RELATIVISTIC EFFECTS IN THE DEUTERON BINDING ENERGY

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(Received November 8, 1983)

The binding energy of a two nucleon system is calculated from the relativistic Weinberg equation with spin, and from its nonrelativistic limit. We get that the interaction of nucleons inside deuteron is largely different from that of nonrelativistic particles, in spite of the fact that the binding energy is very small. The relativistic interaction is much less attractive than the nonrelativistic one. An inclusion of spin introduces significant dynamical effects. The light front dynamics, used in our calculations, has unique advantages over other existing approaches. The practical virtues of this scheme, including the invariant spinor representation, are presented in full detail.

PACS numbers: 21.10.Dr, 11.80.-m, 27.10.+h, 13.75.Cs

1. Introduction

The nucleon-nucleon interaction is a central problem of nuclear physics. It is widely believed that the low energy properties of nucleon-nucleon forces may be explained by the mesonic theory. However, it is not a fundamental theory. For example, the nucleon--meson vertices still have not been adequately described and the renormalizable relativistic theory including vector mesons has not been formulated. So, many approximations are usually made in physical calculations which are beyond the theoretical control. One of the most important of those approximations is the nonrelativistic description of low energy phenomena. In this paper we investigate this approximation on the example of the deuteron. The deuteron is a loosely bound state of two nucleons and its binding energy is usually calculated from the nonrelativistic Schrödinger equation. When one tries to take into account the structure of nucleons or the retardation of the binding forces one comes across unsolved problems of contemporary physics. Nevertheless it is very interesting to find the difference between the deuteron binding energy as predicted by the nonrelativistic theory and as predicted by the most satisfactory relativistic one. In this paper we develop

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the relativistic description based on the light front quantum field theory. Our choice of the light front quantization is motivated by the unique simplification of the structure of the Poincaré generators. When a system evolves in a time t then at least four out of ten generators depend on the interaction [1]. E.g. in the usual old fashioned theory the Hamiltonian and the three Lorentz boosts are interaction-dependent while three momenta and three angular momenta are interaction-independent. Thus one may easily find eigenstates of the angular momentum but nothing is known about the shape of the wave function for the bound state in motion. On the contrary, in the light front dynamics, when a system evolves in $x^+ = t+z$, only three generators are interaction-dependent. This is the unique property of the front form of dynamics that has as many as seven generators interaction-independent while on other forms at most six of them are interaction-independent [1]. The additional seventh interaction-independent generator is the boost generator along the direction of the front. Therefore, in the light front dynamics it is possible to find the wave function of the moving bound system. This simplification plays the crucial role in the recent success of the QCD calculations of the hadronic, including deuteron, electromagnetic properties at high energies [2, 3]. The problem of the angular momentum of deuteron may be partly solved using the nonrelativistic limit [4]. Another important reason to choose the light front form of dynamics is that in the x^+-ordered perturbation theory fewer diagrams correspond to a given Feynman diagram than in the t-ordered one. E.g., in the scalar theory, for the Feynman box from Fig. 1a there are $24 = 4!$ t-ordered old fashioned diagrams and only 6 diagrams in the x^+-ordered formulation (see Fig. 1b). All diagrams containing creation of particles from vacuum or their annihilation into vacuum are absent in the light front dynamics.

Among other important advantages of the front form we enumerate the cluster property, and the resemblance to the nonrelativistic theory. The cluster property means that the two body interaction is not unphysically influenced by the presence of a third body when the third body is very far away. Only the light front dynamics, among the old fashioned theories, and three dimensional reductions of the Bethe-Salpeter-like equations, simply satisfies the cluster condition [5]. The two nucleon scattering amplitude is determined in the light front dynamics by the Weinberg equation [6], the relativistic counterpart of the Lippmann-Schwinger equation, or the equal x^+-projection of the Bethe-Salpeter equation [7]. When expressed in terms of the proper relative momenta of nucleons [8] from their center of mass frame, the Weinberg equation becomes formally equivalent to the Schrö-

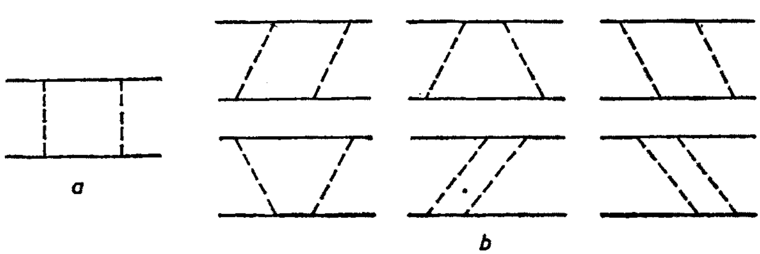

Fig. 1. The box diagram in the spinless theory: a — the Feynman covariant diagram, b — the corresponding x^+ — ordered diagrams

