 \to Q$ with $(Q, C) = (0, 0) \equiv \nu$ or $(-1/3, 4/3) \equiv \mathrm{d}$, respectively) is given by the formula

$$\begin{aligned} J_\mu(x) &= \int_{-\infty}^{+\infty} dq\, \bar{\psi}_{QC}(x, q)\Gamma_\mu \psi_{Q\mp 1C}(x, q) \\ &= \sum_n \bar{\psi}_{nQC}(x)\Gamma_\mu \psi_{nQ\mp 1C}(x) \\ &= \sum_{n'n} \bar{\psi}^{\mathrm{D}}_{n'QC}(x)\, {}_C\langle n'|K^{-1}_{QC}|n\rangle_C\, \Gamma_\mu \psi^{\mathrm{D}}_{nQ\mp 1C}(x) \\ &= \sum_n \bar{\psi}^{K}_{nQC}(x)\Gamma_\mu \psi^{\mathrm{D}}_{nQ\mp 1C}(x), \end{aligned} \tag{68}$$

where

$$K_{QC} = U^{-1}_{Q\mp 1C} U_{QC} \tag{69}$$

is the generalized Cabibbo-Kobayashi-Maskawa (CKM) unitary operator which becomes the usual CKM unitary operator [5] when $(Q, C) = (-1/3, 4/3) \equiv \mathrm{d}$ (corresponding to interfamily transitions $\mathrm{u} \to \mathrm{d}$) and if three generations $n = 0, 1, 2$ are assumed. In Eq. (68) the unitary operator (69) mixes the *dressed* n'-states in the family ν or d, leading to the CKM-like rotated wave function (or quantum field)

$$\psi^{K}_{nQC}(x) = \sum_{n'} {}_C\langle n|K_{QC}|n'\rangle_C \psi^{\mathrm{D}}_{n'QC} \tag{70}$$

with $(Q, C) = (0, 0) \equiv \nu$ or $(-1/3, 4/3) \equiv \mathrm{d}$, respectively. As is well known, this mixing is invisible for leptons if all neutrino masses $M_{\nu_n} \equiv m_{\nu_n \mathrm{exp}}$ are zero or, more generally, are degenerated [6], since then the mass matrix M_ν of ν family commutes (trivially) with the CKM-like matrix K_ν for leptons and hence $\bar{\psi}_\nu M_\nu \psi_\nu = \bar{\psi}^D_\nu(\delta_{n'n} M_{\nu_n})\psi^D_\nu = \bar{\psi}^K_\nu(\psi_{n'n} M_{\nu_n})\psi^K_\nu$. Note that in our model the operatoı M_{QC} as well as the operators U_{QC} and K_{QC} are independent of the Dirac γ_5 matrix. In the representation given by n-states of the bare particle all three are real matrices (cf. Eq.(44)) and, therefore, the two latter are orthogonal (as being unitary).

As to the derivation of Eq. (68) we should like to emphasize that the Q-independence of $|n\rangle = |n\rangle_C$ plays there a crucial role because otherwise we would get the bilinear form in the generations $n = 0, 1, 2, \ldots$:

$$J_\mu(x) = \sum_{n'n} \bar{\psi}_{n'QC}(x)\; {}_{QC}\langle n'|n\rangle_{Q\mp 1C} \Gamma_\mu \psi_{nQ\mp 1C}(x), \tag{71}$$

where in general ${}_{QC}\langle n'|n\rangle_{Q\mp 1C} \neq 0$ for $n' \neq n$. The reason is that in this case $|n\rangle_{QC}$ and $|n\rangle_{Q\mp 1C}$ would be eigenstates of two different operators N_{QC} and $N_{Q\mp 1C}$. Fortunately, in our model the operator N is independent of Q: $N = N_C$ (and approximately of C: $N = N_C$ is a gentle function of C). This remark reveals, however, a possible additional mechanism (beside that given by the non-diagonal mass matrix) of spoiling the separate conservation laws for sequential leptonic and quarkonic numbers foı the generations $n = 0, 1, 2, \ldots$ (and for $C = 0$ and $C = 4/3$). Such a mechanism might be provided in our model by introducing a slight Q-dependence of the operator $N = N_{QC}$ related to a gentle function $\lambda = \lambda(Q, C)$.

In the case of quarks the CKM unitary operator can be estimated in our model from Eq. (59) applied to u and d families. Then using its definition $K_d = U_u^{-1} U_d$ we calculate up to the second order in g_u and g_d:

$$K_d \simeq \begin{pmatrix} 1-\frac{1}{2}(s_d-s_u)^2 & , \quad s_d-s_u & , \quad 0.022s_d^2-0.30s_d s_u+0.27s_u^2 \\ -(s_d-s_u) & , \quad 1-\frac{1\cdot 1}{2}(s_d-s_u)^2, & 0.30(s_d-s_u) \\ 0.27s_d^2-0.30s_d s_u+0.022s_u^2, & -0.30(s_d-s_u), & , \quad 1-\frac{0.088}{2}(s_d-s_u)^2 \end{pmatrix}, \tag{72}$$

where $s_u = g_u \kappa_{\exp u}/m_c$ and $s_d = \xi g_d \kappa_{\exp d}/m_s$. Taking the experimental value of Cabibbo angle as an input, $s_d - s_u = \sin\theta_C = 0.219$, and putting $m_c \simeq 1.5$ GeV/c^2, $m_s \simeq 0.38$ GeV/c^2 and $\kappa_{\exp} \simeq (0.51$ GeV/$c^2)Q^2$ we can evaluate

$$s_u \simeq \pm 0.022, \qquad s_d = \begin{cases} 0.24 \\ 0.20 \end{cases}, \qquad \xi = \begin{cases} 4.2 \\ 3.5 \end{cases} \tag{73}$$

if $s_u s_d > 0$ or < 0, respectively. Then the "corrected mass" of s quark is $m_s/\xi = 90$ MeV/c^2 or 110 MeV/c^2. Here, we used the previous estimate $s_u \simeq \pm 0.022$ and $s_d \simeq \pm 0.057\xi$, corresponding to $m_u \simeq 4$ MeV/c^2 and $m_d \simeq 7$ MeV/c^2 but containing the extra parameter ξ.

