ON THE UNIFIED AFFINE ELECTROMAGNETISM AND GRAVITATION THEORIES

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It is shown, by a different method than used originally, that the gauge theory proposed by Ferraris and Kijowski is an interacting gravitational and electromagnetic fields theory. Using a new method, the affine Lagrangians of the linear Maxwell-Einstein and of the Born-Infeld-Einstein electrodynamics are derived. Two theorems relating the affine and the matter Lagrangians for the electromagnetic field are proved.

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1. Introduction

It was shown by Ferraris and Kijowski [3] that the Einstein-Maxwell theory may be formulated as a gauge theory, with an invariant Lagrangian depending on a GL(4, R) connection, obtaining thus a unified description of the gravitational and electromagnetic interactions. The spirit of the proof presented by Ferraris and Kijowski [3] may seem to be somewhat inconsistent with the unification idea, since it requires the splitting of the connection into different parts, which acquire "an autonomous existence" in the derivation of the equivalence of the unified theory with the standard one. A derivation which avoids such a splitting is presented in Section 2.

The Lagrangian of the unified theory was derived by Ferraris and Kijowski ([1, 2]). They used an ingeneous method which makes appeal to coordinates adapted both to the gravitational and to the electromagnetic field. It seemed interesting to find a method of deriving the affine Lagrangian (the Lagrangian from which the metric has been eliminated by performing a Legendre transformation, see Kijowski [4]) which avoids the use of such coordinates, with the hope that such a method could be applied to other field theories, such as the Yang-Mills theory, for example. Although this last hope revealed too optimistic, a systematic method of calculating the affine Lagrangian for any non-linear Einstein-

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-Maxwell theory has been obtained. It is presented in its application for the linear and for the Born-Infeld electrodynamics in Sections 4 and 5.

The comparison of the results of Section 5 with the results of Ferraris and Kijowski [2] suggested the existence of very simple relations between the affine and the matter Lagrangians. These relations are proved in Section 6, in the case of the electromagnetic field. It is also shown, that the theorems proved in Section 6 do not generalize neither to the scalar field, noı to the Yang-Mills fields cases.

2. The unified theory of electromagnetism and gravitation

The most general coordinate-invariant first order Lagrangian for a GL(4, R) connection field theory must be of the form

$$L(\Gamma^{\lambda}_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu,\varrho}) = L(Q^{\lambda}_{\mu\nu}, R^{\lambda}_{\mu\nu\varrho}, Q^{\lambda}_{\mu\nu;\varrho}),$$

where

$$Q^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\lambda}_{\mu\nu} \tag{1}$$

is the torsion of the connection,

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}_{\beta\gamma} \tag{2}$$

is the curvature tensor of the connection $\Gamma^{\lambda}_{\mu\nu}$, and L is a density of weight 1 (greek indices run from 1 to 4, a coma or a ∂ denotes a partial derivative, a semi-colon or a nabla (∇) denotes covariant differentiation with respect to the connection $\Gamma^{\lambda}_{\mu\nu}$: $X^{\mu}{}_{;\nu} = X^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\lambda\nu}X^{\lambda}$). If we restrict the Lagrangian to depend only upon the curvature tensor, we can define the canonical momentum conjugated to $\Gamma^{\lambda}_{\mu\nu}$ as

$$\pi^{\mu\nu\varrho}_{\lambda} = \frac{\partial L(R^{\alpha}_{\beta\gamma\delta})}{\partial \Gamma^{\lambda}{}_{\mu\nu,\varrho}} = 2\,\frac{\partial L(R^{\alpha}_{\beta\gamma\delta})}{\partial R^{\lambda}{}_{\mu\varrho\nu}}\,. \tag{3}$$

From the definition of $\pi^{\mu\nu\varrho}_{\lambda}$, and the invariance of the Lagrangian, it follows that $\pi^{\mu\nu\varrho}_{\lambda}$ is a tensor density of weight 1, antisymmetric in the last two indices. The Euler-Lagrange equations for such a theory

$$\partial_{\lambda}\frac{\partial L}{\partial \Gamma^{\mu}_{\nu\varrho,\lambda}} = \frac{\partial L}{\partial \Gamma^{\mu}_{\nu\varrho}}$$

may be written in a tensor form

$$\nabla_{\alpha}\pi^{\mu\nu\alpha}_{\lambda} = Q^{\alpha}_{\alpha\beta}\pi^{\mu\nu\beta}_{\lambda} + \tfrac{1}{2}\,Q^{\nu}_{\alpha\beta}\pi^{\mu\beta\alpha}_{\lambda}. \tag{4}$$

For the description of the electromagnetic field we need a two-covariant antisymmetric tensor $F_{\mu\nu}$, which fulfills the "first pair" of Maxwell-equations:

$$F_{[\mu\nu,\varrho]} = 0. \tag{5}$$

The following contraction

$$F_{\mu\nu} = R^{\alpha}_{\alpha\mu\nu} \tag{6}$$

of the curvature tensor has the required property (5), since, from Bianchi's identity

$$R^{\alpha}_{\beta[\mu\nu;\varrho]} = -R^{\alpha}_{\beta\sigma[\mu}Q^{\sigma}_{\nu\varrho]},$$

one can verify that equation (5) is fulfilled identically.

One needs a symmetric tensor to describe the gravitational field. As proposed by Ferraris and Kijowski in [3], the following quantity[1]:

$$\pi^{\mu\nu} = \frac{\partial L}{\partial K_{\mu\nu}}, \tag{7}$$

where

$$K_{\mu\nu} = \tfrac{1}{2}(R^{\alpha}_{\mu\alpha\nu}+R^{\alpha}_{\nu\alpha\mu}) \tag{8}$$

is the symmetric part of the Ricci tensor, will be interpreted as the contravariant metric density

$$\pi^{\mu\nu} = -\sqrt{-\det g_{\alpha\beta}}\, g^{\mu\nu} \tag{9}$$

(the units $8\pi G = c = \hbar = 1$ are used).