dinger equation with a nonlocal and energy dependent potential. Thus the whole relativity is contained in the Weinberg potential (kernel or driving term). The Weinberg equation has the nonrealitivistic limit which is the Schrödinger equation and it is possible to compare the predictions of the full relativistic equation and its nonrelativistic reduction. In the case of a model spinless theory $:\varphi^2:\varphi_0$ such a comparison has revealed [9] large differences, about 50% in predictions for the binding energy of the lowest bound state. In this paper we investigate a field theory model of deuteron, including the spin of nucleons, with a coupling of nucleons to pions via the interaction term $\bar{\psi}i\gamma^5\boldsymbol{\tau}\psi\boldsymbol{\phi}$. There is an important difficulty in the introduction of spin into the light front dynamics. The new singular terms, called seagulls, are present in the interaction Hamiltonian in the local field theory. On the other hand, for composite nucleons we have to introduce form factors into nucleon-meson vertices. There is no simple way to introduce in to the theory the phenomenological form factors from the very beginning. To answer what is going on with the seagull terms for composite nucleons we use the connection of the light front dynamics with the old fashioned perturbation theory in the infinite momentum frame. This connection is a very interesting subject in itself (especially some kinematic results for spinors). It is also relevant to current attempts to calculate the deep inelastic electron-hadron scattering in the bag-like models. In Section 2 we briefly describe that connection and give the argument for the neglect of the seagull terms in the presence of form factors. Section 3 is the presentation of the Weinberg equation and the discussion of the binding energy of two nucleon bound state as predicted by the Wienberg equation and the Schrödinger equation.

2. *The infinite momentum frame and the frame invariant light front dynamics*

Briefly speaking, the light front dynamics in any frame of reference provides the perturbation rules which are as simple as the old fashioned perturbation rules, in the infinite momentum frame [10]. In other words the well known advantages of the infinite momentum frame may be achieved in the laboratory, when the evolution of the system is parametrized by x^+ instead of time t. The light front rules are invariant under three independent Lorentz boosts and, in fact, may be used in any frame of reference within reach of that family of boosts. For a more detailed analysis we refer the reader to the reference [11]. The most useful formulae are given in the Appendix A. Here we quote only the facts which are necessary for discussing the role of form factors in the light front dynamics.

When passing from the old fashioned perturbation rules in the infinite momentum frame in the scalar theory [10] to the theory with fermions coupled to scalar bosons [12] a new feature emerges. In the infinite momentcm frame the on mass shell momenta of particles are labelled by their longitudinal components denoted by η and by their transverse components. In the spinless case all intermediate particles have to have positive η's, i.e. they have to move forward along the infinite momentum of the total system of interacting bodies. When the spin is included it is possible that some particles may have η's negative as well. This possibility arises from the fact that the spin factors are momentum dependent and may compensate a large denominator in certain cases with backward moving particles. Therefore, there are additional diagrams in the spinor theory as compared to the scalar one. The common feature of these additional diagrams is that the line of a particle with $\eta < 0$ may be extended over only one intermediate state [13]. The detailed analysis of the $\bar{\psi} i \gamma^5 \psi \varphi$ theory is given in reference [6].

The above mentioned additional diagrams have lines with $\eta > 0$, like in the scalar theory, and the lines with $\eta < 0$ extended only over one intermediate state. It may be shown that these singular intermediate states are equivalent to the additional two fermions — two bosons singular vertices, coming from the contraction of the fermion lines with $\eta < 0$ to the points [6]. In fact, the information about other particles in the singular intermediate states is lost in the infinite momentum limit. Thus the infinite momentum old fashioned perturbation rules in the spinor theory may be obtained from the scalar rules by an addition of new two fermion-two boson vertices, and spinor factors.

The phenomenological form factors may be introduced into the meson-nucleon vertices in many ways. Usually a form factor depends on different invariants like s, t, u for the N–N scattering. It turns out that for vertices joining lines with different signs of η always at least one of the invariants goes to infinity like square of the infinite momentum [14, 6]. Therefore, the form factors provide additional damping of the diagrams with negative η's. This damping is not compensated in any way. Thus the form factors kill the new singular vertices in the old fashioned perturbation theory in the infinite momentum frame.

We are going to use the light front dynamics which preserves the simplicity of the infinite momentum rules and is much more universal than the infinite momentum scheme. The basic reference for the light front quantization is a series of papers written by Yan et al. [15]. Instead of the time ordered product the x^+-ordered product is used in the quantum theory and the creation and annihilation operators are expressed by fields on the light front instead of the constant time hyperplane, as usual. In the scalar case the old fashioned rules in the light front dynamics and in the infinite momentum coincide when we substitute the longitudinal fraction η_i of i'th particle momentum by the $x_i = p_i^+/P^+$, the fraction of the plus component of the total momentum of the system. The η's are frame dependent while x's are invariant under the three Lorentz boosts. That is the reason for the utility of the light front rules in contrast to the very specific infinite momentum rules. When passing to the spinor case it becomes apparent that the interaction Hamiltonian is not simply related to the interaction Lagrangian, and it contains new singular terms in the local field theory $\bar{\psi} i \gamma^5 \psi \varphi$. The point is that these additional terms give vertices which we observed previously in the infinite momentum rules.

The last thing we have to notice is that the very special representation of the Dirac spinors has to be used to see the identity of the singular vertices of the infinite momentum frame with those in the light front rules [6]. In that particular representation both sets of vertices are equivalent by substitution of η's by x's [6]. Moreover, in this representation the three Lorentz boosts of the light front transform the spinors witohut the Wigner rotation [6, 11] (see also Appendix A). This is another example of the simplicity of the light front rules, as compared to other formulations. When we have established that the light front seagulls come from the backward moving particles and that the form factors damp the backward moving particles, we simply introduce form factors into the light front rules and forget the seagulls.