Then from Eq. (72) we obtain

$$K_d \simeq \begin{bmatrix} 0.98 , & 0.22 , & \begin{Bmatrix} -0.00018 \\ 0.0023 \end{Bmatrix} \\ -0.22 , & 0.97 , & 0.066 \\ \begin{Bmatrix} 0.014 \\ 0.012 \end{Bmatrix}, & -0.066, & 1.00 \end{bmatrix} . \tag{74}$$

Comparing the matrix (74) with its standard form [5] (for real matrix elements)

$$K_d = \begin{pmatrix} c_1 , & s_1 c_3 , & s_1 s_3 \\ -s_1 c_2, & c_1 c_2 c_3 + s_2 s_3, & c_1 c_2 s_3 - s_2 c_3 \\ -s_1 s_2, & c_1 s_2 c_3 - c_2 s_3, & c_1 s_2 s_3 + c_2 c_3 \end{pmatrix}, \tag{75}$$

where $s_i = \sin \theta_i$ and $c_i = \cos \theta_i$ ($i = 1, 2, 3$ and $s_1 c_3 = \sin \theta_C$), we can conclude that up to the second order in g_u and g_d

$$s_1 \simeq 0.22, \quad s_2 \simeq \begin{Bmatrix} -0.065 \\ -0.054 \end{Bmatrix}, \quad s_3 \simeq \begin{Bmatrix} -0.00082 \\ 0.011 \end{Bmatrix} . \tag{76}$$

As a check, it is worthwhile to observe that the orthogonality condition is satisfied quite well for the matrix (74), though it was calculated perturbatively. In fact,

$$K_d^{-1} \simeq \begin{bmatrix} 0.98 , & -0.22 , & \begin{Bmatrix} 0.015 \\ 0.012 \end{Bmatrix} \\ 0.22 , & 0.97 , & \begin{Bmatrix} -0.064 \\ -0.065 \end{Bmatrix} \\ \begin{Bmatrix} 0.00056 \\ 0.0028 \end{Bmatrix}, & 0.067, & 1.00 \end{bmatrix} \tag{77}$$

(and Det $K_d \simeq 1.0005$ in both cases), so that $K_d^{-1} \simeq K_d^T$.

An algebraic relationship of s_i ($i = 1, 2, 3$) with s_u and s_d follows from the comparison of the matrix (72) valid up to the second order in g_u and g_d with the standard form (75):

$$s_1 \simeq s_d - s_u, \quad s_2 - s_3 \simeq -0.30(s_d - s_u), \quad s_2 + s_3 \simeq -0.25(s_d + s_u). \tag{78}$$

From the definition of s_u and s_d and from Eqs. (50) and (51) for κ_{exp} and g^2 we can write

$$s_u = \pm x \sqrt{\frac{m_u}{m_c}}, \quad s_d = \xi x \sqrt{\frac{m_d}{m_s}} = \sqrt{\frac{m_d}{m_s (\xi x)^{-2}}}, \tag{79}$$

where

$$x = \sqrt{\frac{\kappa_{exp\,u}}{m_c - \kappa_{exp\,u}}} = \sqrt{\frac{\kappa_{exp\,d}}{m_s - \kappa_{exp\,d}}} \simeq \sqrt{\frac{2}{\lambda_{exp}^2 - 1}} \simeq 0.42. \tag{80}$$

Here, $\xi x \simeq 1.8$ or 1.5 and another "corrected mass" of s quark $m_s(\xi x)^{-2} \simeq 120$ MeV/c^2 or 180 MeV/c^2 may be considered.

9. *Conclusion*

The model described in this paper suggests that the puzzling phenomenon of fermionic generations may, in fact, be caused by "quasiacoustic" excitations of fermionic internal charge distribution kept in equilibrium by joint action of internal electromagnetic forces and Poincaré stresses (the latter representing the effect of other internal forces). However, the way in which these "quasiacoustic" excitations may be generated is another essential ingredient of the model, where the mass operator (or hamiltonian) is defined only implicitly through a closed logical loop: mass depends on mass which depends on mass which ... . Nevertheless, the specific, quasiharmonic-oscillator model proposed in the paper is exactly solvable, giving explicitly the exponential mass spectrum $m_n - m_0 = \varepsilon Q^2 N_n$ $(n = 0, 1, 2, \ldots)$, where $N_n = (\lambda^{2n}-1)(\lambda^2-1)^{-1}$. The fitted λ is ca. 4 and 3.5 and the fitted ε ca. 105 MeV/c^2 and 3.4 GeV/c^2 for charged leptons and quarks, respectively. The excitations having energies $\varepsilon c^2 Q^2 N_n$ are here "quasiphonons". Since $m_n - m_0 \sim Q^2$, all of them are excitations of an electromagnetic mode of internal oscillations (which is largely modified in the case of quarks by internal colour forces).

When an initial stress appears in the quasiharmonic oscillator, the model is perturbed and gives a mixing of the previous unperturbed n-states that may be consequently interpreted as Cabibbo-like mixing of fermionic generations $n = 0, 1, 2, \ldots$.

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