To obtain a theory without other fields than the electromagnetic and the gravitational fields, it will be assumed that the Lagrangian depends only upon $K_{\mu\nu}$ and $F_{\mu\nu}$

$$L = L(K_{\mu\nu}, F_{\mu\nu}). \tag{10}$$

The electromagnetic induction density field $\hat{F}^{\mu\nu}$ is defined as usually by[1]

$$\hat{F}^{\mu\nu} = -2\frac{\partial L}{\partial F_{\mu\nu}}. \tag{11}$$

It follows from (10), that the momentum $\pi^{\mu\nu\alpha}_{\lambda}$ has the following form[1]

$$\pi^{\mu\nu\alpha}_{\lambda} = 2\pi^{\mu[\nu}\delta^{\alpha]}_{\lambda}+\hat{F}^{\nu\alpha}\delta^{\mu}_{\lambda}. \tag{12}$$

From (4) and (12) one obtains the equations of motion of the theory

$$2\nabla_{\alpha}\pi^{\mu[\nu}\delta^{\alpha]}_{\lambda}+\nabla_{\alpha}\hat{F}^{\nu\alpha}\delta^{\mu}_{\lambda} = 2Q^{\alpha}_{\alpha\beta}\pi^{\mu[\nu}\delta^{\beta]}_{\lambda}+Q^{\nu}_{\lambda\alpha}\pi^{\mu\alpha}+Q^{\alpha}_{\alpha\beta}\hat{F}^{\nu\beta}\delta^{\mu}_{\lambda}+Q^{\nu}_{\alpha\beta}\hat{F}^{\beta\alpha}\delta^{\mu}_{\lambda}/2. \tag{13}$$

Contracting μ with λ in (13) yields

$$\nabla_{\alpha}\hat{F}^{\nu\alpha} = Q^{\alpha}_{\alpha\beta}\hat{F}^{\nu\beta}+\tfrac{1}{2}Q^{\nu}_{\alpha\beta}\hat{F}^{\beta\alpha}. \tag{14}$$

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[1] In this paper, the convention that $\partial L/\partial K_{\mu\nu}$ is one half of the usual derivative of L with respect to the set of independent variables $\{K_{\mu\nu}, \mu \leqslant \nu\}$ ($K_{\mu\nu}$ is symmetric by its definition) is used (as in [1] and [2]), therefore $dL = \pi^{\mu\nu}dK_{\mu\nu}+\pi_A d\phi^A$ and not $\pi^{\mu\nu}dK_{\mu\nu}/2$ (note the change of convention between [1], [2] and [4]). This remark applies also to differentiation with respect to $F_{\mu\nu}$, $g_{\mu\nu}$, etc.

Taking into account that $\hat{F}^{\alpha\beta}$ is a tensor density, one obtains

$$\partial_\alpha \hat{F}^{\nu\alpha} = 0. \tag{15}$$

Equations (11), (15) and (5) show that the fields $F_{\alpha\beta}$ and $\hat{F}^{\alpha\beta}$ fulfill the full set of Maxwell equations, foı a possibly nonlinear electrodynamics. If one takes into account Eq. (14), equation (13) becomes

$$\nabla_\alpha \pi^{\mu\alpha}\delta^\nu_\lambda - \nabla_\lambda \pi^{\mu\nu} = Q^\alpha_{\alpha\beta}\pi^{\mu\beta}\delta^\nu_\lambda - Q^\alpha_{\alpha\lambda}\pi^{\mu\nu} + Q^\nu_{\alpha\lambda}\pi^{\mu\alpha}. \tag{16}$$

Since every connection may be written in the form

$$\Gamma^\lambda_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\} + C^\lambda_{\mu\nu}, \tag{17}$$

where $\{{}^{\lambda}_{\mu\nu}\}$ is the Christoffel symbol built from $g_{\mu\nu}$ and its derivatives, equations (16) are algebraic-constraints equations, which fix the form of $C^\lambda_{\mu\nu}$. The electıomagnetic field is determined by the trace of the connection $A_\mu = \Gamma^\lambda_{\lambda\mu}$: $F_{\mu\nu} = R^\alpha_{\alpha\mu\nu} = \partial_\mu \Gamma^\alpha_{\alpha\nu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Equations (16) must deteımine $C^\lambda_{\mu\nu}$ uniquely through the metric connection and A_μ, without imposing any constraints on them (constraints on A_μ would imply a theory poorer than electrodynamics, non-uniqueness of $C^\lambda_{\mu\nu}$ would imply the undesired existence of supplementary fields). Let us show that this is indeed the case. A contraction of ν with λ in (16) gives

$$\nabla_\alpha \pi^{\mu\alpha} = 2Q^\alpha_{\alpha\beta}\pi^{\mu\beta}/3.$$

Therefore

$$\nabla_\lambda \pi^{\mu\nu} = Q^\nu_{\lambda\alpha}\pi^{\mu\alpha} + Q^\alpha_{\alpha\lambda}\pi^{\mu\nu} - Q^\alpha_{\alpha\beta}\pi^{\mu\beta}\delta^\nu_\lambda/3. \tag{18}$$

It will be assumed, that $\pi^{\mu\nu}$ is non-degenerate. If we denote by

$$Y_\beta = C^\alpha_{\alpha\beta} \qquad Z_\beta = C^\alpha_{\beta\alpha}$$

and use $\pi^{\mu\nu}$, and its inverse, $\pi_{\mu\nu}$, to upper and lower indices, (17) inserted into (18) gives

$$C^\mu{}_{\nu\lambda} = -C_{\nu\lambda}{}^\mu + Z_\lambda \delta^\mu_\nu + (Y^\mu - Z^\mu)\pi_{\nu\lambda}/3.$$