3. *The Weinberg equation for a two nucleon system*

The two nucleon scattering in the one meson exchange approximation in the light front dynamics is represented by the infinite sum of diagrams from Fig. 2a. This sum may be written as an integral equation depicted in Fig. 2b and symbolically written as follows

$$M = V+VGM. \tag{1}$$

This equation is called the Weinberg equation [10] and the kernel V shown in full details in Fig. 3, is called the Weinberg potential.

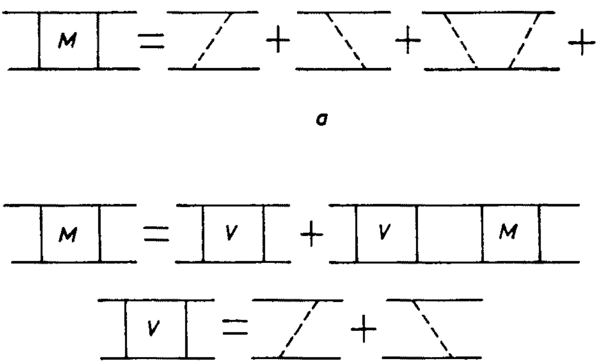

Fig. 2. a — The one meson exchange approximation for the nucleon-nucleon scattering amplitude M, b — the graphical representation of the corresponding Weinberg equation

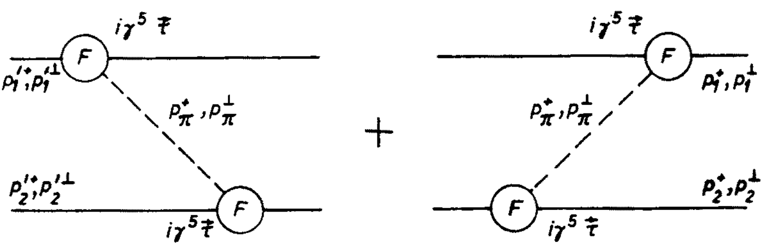

Fig. 3. The Weinberg potential for one pion exchange

Instead of the energy denominators we have here P^- denominators. When we introduce the relative and total four momenta of nucleons $q = \frac{1}{2}(p_1-p_2)$ and $P = p_1+p_2$, respectively, then the denominators may be combined to one denominator which is equal to

$$(q'-q'')^2+\mu^2+|q'^+-q''^+|\tfrac{1}{2}(P'^-+P''^--2P^--i\delta). \tag{2}$$

According to the light front rules we have (spin labels omitted) the following Weinberg equation

$$\begin{aligned} M(q'^+, \mathbf{q}'^\perp; q^+, \mathbf{q}^\perp) = {} & V(q'^+, \mathbf{q}'^\perp; q^+, \mathbf{q}^\perp) - \tfrac{1}{2}(2\pi)^{-3}\int d^2q''^\perp dq''^+ \theta(q''^+ + P^+/2) \\ & \times \theta(-q''^+ + P^+/2) V(q'^+, \mathbf{q}'^\perp; q''^+, \mathbf{q}''^\perp) \\ & \times [p_1''^+ p_2''^+ (P''^- - P^- - i\delta)]^{-1} M(q''^+, \mathbf{q}''^\perp; q^+, \mathbf{q}^\perp). \end{aligned} \tag{3}$$

Now we introduce a parametrization by the relative three momentum $\boldsymbol{k}(k = |\boldsymbol{k}|)$ of nucleons [8] in their rest frame of reference as given in the Appendix B. The Jacobian of such a transformation is equal to

$$\frac{\partial(q''^{+}, \boldsymbol{q}''^{\perp})}{\partial(k'', \theta, \varphi)} = k''^{2} \sin\theta'' \cdot 4p_1''^{+} p_2''^{+}/M''M, \tag{4}$$

where $M = 2(m^2+k^2)^{1/2}$ denotes the total energy of two nucleons. The free propagator G from equations (1) and (3) becomes

$$P''^{-} - P^{-} - i\delta = 4(k''^{2} - k^2 - i\delta')/M \tag{5}$$

and the denominator from Eq. (2) is equal [9] to $A+\alpha$, where

$$A = \mu^2 + k'^2 + k''^2 - 2k'k''(\cos\theta' \cos\theta'' + \sin\theta' \sin\theta'' \cos(\varphi' - \varphi'')),$$

$$\alpha = 2|k' \cos\theta'/M' - k'' \cos\theta''/M''|(k'^2 + k''^2 - 2k^2)$$
$$- k' \cos\theta' k'' \cos\theta''(M' - M'')^2/M'M''. \tag{6}$$