A contraction over μ and ν gives

$$Y_\lambda = 4Z_\lambda.$$

Therefore

$$C_{\mu\nu\lambda} + C_{\nu\lambda\mu} = (\pi_{\mu\nu}Y_\lambda + \pi_{\nu\lambda}Y_\mu)/4. \tag{19}$$

Taking three cyclic permutations of (19), subtracting the last of equations so obtained from the sum of the first and the second gives

$$C^\mu_{\nu\lambda} = \delta^\mu_\nu Y_\lambda/4,$$

which shows that $\Gamma^{\lambda}_{\mu\nu}$ is of the form

$$\Gamma^{\lambda}_{\mu\nu} = \{^{\lambda}_{\mu\nu}\}+\delta^{\lambda}_{\mu}Y_{\nu}/4 = \{^{\lambda}_{\mu\nu}\}+\delta^{\lambda}_{\mu}(A_{\nu}-\{^{\lambda}_{\lambda\nu}\})/4 \tag{20}$$

with A_{μ} — an arbitrary field.

To obtain a theory equivalent to the Einstein-Maxwell linear electrodynamics (linear in the sense that $\hat{F}^{\mu\nu} = \sqrt{-\det g_{\alpha\beta}}\, g^{\mu\varrho}g^{\nu\sigma}F_{\varrho\sigma}$) one has to find a suitable Lagrangian. Since $\Gamma^{\lambda}_{\mu\nu}$ is built from the Christoffel symbols of the metric and the potential A_{μ}, and since

$$K_{\mu\nu} = \dot{K}_{\mu\nu},$$

(where $\dot{K}_{\mu\nu}$ is the (symmetric) Ricci tensor of the curvature of $\{^{\lambda}_{\mu\nu}\}$), this theory may be looked upon as the theory of two interacting fields — the metric connection $\{^{\lambda}_{\mu\nu}\}$ and a gauge field A_{μ}. As was shown by Kijowski in [4], the affine formulation of such a theory may be obtained from the standard version by performing a Legendre transformation, which exchanges $g_{\mu\nu}$ with $K_{\mu\nu}$, equation (7) being the solution of the Einstein equations,

$$K_{\mu\nu}-(g^{\alpha\beta}K_{\alpha\beta})g_{\mu\nu}/2 = T_{\mu\nu}(g_{\alpha\beta}, F_{\alpha\beta}) \tag{21}$$

considered as algebraic equations, with respect to $g_{\mu\nu}$

$$g_{\mu\nu} = g_{\mu\nu}(F_{\alpha\beta}, K_{\varrho\sigma}).$$

The next three sections will be devoted to solving Eq. (21) with respect to $g_{\mu\nu}$. The formulas presented in Section 3 are used in Sections 4 and 5 to derive the affine Maxwell--Einstein and Born-Infeld-Einstein Lagrangians.

3. *Invariant polynomials*

Let V be an n-dimensional vector space. To every linear transformation

$$\mathbf{A} : V \to V,$$

we can assign the following n invariants $f_i(\mathbf{A})$:

$$\det(A+\lambda I) = \Sigma f_i(A)\lambda^i, \tag{22}$$

where A is the matrix of $\mathbf{A}$ in some basis, and I is the identity matrix. In what follows, we will write $f_i(A)$ instead of $f_i(\mathbf{A})$, in order to make the life easier to the printer.

From (22) one obtains

$$f_n(A) = 1 \quad f_{n-1}(A) = \mathrm{Tr}\, A = \text{trace of } A,$$
$$f_0(A) = \det A. \tag{23}$$

From the identity

$$A^{-1}+\lambda I = A^{-1}(I+\lambda A)$$

one obtains immediately

$$f_i(A^{-1}) = f_{n-i}(A)/\det A. \tag{24}$$

We will be chiefly concerned with 4×4 matrices which can be written in the following way

$$A^{\mu}{}_{\nu} = B^{\mu\varrho}C_{\varrho\nu}$$

or, shortly

$$A = BC, \tag{25}$$

where C is an antisymmetric covariant tensor and B a symmetric contravariant one. From the symmetry properties of B and C it follows that

$$\mathrm{Tr}\, A = 0.$$

If B and C aie inveitible, then

$$A^{-1\mu}{}_{\nu} = C^{-1\mu\alpha}B^{-1}{}_{\alpha\nu}.$$

Since C^{-1} is antisymmetric, and B^{-1} is symmetric, it follows that

$$\mathrm{Tr}\, A^{-1} = 0$$

and (24) shows that

$$f_1(A) = 0 \tag{26}$$

therefore

$$\det(A+\lambda I) = \lambda^4+f_2(A)\lambda^2+\det A = \det(A-\lambda I). \tag{27}$$

From

$$\begin{aligned}\det(A^2+\lambda I) &= \det\{(A+i\sqrt{\lambda}I)(A-i\sqrt{\lambda}I)\}\\ &= \det(A+i\sqrt{\lambda}I)^2 = (\lambda^2-f_2(A)\lambda+\det A)^2\end{aligned} \tag{28}$$

it follows that

$$\begin{aligned}f_2(A) &= -\mathrm{Tr}\,(A^2)/2,\\ f_2(A^2) &= f_2^2(A)+2\det A,\\ f_1(A^2) &= \mathrm{Tr}\,(A^2)\det A.\end{aligned} \tag{29}$$

From (24) and (29) one obtains

$$\mathrm{Tr}\,(A^{-2}) = f_3(A^{-2}) = \mathrm{Tr}\,(A^2)/\det A. \tag{30}$$

To invert Einstein equations, the Hamilton-Cayley theorem

$$\sum f_i(A)A^i(-1)^i = 0 \tag{31}$$

will be applied. When A is of the form (25), formula (31) reads

$$A^4-\mathrm{Tr}\,(A^2)A^2/2+\det A\cdot I = 0. \tag{32}$$

When $\det A \neq 0$, (32) may be used to express A^{-1} as a polynomial in A:

$$A^{-1} = (-A^3+\mathrm{Tr}\,(A^2)A/2)/\det A. \tag{33}$$

4. *The unified linear Maxwell-Einstein theory*

The metric-affine Lagrangian for the Einstein-Maxwell theory is

$$L = -\sqrt{-\det g_{\alpha\beta}}\,(g^{\mu\nu}K_{\mu\nu}+g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}/4). \tag{34}$$

The affine Lagrangian is obtained from the Lagrangian (34) by solving Einstein equations for $g_{\mu\nu}$

$$2K_{\mu\nu} = F_{\mu\alpha}g^{\alpha\beta}F_{\beta\nu}-(g^{\alpha\varrho}g^{\beta\sigma}F_{\alpha\beta}F_{\sigma\varrho})g_{\mu\nu}/4. \tag{35}$$

In order to find the affine Lagrangian, we will assume that $F_{\mu\nu}$ is invertible (since the Lagrangian is the generating function of the dynamics in the sense of Kijowski and Tulczyjew [5], its singularity at a zero measure set is irrelevant). Multiplying (35) by $F^{-1\varrho\mu}$ from the left, one obtains the matrix equation

$$B = A-SA^{-1}, \tag{36}$$

where

$$A = g^{-1}F \quad (A^{\mu}{}_{\nu} = g^{\mu\alpha}F_{\alpha\nu}),$$

$$B = 2F^{-1}K \quad (B^{\mu}{}_{\nu} = 2F^{-1\mu\alpha}K_{\alpha\nu}),$$

$$4S = g^{\alpha\varrho}g^{\beta\sigma}F_{\alpha\beta}F_{\sigma\varrho} = \mathrm{Tr}(A^2). \tag{37}$$

We will also adopt the following notation:

$$R_0 = \sqrt{|\det B|}, \qquad R = \sqrt{|\det A|} \quad (R^2 = -\det A),$$

$$S_0 = \mathrm{Tr}(B^2)/4 \tag{38}$$

(since $A = g^{-1}F$, $\det g < 0$, and $\det F > 0$, because F is antisymmetric, it follows that $\det A < 0$). From (36) and (28) one obtains

$$\det B = \det\{A^{-1}(A^2-SI)\} = -(R^2+S^2)^2/R^2, \tag{39}$$

which shows that $\det B < 0$. Equation (39) gives

$$R_0 = R+S^2/R. \tag{40}$$

From (36) one easily obtains

$$B^2 = A^2-2SI+S^2A^{-2}. \tag{41}$$

From (41) and (30) one obtains

$$S_0 = \mathrm{Tr}\,(B^2)/4 = -S-S^3/R^2. \tag{42}$$

Introducing

$$x_0 = S_0/R_0, \qquad x = S/R, \qquad y = S_0/S \tag{43}$$

$$\Rightarrow S = S_0/y, \qquad R = S_0/xy \tag{44}$$

one obtains the following set of equations

$$xy = (1+x^2)x_0,$$
$$y = -(1+x^2). \tag{45}$$

The elimination of y gives

$$(x+x_0)(x^2+1) = 0. \tag{46}$$

The only real solution of this equation is

$$x = -x_0.$$

Which gives

$$S/R = -S_0/R_0,$$
$$S = -S_0R_0^2/(R_0^2+S_0^2), \tag{47}$$
$$R = R_0^3/(R_0^2+S_0^2).$$

The Lagrangian of the affine theory may be obtained from formula (34), by taking into account that

$$g^{\mu\nu}K_{\mu\nu} = 0.$$

From (34) and (47) one obtains

$$L = \sqrt{-\det g}\,\mathrm{Tr}\,(A^2)/4 = \sqrt{-\det g}\,S = -4\sqrt{|\det K|}\,S_0/R_0^2 \tag{48}$$

and we have used the relation

$$g = 2KB^{-1}A^{-1} \tag{49}$$

to eliminate $\det g$.

The formula obtained by Ferraris and Kijowski may be derived from (48), if one notices, using (30), that

$$S_0 = \mathrm{Tr}\,B^2/4 = -\mathrm{Tr}\,(B^{-2})R_0^2/4,$$

which gives finally, in matrix notation

$$L = \sqrt{|\det K|}\,\mathrm{Tr}\,(K^{-1}FK^{-1}F)/4 \tag{50}$$

or, in index notation,

$$L = \sqrt{-\det K_{\varrho\sigma}}\,K^{-1\alpha\beta}F_{\beta\mu}K^{-1\mu\nu}F_{\nu\alpha}/4 \tag{51}$$

($\det K$ is negative in virtue of (49)).

Since we have the Lagrangian of the affine theory, it could be possible to make appeal to the formalism presented by Kijowski in [4] to deduce that this theory is equivalent to the standard Einstein-Maxwell theory. One can avoid this by showing explicitly, that the metric density obtained from (51) by variation of L with respect to $K_{\mu\nu}$ is a solution of Einstein equations with respect to $g_{\mu\nu}$, and that the electromagnetic induction field $\hat{F}^{\mu\nu}$

is just the electromagnetic field tensor $F_{\mu\nu}$ with indices raised with the help of $g^{\alpha\beta}$, and multiplied by the square root of the determinant of g:

$$\hat{F}^{\mu\nu} = -2\frac{\partial L}{\partial F_{\mu\nu}} = \sqrt{-\det g_{\alpha\beta}}\, g^{\mu\varrho}g^{\nu\sigma}F_{\varrho\sigma}. \tag{52}$$

First it will be shown, that the equations

$$2\pi^{\mu\nu} = -2\sqrt{-\det g_{\alpha\beta}}\, g^{\mu\nu} = 2\frac{\partial L}{\partial K_{\mu\nu}} = \sqrt{-\det K_{\alpha\beta}}\, \{K^{-1\mu\alpha}F_{\alpha\varrho}K^{-1\varrho\sigma}F_{\sigma\beta}K^{-1\beta\nu}$$

$$+(K^{-1\varrho\alpha}K^{-1\beta\sigma}F_{\alpha\beta}F_{\sigma\varrho})K^{-1\mu\nu}/4\} \tag{53}$$

are just Einstein equations (21) written another way round. Equation (53), multiplied by $F_{\nu\chi}$ from the right, in matrix notation gives

$$A = RR_0(B^{-3}+S_0B^{-1}/R_0^2), \tag{54}$$

where $\det g$ and $\det K$ have been eliminated with the help of equation (49). From (54) one can calculate $\det A$ and $\operatorname{Tr} A^2$, obtaining equations (47) (one uses the Cayley-Hamilton theorem to express B^{-6} and B^{-4} through B^{-2} and the identity matrix). These equations may be solved for R_0 and S_0 to give equations (40) and (42). Therefore R_0 and S_0 in (54) are known functions of R and S.