The Weinberg equation then reads as follows

$$\langle k' \cos\theta' 0 s_1' s_2' | M | k 1 0 s_1 s_2 \rangle = \langle k' \cos\theta' 0 s_1' s_2' | V | k 1 0 s_1 s_2 \rangle$$
$$-(2\pi)^{-3} \sum_{s_1'' s_2''} \int k''^2 dk'' \sin\theta'' d\theta'' d\varphi'' \langle k' \cos\theta' 0 s_1' s_2' | V | k'' \cos\theta'' \varphi'' s_1'' s_2'' \rangle$$
$$\times (k''^2/m - k^2/m - i\delta)^{-1} \langle k'' \cos\theta'' \varphi'' s_1'' s_2'' | M | k 1 0 s_1 s_2 \rangle, \tag{7}$$

where the potential V is given by

$$\langle k' \cos\theta' \varphi' s_1' s_2' | V | k'' \cos\theta'' \varphi'' s_1'' s_2'' \rangle$$
$$= f^2/\mu^2 I F(t, u)(M')^{-1/2} 2m \langle s_1' s_2' | W | s_1'' s_2'' \rangle (A + \alpha - i\delta')^{-1} (M'')^{-1/2}. \tag{8}$$

This potential corresponds to the interaction Lagrangian equal to $-\dfrac{2mf}{\mu} : \bar{\psi} i \gamma^5 \boldsymbol{\tau} \psi : \boldsymbol{\phi}$, with m denoting the nucleon mass, μ is a pion mass, and the coupling constant f determines the value of the binding energy. I denotes the isospin factor equal to 1 or -3. The form factor $F(t, u)$ is chosen in the form [16]

$$F(t, u) = \lambda^4[(\lambda^2 - t)(\lambda^2 - u)]^{-1}, \tag{9}$$

with

$$2t = (p_1' - p_1'')^2 + (p_2' - p_2'')^2,$$
$$2u = (p_1' - p_2'')^2 + (p_2' - p_1'')^2, \tag{10.a}$$

or, equivalently,

$$-t = k'^2 + k''^2 - 2k'k''[\tfrac{1}{2}(M'/M'' + M''/M') \cos\theta' \cos\theta'' + \sin\theta' \sin\theta'' \cos(\varphi' - \varphi'')],$$
$$-u = k'^2 + k''^2 + 2k'k''[\tfrac{1}{2}(M'/M'' + M''/M') \cos\theta' \cos\theta''$$
$$+ \sin\theta' \sin\theta'' \cos(\varphi' - \varphi'')]. \tag{10.b}$$

The spin amplitudes $\langle s_1's_2'|W|s_1''s_2''\rangle$ can be written in the form

$$\langle s_1's_2'|W|s_1''s_2''\rangle = -U_{s_1'}^{+}\sigma(\gamma_1 k_-'' - \gamma_1^{-1} k_-')U_{s_1''}U_{s_2'}^{+}\sigma(\gamma_2 k_+'' - \gamma_2^{-1} k_+')U_{s_2''}, \quad (11)$$

where we have denoted

$$k_\pm = (k^\perp, \pm m),$$

and the relativistic factors γ_i are equal

$$\gamma_i = (p_i'^{+}/p_i''^{+})^{1/2}, \quad \text{or} \quad \gamma_{1,2} = \{[1 \pm 2k' \cos\theta'/M']/[1 \pm 2k'' \cos\theta''/M'']\}^{1/2}. \quad (12)$$

The Pauli spinors U_s carry the invariant spin indices of the particular spinor representation from Appendix A. Note that the relativistic Weinberg equation (7) is formally equivalent to the Lippmann-Schwinger nonrelativistic equation. The "minimal relativity" factors are included in the Weinberg potential. The free propagator (5) is quadratic, as in the Schrödinger approach. In the nonrelativistic limit, when the nucleon masses become very large and we neglect the terms of the order of $k/m \gg 1$, the Weinberg potential is equal to the Yukawa potential

$$V = -f^2/\mu^2 F(t_{NR}, u_{NR})\sigma_1(k''-k')A_{NR}^{-1}\sigma_2(k''-k'), \quad (13)$$

where

$$k = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)k,$$

$$-t_{NR} = (k''-k')^2,$$

$$-u_{NR} = (k''+k')^2, \quad A_{NR} = \mu^2 - t_{NR}. \quad (14)$$

Now, it is possible to compare the predictions for the lowest bound state binding energy given by the full Weinberg equation (7) and by its nonrelativistic limit, i.e. the Schrödinger equation with the Yukawa potential (13). We have used the method from Refs. [9], [17] to find the coupling constant $f^2/4\pi$, which corresponds to the assumed binding energy $\varepsilon = 2m - s^{1/2}$ of the lowest bound state. The essence of this method is to find the eigenvalue of the integral operator from the Weinberg equation by the successive iterations of that operator. The main numerical problem comes from the fact that the Weinberg potential off energy shell does not conserve the orbital label l of spherical harmonics as functions of the angles (θ, φ) introduced in Eq. (4) and Appendix B. Only the projection m on the light front direction is conserved. The calculations may be done as in the Ref. [9] although there are, roughly speaking, 4×4 times larger matrices due to the presence of spin. The partial wave decomposition is briefly presented in the Appendix C. The lack of the Wigner rotation in the invariant spinor representation, plays the essential role there.