From the Cayley-Hamilton theorem, one can express B^{-3} through B and B^{-1}, which gives

$$A = RB/R_0+SB^{-1}. \tag{55}$$

From (55) one can obtain

$$A^2 = RR_0B^{-2}. \tag{56}$$

From (55) and (56) one can calculate A^3, which allows us to evaluate A^{-1} with the help of (33)

$$A^{-1} = B^{-1}-SB/R_0R. \tag{57}$$

Eliminating B^{-1} from (55) and (57) gives Einstein equations in their matrix form (36). This shows that the metric defined through equation (53) is a solution of Einstein equations. It may also be shown (using the above procedure in the reverse order, and expressing R and S through R_0 and S_0 with the use of (40) and (42)), that every solution of Einstein equations is of the form (54), showing that equations (54) are equivalent to Einstein equations (as has been shown by Kijowski in [4] from general considerations).

All it remains to show is, that the tensor density

$$\hat{F}^{\mu\nu} = -2\frac{\partial L}{\partial F_{\mu\nu}} = \sqrt{-\det K_{\alpha\beta}}\, K^{-1\mu\alpha}F_{\alpha\beta}K^{-1\beta\nu}$$

is the usual electromagnetic induction field:

$$\hat{F}^{\mu\nu} = \sqrt{-\det g_{\alpha\beta}}\, g^{\mu\alpha}F_{\alpha\beta}g^{\beta\nu}. \tag{58}$$

From (56) one obtains

$$K^{-1}FK^{-1} = 4B^{-2}F^{-1} = 4g^{-1}Fg^{-1}/R_0R. \tag{59}$$

From (49) and (59), one easily obtains equation (58).

5. Born-Infeld-Einstein electrodynamics

The most general, gauge and coordinate invariant, first order Lagrangian describing the electromagnetic field can be written in the form

$$L_m = \sqrt{-\det g_{\alpha\beta}}\, f(R, S), \tag{60}$$

where f is any suitably differentiable scalar function of the two invariants

$$S = \mathrm{Tr}\,(g^{-1}Fg^{-1}F)/4,$$
$$R = \sqrt{-\det (g^{-1}F)}$$

(the 1/4 factor is chosen for later convenience). Since (in matrix notation)[1]

$$\frac{\partial S}{\partial g^{-1}} = Fg^{-1}F/2,$$
$$\frac{\partial R}{\partial g^{-1}} = Rg/2, \tag{61}$$

the gravitational field equations resulting from the Lagrangian

$$L = -\sqrt{-\det g_{\alpha\beta}}\; g^{\mu\nu}K_{\mu\nu} + L_m$$

are

$$K = \left\{\frac{\partial f}{\partial S}Fg^{-1}F + \left(f - 2\frac{\partial f}{\partial S}S - \frac{\partial f}{\partial R}R\right)g\right\}\Big/2. \tag{62}$$

The affine Lagrangian of such a theory can be obtained from

$$L = -\sqrt{-\det g_{\alpha\beta}}\,(g^{\mu\nu}K_{\mu\nu} - f) = -\sqrt{-\det g_{\alpha\beta}}\left(f - 2\frac{\partial f}{\partial R}R - 2\frac{\partial f}{\partial S}S\right), \tag{63}$$

where $\det g_{\alpha\beta}$, R and S have to be expressed through the invariants of K and F. The Born-Infeld theory is obtained if one takes the electromagnetic field Lagrangian to be of the following form[2]:

$$L_m = \{\sqrt{-\det g_{\alpha\beta}} - \sqrt{-\det (g_{\alpha\beta} + bF_{\alpha\beta})}\}/b. \tag{64}$$

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[2] The original reference is M. Born, L. Infeld, *Proc. Roy. Soc.* **A144**, 425 (1934). Reprints of papers of Born and Infeld may be found in *Leopold Infeld, his life and scientific work*, ed. E. Infeld, PWN 1978, Warsaw.

In the limit $b \to 0$, one obtains the usual electrodynamics. Since $g_{\mu\nu}$ is dimensionless, and b (which therefore has dimensions *length²/charge*) plays the role of a fundamental constant, b could be taken as $\lambda_{PW}^2/e = G\hbar/c^3e \approx 2/15$ in our units, where λ_{PW} is the Planck--Wheeler length.

Introducing

$$A' = bg^{-1}F, \quad B' = 2bF^{-1}K,$$
$$S' = \mathrm{Tr}\,(A'^2)/4, \quad R'^2 = -\det A', \quad S_0' = \mathrm{Tr}\,(B'^2)/4,$$
$$R_0'^2 = -\det B', \tag{65}$$

the function f takes the form

$$f = h/b^2, \quad h = 1-e, \quad e = (1-2S'-R'^2)^{1/2} \tag{66}$$

and Einstein equations read

$$B' = (A' - hA'^{-1})/e. \tag{67}$$

Calulating the determinant of B' one obtains

$$R_0'/2 = (h-S')/R'e \tag{68}$$

($h-S'$ is non-negative, for R' and S' such that e is real). Calculating $\mathrm{Tr}\,B'^2$ gives

$$S_0' = 2(S'-h)(1+S'/R'^2)/e^2.$$

Making use of (68), one obtains

$$S_0'/R_0' = -(R'^2+S')/R'e. \tag{69}$$

These equations are complicated to solve directly for R' and S' (nevertheless it can be done analytically), however one does not need to solve them if one wants just to obtain the affine Lagrangian. The Lagrangian can be calculated from (63)

$$L = 4b^2\sqrt{-\det K_{\mu\nu}}\,(h+R'^2)/R_0'R'e.$$

Subtracting (69) from (68), one obtains

$$(h+R'^2)/R'e = R_0'/2 - S_0'/R_0',$$

which gives, using (30)

$$L = \sqrt{-\det K_{\mu\nu}}\,\{2b^2 + \mathrm{Tr}\,(K^{-1}FK^{-1}F)/4\}. \tag{70}$$

It is remarkable that the Lagrangian (70) is just the sum of the linear electrodynamics Lagrangian, and a Lagrangian corresponding to a vacuum Einstein theory with a cosmological constant.

From (70) one can obtain the relation between $g_{\mu\nu}$ and $K_{\mu\nu}$

$$\sqrt{-\det g}\,g^{-1} = \sqrt{-\det K}\,\{K^{-1}FK^{-1}FK^{-1}/2 - (b^2 + \mathrm{Tr}\,(K^{-1}FK^{-1}F)/8)K^{-1}\}. \tag{71}$$

A detailed examination of these equations reveals that they are more general than Einstein-Born-Infeld equations (67) (see Appendix). The affine Lagrangian (70) corresponds to the standard Born-Infeld Lagrangian (64), for R_0 and S_0 fulfilling the condition

$$4(R_0^2+S_0^2)-R_0^4 > 0. \tag{72}$$

The matter Lagrangian for

$$4(R_0^2+S_0^2)-R_0^4 < 0 \tag{73}$$

is

$$L_m = \sqrt{-\det g}\,(-1-e)/b^2. \tag{74}$$

This Lagrangian leads to the same relation between $\hat{F}^{\mu\nu}$ and $F_{\mu\nu}$ as in the standard Born-Infeld theory, but Einstein equations derived from the Lagrangian (74) differ from Einstein equations for the standard Born-Infeld theory by a cosmological constant $\lambda = 2/b^2$.

6. Two theorems on the affine Lagrangians for a general electromagnetism theory

It is worthwhile to compare the Lagrangians considered in the previous section, with the matter and affine Lagrangians for a linear Einstein-Maxwell electrodynamics with a cosmological constant. The affine Lagrangian for this theory has been derived in [2]. The standard Lagrangian is

$$L = -\sqrt{-\det g_{\alpha\beta}}\,\{g^{\mu\nu}K_{\mu\nu}+g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}/4-\lambda\} \tag{75}$$

and its affine equivalent is (for $(\vec{E}^2+\vec{B}^2)/2-\lambda^2 > 0$)

$$L = 2\sqrt{-\det K_{\alpha\beta}}\,\{1-(1-\lambda S_0-\lambda^2 R_0^2/4)^{1/2}\}/\lambda. \tag{76}$$

This is the "Born-Infeld Lagrangian with $g_{\mu\nu}$ replaced by $K_{\mu\nu}$ and b^2 replaced by $\lambda/2$". For a zero cosmological constant, the electromagnetic field Lagrangian is

$$L = -\sqrt{-\det g_{\alpha\beta}}\,g^{\alpha\beta}g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu}/4 \tag{77}$$

and its affine equivalent is

$$L = -\sqrt{-\det K_{\alpha\beta}}\,K^{-1\alpha\beta}K^{-1\mu\nu}F_{\alpha\mu}F_{\beta\nu}/4, \tag{78}$$

which is "the linear Maxwell Lagrangian with $g_{\mu\nu}$ replaced by $K_{\mu\nu}$". Since the affine formulation and the metric affine formulation of the theory are symmetric in some sense (the metric is obtained by varying the affine Lagrangian with respect to the symmetric part of the Ricci tensor, while the symmetric part of the Ricci tensor is obtained by varying the matter Lagrangian with respect to the metric), comparison of equations (77) and (78) suggests, that the affine Lagrangian of the theory, for a theory for which $g^{\alpha\beta}K_{\alpha\beta} = 0$ identically, can be obtained by replacing $g_{\mu\nu}$ by $K_{\mu\nu}$ in the matter Lagrangian. Similarly, the comparison of equations (75) and (76) (with λ replaced by $2b^2$) with equations (64), (66) and (70) suggests, that the knowledge of the metric affine matter Lagrangian $L_1(g_{\mu\nu}, \varphi^A)$ (φ^A — some matter fields) and its affine equivalent $L_2(K_{\mu\nu}, \varphi^A)$ allows us to find the affine

Lagrangian for a metric affine theory described by the matter Lagrangian $L_2(g_{\mu\nu}, \varphi^A)$, as being $L_1(K_{\mu\nu}, \varphi^A)$. Let us show, that the second of these hypotheses is true in the case of the electromagnetic field.