The results of our calculations for the dependence of the model deuteron binding energy ε on the coupling constant $f^2/4\pi$ are presented in Figs. 4–6 and in Table I. As a measure of the relativistic effect in the nonrelativistic system we take the ratio

$$R = \frac{f_W^2 - f_S^2}{f_S^2},$$

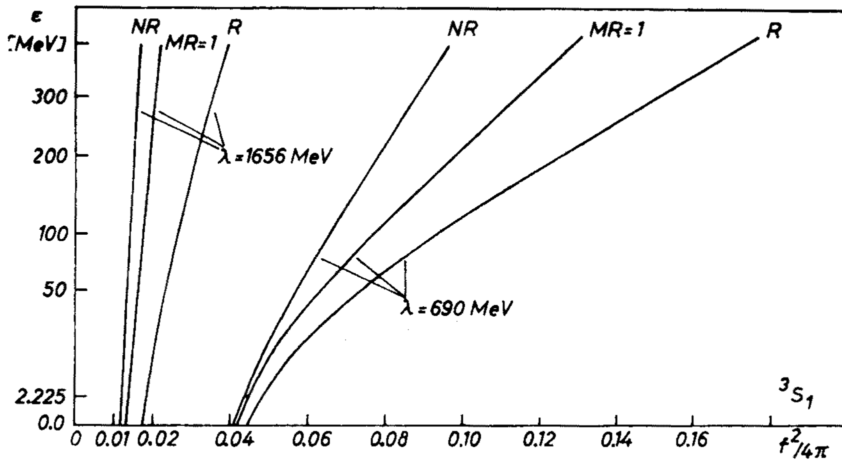

Fig. 4. The binding energy ε in the triplet state $^3S_1$, as a function of the coupling constant $f^2/4\pi$, from the relativistic Weinberg equation (R), from the Weinberg eq. with the "minimal relativity" factors put equal to 1 (MR = 1) and from the nonrelativistic Schrödinger equation (NR), for two cut-off masses λ

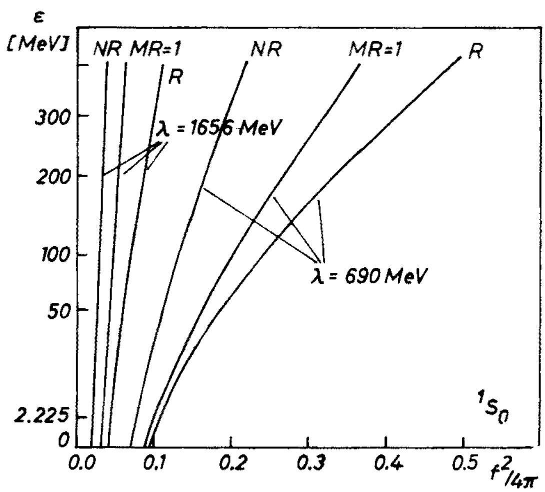

Fig. 5. The binding energy ε in the singlet state $^1S_0$, as a function of $f^2/4\pi$

TABLE I

The values of the coupling constant $f^2/4\pi$ giving the deuteron binding energy according to the relativistic, and the nonrelativistic equations. For two values of the cut-off mass λ

Equation \ λ	1656 MeV	690 MeV
R	0.019	0.048
NR	0.013	0.045

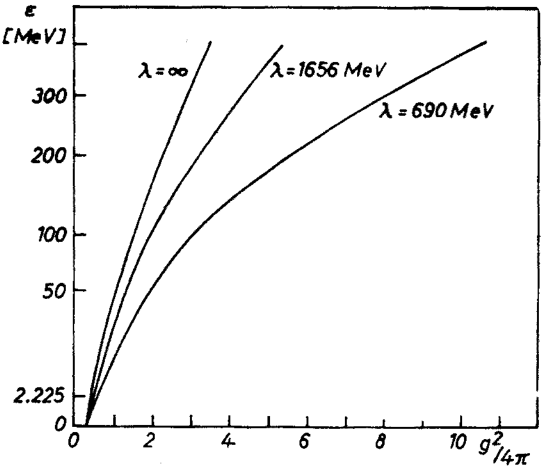

Fig. 6. The role of the nucleon-meson vertex form factor in the scalar Weinberg equation. The curve labelled $\lambda = \infty$ represents the results without cut-off, first obtained by Danielewicz and Namysłowski, Ref. [9]

where f_W and f_S denote the coupling constants predicted by the Weinberg and the Schrödinger equations respectively, for the same binding energy. The function $\varepsilon(f^2/4\pi)$ for the $^3S_1$ bound state is plotted in Fig. 4 with the obvious labels. MR = 1 means that we replaced the "minimal relativity" factors $\frac{2m}{(M'M'')^{1/2}}$ by 1 in the Weinberg equation.

For the deuteron binding energy $\varepsilon = 2.225$ MeV (channel $^3S_1$) we obtain $R = 50\%$ for the cut-off mass $\lambda_1 = 1656$ MeV and $R = 7\%$ for the cut-off mass $\lambda_2 = 690$ MeV. For larger binding energy of the order of 100 MeV we have $R = 75\%$ for λ_1a nd $R = 50\%$ for λ_2. These effects are slightly larger than in the scalar equation [9]. The "minimal relativity" factors have larger contribution to the relativistic effects in the spin case than in the scalar case because the relativistic spin modifications $\gamma k\sigma$ diminish the repulsive effects of the term α in the Weinberg potential and the effects of the factor $(M'/M''+M''/M')$ in the nucleon-meson vertices.

The relativistic corrections grow with the cut-off mass λ. The coupling constants giving the deuteron binding energy according to the relativistic Weinberg and the Schrödinger equations, are given in Table I.

For completeness we present the similar curves for the singlet bound state (channel $^1S_0$) in Fig. 5. The relativistic effects are also about $R = 50\%$. The role of "minimal relativity" factors is smaller than in the triplet case.

Another measure of the relativistic effects is provided by the comparison between the binding energy as predicted by the Schrödinger and the Weinberg equations for the same coupling constant. The binding energy is much larger nonrealtivistically than relativistically. The difference may be even of the order of magnitude as it is visible in Figs. 4 and 5, and is much larger than that observed in the scalar model [9]. Generally speaking, the relativistic effects are considerably large and they lower down the attraction of nucleons.

The role of spin is also visible in comparison of Figs. 4 and 5, with Fig. 6, where we plotted $\varepsilon(g^2/4\pi)$ for the scalar Weinberg equation with and without the meson-nucleon form factors in the pontential. The inclusion of spin causes that the sensitivity for the cut-off mass λ in adjusting the coupling constant to the binding energy grows about ten times. Also taking into account the spin of nucleons gives about five times larger sensitivity of the binding energy to the change in the coupling constant.

In conclusion we stress that the relativistic effects in the nonrelativistic systems like the deuteron are large. The proper relativistic treatment has to include the spin of nucleons, producing significant dynamical effects. Although the binding energy is small the motion and interaction of nucleons inside deuteron is much different from that of nonrelativistic particles.

We are grateful to Professor J.M. Namysłowski for his continuous guidance of our study of the light front dynamics and for his stimulating remarks. We would also like to thank Dr P. Danielewicz for many useful discussions and express our gratitude to Dr J. Tarasiuk for providing us with his computer procedures.

APPENDIX A

Invariant spinor representation

The equal time dynamics and the equal x^+ dynamics are perturbatively identical in the infinite momentum frame when we identify [11]

$$(\eta = p_3/P_3)_{\mathrm{IMF}} \equiv (p^+/P^+ = x)_{\mathrm{IMF}}.$$

The direction of the light front is along the direction of the infinite momentum. Contrary to the equal time dynamics, the equal x^+ dynamics is invariant under the Lorentz boosts along the front or along the large total momentum. This additional invariance allows us to go back to any frame of reference, with the finite total momentum in the light front dynamics, and to preserve the crucial simplifications which appear in the infinite momentum frame. The procedure of going back from the infinite momentum frame to a given frame of reference, is not possible in the equal time dynamics.

In the fast moving frame we denote the components of tensors by the subscript v. The fast moving frame approaches the infinite momentum frame when its velocity approaches the velocity of light. This limit is symbolically denoted by c. The velocity v is measured relatively to the frame of reference of the observer. We call this frame the laboratory frame and denote there the components of tensors by the subscript l. The laboratory frame is connected with the fast moving frame by the boost along the front, denoted by $L(v)$. The matrix $L(v)$ of that boost acts in the following way on the components of vectors

$$\begin{bmatrix} x_l^- \\ x_l^+ \\ x_l^1 \\ x_l^2 \end{bmatrix} = \begin{bmatrix} e^{\omega} & 0 & 0 & 0 \\ 0 & e^{-\omega} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_v^- \\ x_v^+ \\ x_v^1 \\ x_v^2 \end{bmatrix}, \quad x_l = L(v)x_v, \tag{A.1}$$

where th $\omega = v/c$.

The rotation around the light front direction commutes with $L(v)$, and has the same form in both frames. The remaining two rotations do not commute with $L(v)$. We denote their generators by J^{23} and J^{31}. The infinitesimal spatial rotation in the fast moving frame is

$$R_v(e^{-\omega}\boldsymbol{q}^{\perp}) = \exp\left(-2e^{-\omega}q_1 J^{23} + 2e^{-\omega}q_2 J^{31}\right)$$

and it corresponds to the following mixed Lorentz transformation $R_l(\boldsymbol{q}^{\perp})$ in the laboratory frame

$$R_l(\boldsymbol{q}^{\perp}) = L(v)\cdot R_v(e^{-\omega}\boldsymbol{q}^{\perp})\cdot L^{-1}(v).$$

In the limit $v \to c$ we obtain

$$\begin{bmatrix} x_{l'}^- \\ x_{l'}^+ \\ y_{l'}^1 \\ x_{l'}^2 \end{bmatrix} = \begin{bmatrix} 1 & q_1^2+q_2^2 & 2q_1 & 2q_2 \\ 0 & 1 & 0 & 0 \\ 0 & q_1 & 1 & 0 \\ 0 & q_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_l^- \\ x_l^+ \\ x_l^1 \\ x_l^2 \end{bmatrix}, \qquad x_{l'} = R_l(\boldsymbol{q}^{\perp})x_l. \tag{A.2}$$

The P^- denominators in the x^+-ordered perturbation theory are invariant under transformations (A.2) and under the rotations around the front direction as the old fashioned energy denominators are invariant under the three spatial rotations. The invariance under the transformations (A.1) and (A.2) allows us to boost the bound state wave functions [11, 14, 4] to the desired momentum, by the proper adjustment of v and $\boldsymbol{q}^{\perp}$.