Theorem 1

Let the matter Lagrangian be

$$L_{\text{mat}} = \sqrt{-\det g_{\alpha\beta}}\, f(R, S), \tag{79}$$

where R and S are given by (37) and (38), and let the corresponding affine Lagrangian be

$$L = \sqrt{-\det K_{\alpha\beta}}\, \tilde{f}(\tilde{R}, \tilde{S}), \tag{80}$$

where

$$C^{\mu}{}_{\nu} = K^{-1\mu\alpha}F_{\alpha\nu}, \qquad \tilde{R}^2 = -\det C,$$
$$\tilde{S} = \operatorname{Tr}(C^2)/4. \tag{81}$$

Then the affine Lagrangian for the theory described by the matter Lagrangian

$$L_{\text{mat}} = \sqrt{-\det g_{\alpha\beta}}\, \tilde{f}(R, S) \tag{82}$$

is

$$L = \sqrt{-\det K_{\alpha\beta}}\, f(\tilde{R}, \tilde{S}). \tag{83}$$

Proof:

Formula (62) can be rewritten in the following form

$$B = xA + yA^{-1} \tag{84}$$

where B and A are given by (37), and

$$x = \partial f/\partial S, \qquad y = f - 2S\partial f/\partial S - R\partial f/\partial R. \tag{85}$$

Using the formulas of Section 3, one obtains the relations between the invariants of A and C:

$$4R = \tilde{R}|y^2 + 2xyS - R^2x^2|,$$
$$4\tilde{S}/\tilde{R}^2 = y^2S/R^2 - 2xy - x^2S. \tag{86}$$

The affine Lagrangian is obtained by solving (86) with respect to R and S, and inserting these solutions in (63). It will be shown, that the theory described by the affine Lagrangian (83) leads to the following relations between the invariants of A and C:

$$4\tilde{R} = R|\tilde{y}^2 + 2\tilde{x}\tilde{y}\tilde{S} - \tilde{R}^2\tilde{x}^2|,$$
$$4S/R^2 = \tilde{y}^2\tilde{S}/\tilde{R}^2 - 2\tilde{x}\tilde{y} - \tilde{x}^2\tilde{S}, \tag{87}$$

where

$$\tilde{x} = x(\tilde{R}, \tilde{S}), \qquad \tilde{y} = y(\tilde{R}, \tilde{S}).$$

It will also be shown, that the matter Lagrangian corresponding to the affine Lagrangian (83) is given by the formula

$$L_{\text{mat}} = -\sqrt{-\det K_{\alpha\beta}}\,(f - 2\tilde{R}\partial f/\partial\tilde{R} - 2\tilde{S}\partial f/\partial\tilde{S}). \tag{88}$$

The proof of the theorem follows in a straightforward manner from (87) and (88). Since Eqs. (87) are the same as Eqs. (86) (modulo tildes), the solutions of (86) will be the same functions of their arguments as solutions of (87). Since, moreover, the affine Lagrangian corresponding to the matter Lagrangian (79), obtained from formula (63), and the matter Lagrangian corresponding to the affine Lagrangian (83), obtained from formula (88), are obtained by exactly the same formulas (modulo tildes in the arguments), the insertion of the solutions of (86) and (97) into (63) and (83) respectively, will lead to the same function of the appropriate arguments, which is f.

Let us derive now the formulas (87) and (88). The field equations derived from the Lagrangian (83) are

$$2\pi = \sqrt{-\det K_{\alpha\beta}}\,\{(f-\tilde{R}\partial f/\partial\tilde{R})K^{-1}-\partial f/\partial\tilde{S}K^{-1}FK^{-1}FK^{-1}\}. \tag{89}$$

Using the formulas of Section 3, Eqs. (89) can be written in the following form:

$$2\tilde{R}A = R\{\tilde{x}\tilde{R}^2C^{-1}-\tilde{y}C\}. \tag{90}$$

From (90) one can calculate $\det A$ and $\operatorname{tr} A^2$, obtaining equations (87).

To derive Eq. (88), it must be noted that the matter Lagrangian corresponding to the affine Lagrangian (83) is given by the equation

$$L_{\text{mat}} = L+\sqrt{-\det g_{\alpha\beta}}\, g^{\mu\nu}K_{\mu\nu}.$$

Multiplying (89) by K and calculating the trace gives the formula (88).

As an application of theorem 1, one can obtain without any calculations, using the results of Ferraris and Kijowski [2], the affine Lagrangian corresponding to the matter Lagrangians

$$L_{\text{mat}} = \varepsilon\sqrt{-\det g_{\alpha\beta}}\,(R^2-8b^2S-16b^4)^{1/2}, \tag{91}$$

where b is an arbitrary constant, and $\varepsilon = \pm 1$. In [2] it was shown, that the matter Lagrangian

$$L_{\text{mat}} = -\sqrt{-\det(g_{\mu\nu}+bF_{\mu\nu})}/b^2, \tag{92}$$

which is the Born-Infeld Lagrangian shifted by a cosmological constant $1/b^2$, leads to two different affine Lagrangians

$$L = \varepsilon\sqrt{-\det K_{\alpha\beta}}\,(\tilde{R}^2-8b^2\tilde{S}-16b^4)^{1/2}. \tag{93}$$

From (91), (92), (93) and theorem 1 one finds the affine Lagrangian corresponding to the Lagrangian (91) to be

$$L = -\sqrt{-\det(K_{\mu\nu}+bF_{\mu\nu})}/b^2. \tag{94}$$

The formulas derived in the proof of theorem 1 allow to prove the first hypothesis, in the case of the electromagnetic field.

Theorem 2

Let f and $\tilde{f}$ be as in theorem 1. Let the matter Lagrangian be such, that

$$g^{\mu\nu}T_{\mu\nu} = 0 = g^{\mu\nu}K_{\mu\nu}, \tag{95}$$

or, equivalently,

$$f(R, S) = R\partial f/\partial R + S\partial f/\partial S. \tag{96}$$

Then $\tilde{f}(\tilde{R}, \tilde{S})$ is the same function of its arguments as $f(R, S)$.