If a fermion is described by the spinor

$$u(p_v) = (p_v^0+m)^{1/2}\left[1+(p_v^0+m)^{-1}\boldsymbol{\alpha}\boldsymbol{p}_v\right]u(0) \tag{A.3}$$

in the fast moving frame, then in the laboratory frame it is described by the spinor

$$\tilde{u}(p_l) = S(L(v))u(p_v), \qquad p_l = L(v)p_v,$$

where

$$S(L(v)) = \operatorname{ch}\frac{\omega}{2} - \alpha^3 \operatorname{sh}\frac{\omega}{2}$$

is the spinor representation of the transformation $L(v)$, Eq. (A.1). In the limit $v \to c$ we have [6]

$$\tilde{u}_s(p_l) = (2/p_l^+)^{1/2}\left[m\Lambda_- + (p_l^+ + \boldsymbol{\alpha}^{\perp}\boldsymbol{p}_l^{\perp})\Lambda_+\right[u_s(0) \equiv B(p_l)\cdot u_s(0), \tag{A.4}$$

where $\Lambda_\pm = 1/2(1\pm\alpha^3)$, $p^2 = m^2$, and s denotes the spin projection on the front direction. The spinors (A.4) have the following invariance properties

$$S(L)\tilde{u}_s(p) = \tilde{u}_s(Lp), \qquad S(L) = e^{\omega/2}\Lambda_- + e^{-\omega/2}\Lambda_+,$$

$$S(R)\tilde{u}_s(p) = \tilde{u}_s(Rp), \qquad S(R) = 1+\boldsymbol{\alpha}^{\perp}\boldsymbol{q}^{\perp}\Lambda_+ \tag{A.5}$$

for any boost parameters v and $\boldsymbol{q}^{\perp}$. Thus, there is no Wigner rotation of the light front spin index s. The transformations (A.1) and (A.2) satisfy the following relations

$$R_l(\boldsymbol{q}_1^{\perp}) \cdot R_l(\boldsymbol{q}_2^{\perp}) = R_l(\boldsymbol{q}_1^{\perp}+\boldsymbol{q}_2^{\perp}),$$

$$L(\omega_1) \cdot L(\omega_2) = L(\omega_1+\omega_2),$$

$$R_l(\boldsymbol{q}^{\perp}) \cdot L(v) = L(v) \cdot R_l(\boldsymbol{q}^{\perp}/e^{\omega})$$

and the boost matrix $B(p_l)$ from Eq. (A.4) is represented as follows

$$B(p_l) = S(L) \cdot S(R_1) = S(R_2) \cdot S(L),$$

$$\omega = m/p_l^{+}, \quad \boldsymbol{q}_1^{\perp} = \boldsymbol{p}_l^{\perp}/m, \quad \boldsymbol{q}_2^{\perp} = \boldsymbol{p}_l^{\perp}/p_l^{+}.$$

Therefore, in the light front dynamics it is possible to describe the moving fermions by spinors which do not undergo the Wigner rotations under three independent Lorentz transformations. The light front perturbation theory is invariant under these three boosts.

The spinors (A.4) were independently introduced in Ref. [2] but in an artificial notation having nothing to do with the origin of these spinors.

Finally we present the relation between the conventional spinors $u(p)$ from Eq. (A.3) and the light front spinors $\tilde{u}(p)$ from Eq. (A.4)

$$\tilde{u}_+ = \kappa[(p^+ + m)u_+ + (p^1 + ip^2)u_-]$$

$$\tilde{u}_- = \kappa[-(p^1 - ip^2)u_+ + (p^+ + m)u_-]$$

$$\tilde{v}_+ = \kappa[(p^+ + m)v_+ - (p^1 - ip^2)v_-]$$

$$\tilde{v}_- = \kappa[(p^1 + ip^2)v_+ + (p^+ + m)v_-], \quad \kappa^{-2} = 2p^+(p^0 + m).$$

APPENDIX B

The relative momentum $\boldsymbol{k}$

Our description of the relative three momentum $\boldsymbol{k}$ is based on Ref. [8]. The initial and the final four momenta of nucleons in the Eq. (3) define the frame of reference spanned by the following tetrad

$$\begin{aligned} e_0^{\mu} &= P^{\mu}/(P^2)^{1/2}, \\ e_1^{\mu} &= \varepsilon_1 \varepsilon^{\mu}{}_{\nu\varrho\lambda} P^{\nu} e_2^{\varrho} e_3^{\lambda}, \\ e_2^{\mu} &= \varepsilon_2 \varepsilon^{\mu}{}_{\nu\varrho\lambda} P^{\nu} q^{\varrho} q'^{\lambda}, \\ e_3^{\mu} &= q^{\mu}(-q^2)^{-1/2}, \end{aligned} \tag{B.1}$$

where the constants ε_1 and ε_2 are such that $e_1^2 = e_2^2 = -1$. In the light front perturbation theory the vertices conserve the $+ = 0+3$ and $\perp = 1, 2$ components of momenta and do not conserve their $- = 0–3$ components. Therefore, the center of mass frames of the intermediate and the final nucleons are connected with the basic frame (B.1) by the Lorentz boosts (A.1) along the light front direction e_3. These boosts define the tetrads of the center of mass frames for the intermediate and the final nucleons. The corresponding parameters $\omega = \text{th}^{-1} v/c$ are