Proof:

From (96) it follows, that

$$y = -xS.$$

In this case equations (86) read

$$4R = \tilde{R}x^2(S^2+R^2)$$

$$4\tilde{S}/\tilde{R}^2 = x^2S(S^2+R^2)/R^2. \tag{97}$$

From (97) one obtains

$$\tilde{S}/\tilde{R} = S/R. \tag{98}$$

Formula (63) can be written in the following form

$$R\tilde{f}(\tilde{S}, \tilde{R}) = \tilde{R}f(S, R). \tag{99}$$

Let us introduce

$$\alpha(a = S/R, R) = f(S, R).$$

The homogeneity condition (96) implies

$$\alpha = R\partial\alpha/\partial R \Rightarrow f = \beta(a)R,$$

where β is some function of one variable. The condition (95) implies, that $\tilde{f}$ is also homogeneous of degree one, therefore

$$\tilde{f} = \tilde{\beta}(\tilde{a})\tilde{R}, \quad \tilde{a} = \tilde{S}/\tilde{R}.$$

From (99) one obtains

$$\tilde{\beta}(\tilde{a}) = \beta(a)$$

and (98) implies

$$\tilde{\beta} = \beta.$$

Therefore

$$f = \beta(S/R)R, \quad \tilde{f} = \beta(\tilde{S}/\tilde{R})\tilde{R},$$

which proves theorem 2.

7. *Discussion*

A systematical method of deriving the affine Lagrangian for a unified non-linear theory of interacting gravitational and electromagnetic fields has been presented. It should be noted, that the "adapted coordinates method" presented by Ferraris and Kijowski

([1, 2]) is much simpler in practice, because the equations relating the invariants of $K^{-1\mu\alpha}F_{\alpha\nu}$ and $g^{\mu\alpha}F_{\alpha\nu}$ may be complicated to solve. Moreover, the method presented in this paper cannot be generalized to the case of many electromagnetic fields (Yang-Mills theories), because one has much more invariants than 2 to deal with.

It is easy to check, that the hypotheses emitted in Section 6 are not true for general matter fields. For a scalar field, with a Lagrangian of the form

$$L_{\mathrm{mat}} = \sqrt{-\det g_{\alpha\beta}}\, l(y, \varphi),$$

where

$$y = g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu},$$

the condition for the vanishing of the trace of the energy-momentum tensor implies

$$l = \alpha y^2, \qquad \alpha = \alpha(\varphi).$$

The corresponding affine Lagrangian may calculated to be

$$L = \sqrt{-\det K_{\alpha\beta}}\, 3\sqrt{3}\, \alpha y_0^2, \tag{100}$$

where

$$y_0 = K^{-1\mu\nu}\varphi_{,\mu}\varphi_{,\nu}.$$

Formula (100) is in contradiction with both the first and the second hypothesis. This contradiction suggests, that the first hypothesis should be slightly weakened, to read: the affine Lagrangian, for a theory with a vanishing trace of the energy-momentum tensor, is proportional to the "matter Lagrangian with $g_{\mu\nu}$ replaced by $K_{\mu\nu}$". Computer calculations, with randomly generated values of the fields, for the Yang-Mills theory with different dimensions of the symmetry group, have shown that even this weakened hypothesis is not true in general. More precisely, it has been shown, using the above method, that the straightforward affine generalization of the electromagnetic Lagrangian to the non-abelian symmetry group case, namely

$$L = \alpha\sqrt{-\det K_{\alpha\beta}}\, k_{ij}K^{-1\mu\alpha}\, {}^iF_{\alpha\beta}K^{-1\beta\gamma}\, {}^jF_{\gamma\mu},$$

where k_{ij} is the invariant metric on the group, and α is a constant, depending possibly upon the dimension of the group, does not describe the same dynamics as the Yang-Mills Lagrangian:

$$L_{\mathrm{mat}} = \sqrt{-\det g_{\alpha\beta}}\, k_{ij}g^{\mu\alpha}\, {}^iF_{\alpha\beta}g^{\beta\gamma}\, {}^jF_{\gamma\mu}/4.$$

It remains to find out, whether or not theorems one and two hold for the electromagnetic field only, and, if so, what is the reason for this.

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APPENDIX

Using the same techniques as in Sections 4 and 5, equations (71) can be written in the following form (for notational convenience, primes on A, B, S, R, S_0 and R_0 will be omitted. All relevant quantities appearing in the formulas derived in the appendix should carry a prime):

$$A = RB/R_0-(RS_0/R_0+RR_0/2)B^{-1}. \tag{A1}$$

From (A1) one can obtain

$$\begin{aligned} S/R^2 &= -S_0/R_0^2-S_0(S_0/R_0^2+1/2)^2-1, \\ R &= 4R_0^3/|R_0^4-4(R_0^2+S_0^2)|, \\ e &= \{4(R_0^2+S_0^2+S_0R_0^2)+R_0^4\}/|R_0^4-4(R_0^2+S_0^2)|, \end{aligned} \tag{A2}$$

where $|\ |$ stands for absolute value. From (68) and (69) one obtains, that

$$4(R_0^2+S_0^2)-R_0^4 > 0 \tag{A3}$$

for all S and R such that e is real. One can check, that if (A3) is not satisfied, then (68) is not satisfied ((68) is a necessary condition for the Born-Infeld-Einstein equations to hold). Therefore, the Lagrangian (70) describes the Born-Infeld-Einstein theory only for R_0 and S_0 which satisfy condition (A3). It may be checked, that equations (68) and (69) describe the solutions of equations (A2)–(A3) with respect to R_0 and S_0.

It may be checked, that the matter Lagrangian

$$L_{\text{mat}} = \sqrt{-\det g}\,(-1-e)/b^2 = \sqrt{-\det g}\, h^{(1)}/b^2 \tag{A4}$$

describes the same dynamics as the affine Lagrangian (70), for

$$4(R_0^2+S_0^2)-R_0^4 < 0. \tag{A5}$$

From (A4) one obtains the following relations between the invariants of A and B:

$$\begin{aligned} RR_0 &= -2(h^{(1)}+S)/e, \\ S_0 &= -RR_0(1+S/R^2)/e. \end{aligned} \tag{A6}$$

From (A6) one can calculate the affine Lagrangian to be (70). One can check that condition (A5) is satisfied, for all S and R such that e is real, when Einstein equations hold. A straightforward calculation shows, that equations (A6) describe solutions of equations (A2) with respect to R_0 and S_0, under the condition (A5).

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