$$\omega' = \ln M'/M \text{ and } \omega'' = \ln M''/M, \tag{B.2}$$

where $M' = 2(m^2+k'^2)^{1/2}$ and $M'' = 2(m^2+k''^2)^{1/2}$, with k' and k'' denoting the lengths of the relative three momenta of nucleons in their center of mass frames. Introducing the spherical coordinates we have, respectively

$$(P)_e = (M, 0, 0, 0), \quad (q)_e = (0, 0, 0, k) = (0, \boldsymbol{k}),$$

$$(P')_{e'} = (M', 0, 0, 0), \quad (q')_{e'} = (0, k' \sin\theta', 0, k' \cos\theta') = (0, \boldsymbol{k}'),$$

$$(P'')_{e''} = (M'', 0, 0, 0),$$

$$(q'')_{e''} = (0, k'' \sin\theta'' \cos\varphi'', k'' \sin\theta'' \sin\varphi'', k'' \cos\theta'') = (0, \boldsymbol{k}''). \tag{B.3}$$

The four vectors P', q', and P'', q'' in the basic frame of reference (B.1) have the following components:

$$(P')_e = \left[\tfrac{1}{2}\left(M + \frac{M'^2}{M}\right), 0, 0, \tfrac{1}{2}\left(M - \frac{M'^2}{M}\right)\right],$$

$$(q')_e = \left[\tfrac{1}{2}\left(\frac{M}{M'} - \frac{M'}{M}\right) k' \cos\theta', k' \sin\theta', 0, \tfrac{1}{2}\left(\frac{M}{M'} + \frac{M'}{M}\right) k' \cos\theta'\right],$$

$$(P'')_e = \left[\tfrac{1}{2}\left(M + \frac{M''^2}{M}\right), 0, 0, \tfrac{1}{2}\left(M - \frac{M''^2}{M}\right)\right],$$

$$(q'')_e = \left[\tfrac{1}{2}\left(\frac{M}{M''} - \frac{M''}{M}\right) k'' \cos\theta'', k'' \sin\theta'' \cos\varphi'',\right.$$

$$\left. k'' \sin\theta'' \sin\varphi'', \tfrac{1}{2}\left(\frac{M}{M''} + \frac{M''}{M}\right) k'' \cos\theta''\right].$$

From these equations there follow Eqs. (4), (5), (6), (10) and (12). In the nonrelativistic limit the momentum $\boldsymbol{k}$ plays the role of the relative momentum of nucleons in the Schrödinger equation.

APPENDIX C

Partial wave decomposition

The results of Ref. [8] are extended here to include the spin. The partial wave decomposition of the Weinberg potential appearing in Eq. (8) is defined as follows

$$V(s_1's_2'l'l_z'; s_1''s_2''l''l_z'') = \int d\cos\theta' d\varphi' d\cos\theta'' d\varphi''$$

$$\times Y^*_{l'l_z'}(\theta', \varphi') V(s_1's_2'\theta'\varphi'; s_1''s_2''\theta''\varphi'') Y_{ll}(\theta'', \varphi''). \tag{C.1}$$

The spin labels s_1 and s_2 refer to the spin projection on the light front direction, i.e. they are the spin indices of spinors u_s defined in Eq. (A.4). They are invariant under the relevant boosts (B.2). Therefore, the partial wave decomposition takes the simple form (C.1), although the spherical angles (θ', φ') and (θ'', φ'') are defined in different frames of reference, Eq. (B.3). The Weinberg potential conserves the sum $m = l_z+s_1+s_2 = l_z+s_z$, and the parity of l. For $m = 0$ the Weinberg equation (7) takes the following form

$$M_{l'l}(s's'_zk'; ss_zk) = V_{l'l}(s's'_zk', ss_zk)$$
$$+f^2 \sum_{l''s''s_z''} \int_0^\infty k''^2dk''V_{l'l''}(s's'k'; s''s''_zk'') \cdot G(k', k''; k) \cdot M_{l''l}(s''s''_zk''; ss_zk). \qquad \text{(C.2)}$$

where $G^{-1}(k', k''; k) = -(2\pi)^3\mu^2(k''^2/m-k^2/m-i\delta)$. The integrals over the angles φ'' and φ' in Eq. (C.1) were done analytically, and then the integrals over the angles θ'' and θ' were done numerically, using the Gauss quadratures. The values of the momenta k' and k'' were chosen by the following change of the variables

$$k'' = \kappa(1+x)/(1-x), \qquad x \in]-1, 1[,$$

and using the Gauss quadratures in the x space, up to 32 points. Then the iteration of the kernel matrix in the specified channel was done for the chosen value of $k^2 = \varepsilon^2/4-m\varepsilon$ where ε is a binding energy. In practice, only a few lowest values of l couple, and the values $l = 0, 2, 4$ were sufficient to obtain a very good accuracy. The sufficient number of iterations varied from 10 to 100 and the ratio ϱ of the successive iterations stabilized on six decimal places. The coupling constant $f^2 = \varrho^{-1}$ gives us the pole in the scattering matrix M, corresponding to a bound state, with the binding energy ε.

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